UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
August 2024

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Let X be a Banach space and A, T be bounded linear operators from X to itself. Assume further that A is compact. Show that TA and AT are also compact operators.

2. Let C be a closed convex set in a Hilbert space H. Show that C contains a unique element of minimal norm.

3. For $k \in \mathbb{N}$ let $T_k : \ell^1 \to \ell^\infty$ be the linear operator given by

$$T_k(x_1, x_2, x_3, x_4, \ldots) = \sum_{j=1}^{k} x_j \mathbf{e}_j + \sum_{j=k+1}^{\infty} x_k \mathbf{e}_j$$

where $\mathbf{e}_j = (0, 0, \ldots, 1, \ldots, 0, 0, \ldots)$, i.e. it is the element of ℓ^∞ with a single 1 in the jth position and all other entries are zero.

   (i) Compute $||T_k||$ for each k, and conclude that each T_k is bounded.

   (ii) Prove that the sequence (T_n) is strongly operator convergent.

   (iii) Prove that the sequence (T_n) is not uniformly operator convergent.

4. Let H be a Hilbert space and $A : H \to H$ be a bounded self-adjoint linear operator.

   (a) Prove that

$$||A|| = \sup_{x \neq 0} \frac{|\langle Ax, x \rangle|}{||x||^2}.$$

   (b) Prove that at least one of $||A||$ or $-||A||$ is an element of $\sigma(A)$.

5. Let X be a normed space over $\mathbb{R}$ and $Y \subset X$ be a linear subspace. For $x_0 \in X \backslash Y$ we define the linear functional

$$f : \operatorname{span}\{Y, x_0\} \to \mathbb{R}$$

by

$$f(y + \lambda x_0) = \lambda, \qquad \text{for all } y \in Y, \lambda \in \mathbb{R}.$$

(i) Prove that f is continuous if and only if $x_0 \notin \overline{Y}$.

(ii) Prove that $\overline{Y} = \bigcap\{\operatorname{Ker} g \mid g \in X', Y \subset \operatorname{Ker} g\}$, where X' is the dual of X.

Hint for part (ii): The $\subset$ direction is relatively easy. For the other direction, note that as given in part (i), f satisfies $f|_Y = 0$ and $f(x_0) = 1$. The Hahn-Banach theorem provides an extension of f to the entire space, use this to prove the other direction by contradiction.