Department of Mathematics, University of Utah
Ph.D. Preliminary Examination: Analysis of Numerical Methods, II
Fall 2024

This exam is closed book, closed notes, and no calculators are allowed. You have two hours to complete this exam.

There are 4 problems below. You must complete 3 of them. Each problem is worth 20 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 52 (out of 60) is a *high pass*.
- A score of 48 (out of 60) is a *pass*.

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1. (20 pts) On an equidistant mesh $x_j = jh$, define the operator $\widetilde{D}_0$ as,

$$\widetilde{D}_0 u(x_j) = \frac{u(x_j + h/2) - u(x_j - h/2)}{h},$$

For a given smoooth κ and f, consider the scheme,

$$-\widetilde{D}_0 \left(\kappa(x_j) \widetilde{D}_0 u_j \right) = f(x_j),$$

for the ODE $-\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}u(x)\right) = f(x)$. What order is the local truncation error?

2. (20 pts) Consider the multi-step method,

$$u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} = k\beta_1 f_n + k\beta_2 f_{n-1},$$

where $u_n \approx u(t_n)$, $f_n = f(t_n, u_n)$, $k = t_{n+1} - t_n$, and $u' = f(t, u)$.

(a) (10 pts) Identify the coefficients α, β that yield a scheme of optimal k-order of accuracy, and identify this order of accuracy.

(b) (10 pts) Determine whether or not this scheme is 0-stable and/or A-stable. Is the scheme convergent?

3. (20 pts) Consider the PDE,

$$u_t + a u_x = 0, \qquad u(x, 0) = u_0(x), \qquad a > 0,$$

with periodic boundary conditions over the spatial domain $x \in [0, 1)$. Use the standard notation, $u_j^n \approx u(x_j, t^n)$ with $x_j = jh$ and $t^n = nk$ with $h = \frac{1}{M} > 0$ for some $M \in \mathbb{N}$ and $k > 0$. Consider the following semi-Lagrangian scheme:

$$u_j^{n+1} = p_j(y), \qquad y := X(t^n; x_j, t^{n+1}),$$

where $X(t; x, t_0)$ is the time-t spatial location of the PDE's characteristic curve passing through spatial location x at time t_0, and $p_j(\cdot)$ is the linear interpolant formed by the data (x_{j-1}, u_{j-1}^n) and (x_j, u_j^n).

(a) (10 pts) Write this scheme as an explicit function of the u_j^n values.

(b) (10 pts) Use von Neumann stability analysis to determine a stability condition for this scheme.

4. (20 pts) For the ODE $\boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u})$ with initial condition $\boldsymbol{u}(0)$ and $\boldsymbol{f}$ globally Lipschitz continuous in $\boldsymbol{u} \in \mathbb{R}^M$ uniformly in t, show that forward Euler with initial state $\boldsymbol{u}_0 = \boldsymbol{u}(0)$ is *convergent* to first order. A possibly helpful definition: a scheme is 0-stable if,

$$\max_{n \in [N]} \|\boldsymbol{e}_n\| \leq C \max_{n \in [N]} \|R_n(\boldsymbol{u}(t_n))\|,$$

where $\boldsymbol{e}_n$ is the time-t_n error, and $R_n(\boldsymbol{u}(t_n))$ is the scheme residual using the exact solution.