6520 Fall 2024 Qualifying Exam **Name** ______________________________

ID # u______________________________

Answer at most 5 of the problems below. Each problem is worth 10 points. If you answer more than 5 problems, let me know which 5 you would like me to grade. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a low pass you need to solve *completely* at least two problems and score at least 25 points. Please name any (major) results you use.

Notation: S^n is the n-sphere.

1. Carefully state Van Kampen's theorem. The version for two subsets suffices. Use Van Kampen's theorem to compute $\pi_1(T^2 \vee T^2)$ where T^2 is a torus.

2. Suppose that a complex X is the union of two subcomplexes A and B with A, B, and $A \cap B$ contractible. Prove that X is contractible.

3. (a) Show that $\mathbb{R}\mathrm{P}^2$ has a 2-fold cover which is S^2.

   (b) Show that the Klein bottle has a 2-fold cover which is a torus.

4. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ with no condition on basepoints. Thus there is a natural map $\psi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that ψ is onto if X is path connected.

5. Prove or give a (justified) counterexample. If the first homology group $H_1(X)$ of a connected CW-complex X is trivial then X is contractible.

6. Compute the relative homology groups $H_n(X, A)$ where X is S^2 and A is a finite set of points in X.

7. Show that if $f : S^n \to S^n$ has degree d then the induced map $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

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