UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
August 2024.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. For which $a > 0$ do the surfaces

$$M = \{x^2 + y^2 - z^2 = 1\} \subset \mathbb{R}^3 \text{ and } N = \{x^2 + y^2 + z^2 = a\} \subset \mathbb{R}^3$$

have nonempty transverse intersection?

2. We will identify the set of all real $n \times n$ matrices with $\mathbb{R}^{n^2}$ as usual. Show that $O(n) = \{M \mid M \text{ is an orthogonal } n \times n \text{ matrix}\}$ is a submanifold of $\mathbb{R}^{n^2}$.

3. Let $\omega = dx \wedge dy$ be the 2-form on $\mathbb{R}^3$. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be smooth so that 0 is a regular value. Let $Z = h^{-1}(0)$. Assume that $\frac{\partial h}{\partial z} \neq 0$ at all points of Z. Show that the restriction of ω to Z is a volume form on Z.

4. Let $V, W : \mathbb{R}^4 \to \mathbb{R}^4$ be vector fields on $\mathbb{R}^4$ defined by

$$V(x, y, z, w) = (y, -x, w, -z)$$

and

$$W(x, y, z, w) = (w, z, -y, -x)$$

Is there a nonempty surface $S \subset \mathbb{R}^4$ such that for every $p \in S$ both $V(p)$ and $W(p)$ are tangent to S? Find such a surface or show it does not exist.

5. Let ω be a compactly supported n-form on $\mathbb{R}^n$, $n \geq 1$. Show that the following statements are equivalent:

   (a) $\int_{\mathbb{R}^n} \omega = 0$.

   (b) There exists a compactly supported $(n-1)$-form η on $\mathbb{R}^n$ such that $\omega = d\eta$.

   You are allowed to use computations of de Rham cohomology of Euclidean spaces and spheres but not of the compactly supported cohomology of $\mathbb{R}^n$.

6. Let M be a connected smooth manifold. Show that for all $p, q \in M$ there is a diffeomorphism $f : M \to M$ such that $f(p) = q$.