UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Ordinary Differential Equations

August 16, 2024.

Instructions: This examination consists of seven problems. You are to work four problems. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [25] points. A high-passing score is [75] and a passing score is [60].

Do four problems for full credit

1. [25] Let $(t_0, x_0) \in \Omega$ be a point in an open set $\Omega \subset \mathbf{R} \times \mathbf{R}^d$ and $f : \Omega \to \mathbf{R}^d$ be a continuously differentiable function. Show that for some $\epsilon > 0$ the initial value problem

$$\begin{aligned} \dot{u} &= f(t, u) \\ u(t_0) &= x_0 \end{aligned}$$

has a unique continuously differentiable solution $u(t) : [0, \epsilon] \to \mathbf{R}^d$ such that $(t, u(t)) \in \Omega$ for all $0 \le t \le \epsilon$. [Don't just quote a result. State the theorem and give as complete and detailed a proof as you can.]

2. (a) [2] Let $f : \mathbf{R}^d \to \mathbf{R}^d$ be a continuously differentiable function such that $f(0) = 0$. State the definition: $z(t) = 0$ *is a Liapunov stable solution of* $\dot{u} = f(u)$ *on* $0 \le t < \infty$.

   (b) [15] State conditions on the eigenvalues and eigenvectors of a real $d \times d$ matrix A that hold if and only if $z(t) = 0$ is Liapunov stable solution for $\dot{x} = Ax$ on $0 \le t < \infty$. Prove your conditions.

   (c) [8] Let $A(t)$ be a $T > 0$ periodic $d \times d$ real continuous matrix function. State but don't prove necessary and sufficient conditions on $A(t)$ so that the solution $z(t) = 0$ is Liapunov stable for $\dot{x} = A(t)\, x$ on $0 \le t < \infty$.

3. [25] Let $B(t)$ be a continuous real $d \times d$ matrix function defined for $t \in [0, \infty)$ such that $\int_0^\infty \|B(s)\|\, ds < \infty$. Let A be a real $d \times d$ matrix such that $\Re e\, \lambda < 0$ for all eigenvalues of A. Show that the initial value problem

$$\begin{aligned} \dot{x} &= (A + B(t))x \\ x(0) &= x_0 \end{aligned} \tag{1}$$

has a bounded solution on $[0, \infty)$. [Provide the complete proof. Do not just quote a theorem.

Hint: Use the variation of parameters formula and Gronwall's Inequality. Gronwall's Inequality says that if $c_0 \ge 0$ and if $f(t)$, $\varphi(t)$ are nonnegative continuous functions on the interval $[0, T]$ which satisfy $f(t) \le c_0 + \int_0^t \varphi(s)\, f(s)\, ds$ for all $t \in [0, T]$ then $f(t) \le c_0 \exp\left(\int_0^t \varphi(s)\, ds\right)$ for all $t \in [0, T]$.]

4. (a) [3] For the system $\dot{u} = f(u)$ where f is a continuously differentiable function on $\mathbf{R}^d$, give the definition of an *invariant set*, *attracting set* and ω-limit set.

   (b) [22] Determine the ω-limit sets for this system and prove their existence. Sketch the phase portrait.

$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= -x + y - y^3.\end{aligned}$$

5. Consider Griffith's model for a genetic control system, where x and y are nonnegative concentrations of a protein and messenger RNA, and a and μ are positive constants.

$$\begin{aligned}\dot{x} &= y - \mu x & \qquad x(0) &= x_0 \\ \dot{y} &= a + \frac{x^2}{1+x^2} - y. & \qquad y(0) &= y_0\end{aligned} \tag{2}$$

   (a) [12] Show that the positive quadrant is positively invariant. Show that there is a rectangle in the positive quadrant that is positively invariant.

   (b) [11] Show graphically that there is a region of the a-μ parameter space for which there is a unique stable steady state solution, and a region of parameter space in which there are three steady state solutions. Identify the stability of the solutions. Give some identifying features of these regions.

   (c) [2] What is the bifurcation structure of the solutions as a function of μ?

6. [25] Consider the system

$$\begin{aligned}\dot{x} &= x^2 - xy \\ \dot{y} &= -y + x^2\end{aligned}$$

   Determine the stability at the origin by constructing an approximate center manifold of the form $y = ax^2 + bx^3 + \cdots$ and analyzing the flow on the approximate center manifold. Sketch the phase portrait showing the center and stable manifolds.

7. Let A be a real $d \times d$ matrix. Consider the initial value problem

$$\begin{aligned}\dot{x} &= Ax + h(t) \\ \dot{x}(0) &= x_0\end{aligned} \tag{3}$$

   where $h(t) \in \mathbf{R}^d$ is a real T-periodic continuous function.

   (a) [9] Define the *Poincaré Map* in this context and give an explicit formula for the Poincaré Map in terms of A, h and x_0.

   (b) [8] Under what conditions on A and h is there is a unique T-periodic solution of (3)?

   (c) [8] Compute the derivative of the Poincaré map at the periodic solution. What stability property do you conclude for the periodic solution?