Math 6620 Qualifying Exam, August 2022

Do **any one** of problems 1-2 and **any two** of problems 3-5. Clearly mark the problems you want to be graded. Each complete problem has equal value. A grade of 70% to 85% is a Pass and a grade greater than 85% is a High Pass. No books, notes, or electronic devices may be used during this exam.

1) Suppose $a < b$ and $f \in C^3([a,b])$. Consider the interpolation problem: Find a polynomial $p(x)$ of degree two or less which satisfies the conditions $p(a) = f(a)$, $p'(a) = f'(a)$, and $p(b) = f(b)$.

(i) **Prove** that this problem has a solution and that the solution is unique. (You do not need to find the solution.)

(ii) **Prove** that the error in using $p(x)$ to approximate $f(x)$ for $x \in (a,b)$ is given by the formula

$$f(x) - p(x) = \frac{f^{(3)}(\xi)}{3!}(x-a)^2(x-b), \qquad (1)$$

for some $\xi \in (a,b)$.

2) Suppose you have a program for calculating approximations to a quantity A which takes a value of h as input and produces a value A_h as output. Suppose that

$$A = A_h + a_2 h^2 + a_4 h^4 + a_6 h^6 + O(h^8). \qquad (2)$$

Here, h should be thought of as a small positive number, and a_1, a_2, a_4, and a_6 are fixed nonzero, but otherwise unknown, numbers.

(a) What is the order of the approximation A_h for A?

(b) Use Eq. (2) to derive an approximation B_h to A for which the error $A - B_h = O(h^4)$ and which uses only your program and some simple algebra.

3) Consider the boundary value problem

$$\begin{aligned} u''(x) &= f(x), \qquad \text{for } 0 < x < 1, \\ u(0) &= 0, \qquad u(1) = 0, \end{aligned}$$

where f is a smooth function on $[0, 1]$, and the finite difference scheme

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f(x_j) \qquad j = 1, 2, ..., m,$$

for finding approximations U_j to $u(x_j)$. Here, $x_j = jh$ and $(m + 1)h = 1$.

Analyze the accuracy, stability, and convergence properties of the scheme.

4) For the initial value problem $u'(t) = f(u(t), t)$, $u(0) = \eta$, where $f(u, t)$ is continuous with respect to t and Lipschitz continuous with respect to u, consider the scheme

$$\frac{U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n}{k} = \frac{2}{3} f(U^{n+2}, t_{n+2}),$$

where $t_n = nk$, and U^n is supposed to approximate $u(t_n)$.

(a) Analyze the consistency, zero-stability, and convergence of this scheme.

(b) If you apply this scheme to the initial value problem

$$u'(t) = -10^{12}(u(t) - \cos(t)) - \sin(t), \qquad u(0) = 2,$$

what issues should you consider in choosing the timestep k? Are there time intervals in which k must be very small to get a reasonable solution and others in which it does not? Explain your answers.

5)
a) Consider the initial value problem for the constant-coefficient diffusion equation (with $\beta > 0$)

$$v_t = \beta v_{xx}, \; t > 0$$

with initial data $v(x, 0) = f(x)$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\beta}{h^2} \left\{ u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} \right\}.$$

Analyze the 2-norm stability of this scheme. For which values of $k > 0$ and $h > 0$ is the scheme stable? (Note that there are no boundary conditions here.)

b) Consider the variable coefficient diffusion equation

$$v_t = (\beta(x)v_x)_x, \qquad 0 < x < 1, \; t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \;=\; 0, \qquad v(1,t) \;=\; 0$$

and initial data $v(x, 0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \quad \left\{ \beta_{j-1/2} u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n+1} + \beta_{j+1/2} u_{j+1}^{n+1} \right\}.$$

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem. DO NOT NEGLECT THE FACT THAT THERE ARE BOUNDARY CONDITIONS!