UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination for Math 6410 Ordinary Differential Equations

August 19, 2022.

Instructions:	This examination consists of six problems. You are to work four problems. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [20] points. A passing score is [48], a high pass is [60].

Do four problems for full credit

1. Let $f : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuous function and $x_0 \in \mathbf{R}^d$ and $L \in \mathbf{R}$ be such that

$$|f(t,x) - f(t,y)| \le L|x-y| \qquad \text{for all } x,y \in \mathbf{R}^n \text{ and all } t \in \mathbf{R}.$$

Show that there is $T > 0$ such that the Picard iterates converge to a continuously differentiable solution of the initial value problem

$$\dot{x} = f(t,x)$$
$$x(0) = x_0$$

on $[0,T]$. [Don't just quote a result. State the theorem and give as complete and detailed a proof as you can.]

2. (a) Let $f : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuously differentiable function and $x_0 \in \mathbf{R}^d$ be such that $f(t,x_0) = 0$ for all $t \ge 0$. Define: the constant solution $x(t) = x_0$ on $t \ge 0$ is a *stable* (Liapunov stable) solution of

$$\dot{x} = f(t,x).$$

(b) Determine whether the zero solution $z(t) = 0$ is stable for

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & h(t) \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where

$$h(t) = \frac{\cos t + \sin t}{2 + \sin t - \cos t}.$$

[Hint: the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ae^t - b(2+\sin t) \\ b(2+\sin t - \cos t) \end{bmatrix}$$

where a and b are constants.]

3. Let A be a real matrix such that $\Re e\,\lambda < 0$ for all eigenvalues of A.

   (a) Show that there is $\delta > 0$ such that if $|x_0| \le \delta$, then the initial value problem

$$\begin{aligned} \dot{x} &= Ax + e^{-t}|x|x \\ x(0) &= x_0 \end{aligned} \tag{1}$$

   has a bounded solution on $[0, \infty)$. [Provide the complete proof. Do not just quote a theorem.]

   (b) Is the zero solution of (1) asymptotically stable? Explain.
   [Hint: let $f(t)$, $\varphi(t)$ be nonnegative continuous functions on the interval $J = (\alpha, \beta)$ containing t_0. Let $c_0 \ge 0$. Gronwall's Lemma says that if $f(t) \le c_0 + \left| \int_{t_0}^{t} \varphi(s)\, f(s)\, ds \right|$ for all $t \in J$ then $f(t) \le c_0 \exp \left| \int_{t_0}^{t} \varphi(s)\, ds \right|$ for all $t \in J$.]

4. Describe the bifurcations that occur in the equation as the parameter $a \in \mathbf{R}$ increases.

$$\ddot{x} + (x^2 + \dot{x}^2 - a)\dot{x} + x = 0$$

5. Let $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ have a non-constant, $T > 0$ periodic trajectory $\gamma(t)$ satisfying $\dot{\gamma}(t) = f(\gamma(t))$.

   (a) [1] Define: the periodic solution $\gamma(t)$ is *orbitally stable.*

   (b) [2] Define: the *Poincaré Map* for the orbit γ.

   (c) [17] $\gamma(t) = (2\cos 2t, \sin 2t)$ is a periodic solution to

$$\begin{cases} \dot{x} = -4y + x\left(1 - \dfrac{x^2}{4} - y^2\right), \\ \dot{y} = x + y\left(1 - \dfrac{x^2}{4} - y^2\right). \end{cases}$$

   Determine the orbital stability of $\gamma(t)$ by computing the derivative of its Poincaré Map.

6. Find the stable/center/unstable manifolds of the system

$$\begin{aligned} \dot{x} &= -xy \\ \dot{y} &= -y + x^2 - 2y^2. \end{aligned}$$