

University of Utah, Department of Mathematics
Algebra 2 Qualifying Exam
August 2022

There are five problems on this exam. You may attempt as many problems as you wish; two correct solutions count as a *pass*, and three correct as a *high pass*. Show all your work, and provide reasonable justification for your answers.

1. Let G be a group, and N the subgroup generated by all elements of the form g^2 for $g \in G$. Prove that N is a normal subgroup containing the commutator subgroup of G .
2. Prove that each 7-Sylow subgroup of the symmetric group S_{15} is abelian.
3. Let K be the splitting field of $x^4 - 2x^2 + 4$ over \mathbb{Q} . Determine $\text{Gal}(K/\mathbb{Q})$.
4. Determine the splitting field and the Galois group of $x^9 + 1$ over \mathbb{F}_2 .
5. Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$, with splitting field K . Suppose $\text{Gal}(K/\mathbb{Q})$ is abelian. Prove or disprove: $K = \mathbb{Q}(\alpha)$ for each root α of $f(x)$ in K .