Qualifying Exam
Analysis of Numerical Methods I, August 2021

Instructions: This exam is closed book, no notes, and no electronic devices or calculators are allowed. You have two hours and you will be graded on work for only 3 out of the 4 questions below. All questions have equal weight and a cumulative score of 65% or more on your 3 graded questions is considered a pass. A cumulative score of 80% or greater is considered a high pass. Indicate clearly the work and the 3 questions that you wish to be graded.

Problem 1. (Matrix properties)
For each of the following statements, **either** prove that it is true, **or** show it's false with a counterexample.

a. For $A \in \mathbb{C}^{m\times n}$ any matrix, and $U \in \mathbb{C}^{m\times m}$ any unitary matrix, then $\|UA\|_2 = \|A\|_2$.

b. For $A \in \mathbb{C}^{n\times n}$ any matrix, if $\rho(A) = 0$, then $A = 0$. (Recall that $\rho(A)$ is the spectral radius of A, i.e., the maximum of the moduli of the eigenvalues of A.)

c. For $A \in \mathbb{C}^{m\times n}$ any matrix with $m \geq n$, then the squared singular values of A equal the eigenvalues of A^*A.

Problem 2. (Moore-Penrose pseudo-inverse)
Let $A \in \mathbb{C}^{m\times n}$ have rank r and *reduced* SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$$

The *Moore-Penrose pseudoinverse* of A is defined as

$$A^+ = \widetilde{V}\widetilde{\Sigma}^{-1}\widetilde{U}^*$$

a. In the induced matrix ℓ^2 norm $\|\cdot\|$, is the operation $A \mapsto A^+$ well-conditioned? That is, given an arbitrary but fixed A and a perturbation matrix B, is $||(A+B)^+ - A^+||/||A^+||$ controllable by $||B||/||A||$? Prove it, or give a counterexample.

b. If $m \geq n$ and A has full rank n and $b \in \mathbb{C}^m$ is a given vector, prove that $x = A^+b$ is identical to the least-squares solution z of the over-determined linear system $Az = b$.

Problem 3. (Finite difference formulas)
Given $h > 0$, compute weights $\{w_j\}_{j=0}^2$ for the following one-sided finite difference formula for the first derivative $f^{'}(x)$:

$$\begin{aligned} f'(x) &\approx \sum_{j=0}^{2} w_j f(x + jh) \\ &= w_0 f(x) + w_1 f(x+h) + w_2 f(x+2h) \end{aligned}$$

What is the order of accuracy of your formula?

Problem 4. (Quadrature rules)
Compute the weights $\{A_0, A_1, B_0, B_1\}$ for a 4-point quadrature rule of the form

$$\int_0^1 f(x)dx \approx A_0 f(0) + A_1 f(1) + B_0 f'(0) + B_1 f'(1)$$

that is exact when f is any polynomial of degree 3 or less.