UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Qualifying Examination for Differential Equations,

August 2021

Instructions: This examination has five problems of which you are expected to work three. If you work more than the three required number of problems, then state which problems you wish to have graded. In order to receive maximum credit, solutions to problems must be clearly and carefully presented. All problems are worth 30 points each. A passing score is 70% = 63/90 points.

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1. Consider the differential equation $\frac{dx}{dt} = f(x,t)$, $x \in \mathcal{R}^n$, with initial condition $x(0) = x_0$.
   (a) State a theorem, with conditions on $f(x,t)$, guaranteeing the existence and uniqueness of solutions of the differential equation on some interval $0 < t < T_0$.
   (b) Outline the proof of your theorem, indicating the main steps of the proof (without all the details), indicating where your stated assumptions on $f(x,t)$ are invoked. What are Picard iterates and how do they play a role in your proof?

2. Consider the differential equation

$$\ddot{x} + \eta(x)\dot{x} + U'(x) = 0, \quad \eta(x) > 0.$$

   Suppose that $U(x) \to \infty$ as $x \to \pm\infty$. Find a Lyapunov function for this equation and use this to show that
   (a) there are no periodic solutions.
   (b) minima of $U(x)$ are asymptotically stable steady solutions.
   (c) Under what conditions on $U(x)$ is a minimum of $U(x)$ exponentially stable?

3. Suppose $A(t)$ is a periodic function, $A(t+1) = A(t)$ and $0 < A(t) < 1$. Prove that the differential equation $u' = u(A(t) - u)$ has a unique, positive, stable, periodic solution. Prove that the trivial solution $u(t) = 0$ is unstable. Hint: Use the Poincare map.

4. (a) For a general dynamical system, give the definitions of an *invariant set*, *attracting set* and ω*–limit sets*.
   (b) For the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3,$$

   identify all critical points and their stability, and use the Poincare-Bendixson theorem to show that this system has a periodic orbit in the annular region $1 < x^2 + y^2 < 2$.

5. (a) What are necessary conditions on the parameters a and b so that the boundary value problem

$$u'' + u = \cos(x), \quad u'(0) = a, \quad u'(\pi) = b$$

   has a solution?
   (b) What are the eigenvalues and eigenfunctions for the operator $Lu = u'' + u$ with homogeneous boundary conditions $u'(0) = 0$, $u'(\pi) = 0$? Show that the eigenfunctions for this operator are the same as those for $Lu = u'' - u$ with homogeneous boundary conditions $u'(0) = 0$, $u'(\pi) = 0$. Outline the proof that these eigenfunctions form a complete set. You may invoke a theorem associated with symmetric, compact, linear operators. Be sure to specify the function space on which this result applies.