Linear models (MATH 6010) qualifying exam August, 2021

The value of each question is 20 points. You need to collect 80-90 points for a pass and 91 and above for a high pass. You can try as many problems as you wish.

(1) Let $\rho(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$ be the L^1 norm of $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$. Show that the L^1 norm is not induced by an inner product. (20 points)

(2) Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix and $\mathbf{a} \in R^n$.
Find the necessary and sufficient condition that $(\mathbf{A} + \mathbf{a}\mathbf{a}^\top)^{-1}$ exists. (20 points)
($\cdot^\top$ denotes the transpose of matrices).

(3) Let X_1 and X_2 be jointly normal, with $EX_1 = EX_2 = 0$, $\mathrm{var}(X_1) = \sigma_1^2$, $\mathrm{var}(X_2) = \sigma_2^2$. The correlation between them is ρ.
(i) Find $E(2X_1 + X_2)^2$. (5 points)
(ii) Find $E(X_1^2 X_2^2)$ (5 points)
(iii) Find $E(X_1^4 (2X_1 + X_2)^2)$ (10 points)

(4) We consider the usual linear model $y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \epsilon_i, 1 \le i \le n$. Let $\hat{y}_i, 1 \le i \le n$ and $\hat{\epsilon}_i, 1 \le i \le n$ be the fitted values and the residuals. Assume that the errors are iid normal with mean 0 and variance σ^2.
(i) Compute the joint distribution of the vector of the fitted values $(\hat{y}_1, \ldots, \hat{y}_n)^\top$. (10 points)
(ii) Compute the joint distribution of the vector of the residuals $(\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)^\top$. (10 points)

(5) We consider the model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathcal{E},$$

the coordinates of $\mathcal{E}$ are independent identically distributed normal with zero mean and variance σ^2, $\mathbf{X}$ has full rank, $\mathbf{Y} \in R^n, \boldsymbol{\beta} \in R^p$. We wish to test the null hypothesis

$$\mathbf{A}\boldsymbol{\beta} = \mathbf{c},$$

$\mathbf{A}$ is $q \times p$ with rank q, $\mathbf{c} \in R^q$. Let $\hat{\boldsymbol{\lambda}}_H$ be the estimator for the Lagrange multiplier.
(i) Compute the distribution of $\hat{\boldsymbol{\lambda}}_H$ under the null hypothesis. (10 points)
(ii) Find a matrix $\mathbf{W}$ such that

$$\frac{\hat{\boldsymbol{\lambda}}_H^\top \mathbf{W} \hat{\boldsymbol{\lambda}}_H}{S^2}$$

has an F distribution, where

$$S^2 = \frac{1}{n-p} \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|^2$$

is the estimator for σ^2, $\hat{\boldsymbol{\beta}}$ is the least square estimator. (10 points)

(6) We consider the simple linear regression

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad 1 \le i \le n.$$

We estimate $(\beta_0, \beta_1)^\top$ with the least squares $(\hat{\beta}_0, \hat{\beta}_1)^\top$.
(i) Find the necessary and sufficient condition that $\hat{\beta}_0$ and $\hat{\beta}_1$ are independent. (2 points)
(ii) We wish to test $\beta_0 = \beta_0^*$ and $\beta_1 = \beta_1^*$, where β_0^* and β_1^* are given numbers. We only want to reject if BOTH assumptions are violated, i.e. $\beta_0 \neq \beta_0^*$ AND $\beta_1 \neq \beta_1^*$. Find a test where you can explicitly compute the critical values (you might want to use part (i) of this question). (18 points)