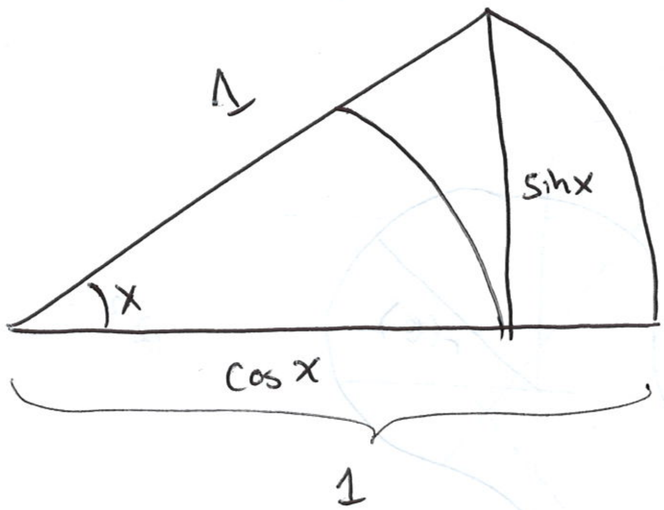
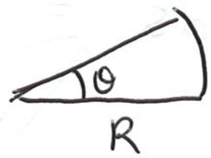
UMPSC Set 1

SOLN IDEAS

(1) Consider a wedge of the unit circle



Note that  has area $\frac{\theta}{2\pi}(\pi R^2)$

so, by the above figure,

$$\frac{x}{2\pi}(\pi \cos^2 x) \le \frac{1}{2} \sin x \cos x \le \frac{x}{2\pi}(\pi 1^2)$$

divide both sides by $\frac{1}{2} x \cos x$ and apply squeeze theorem.

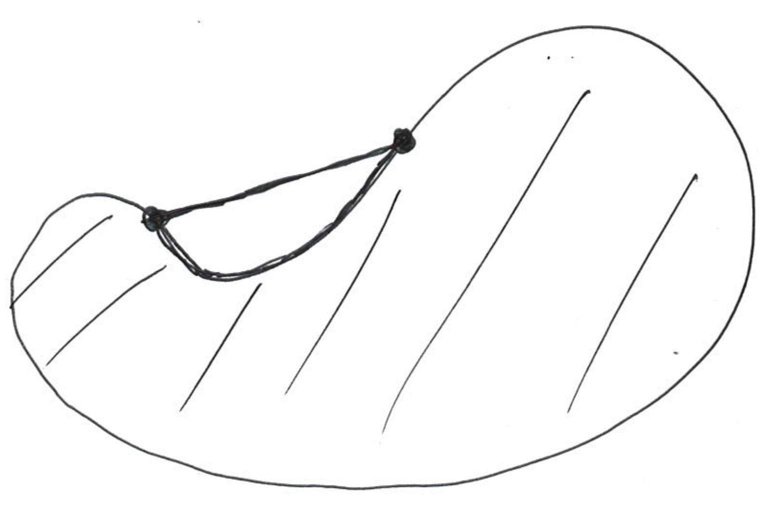
∎

PROBLEM 2

Step 1. (used in Both proofs).

The shape is convex.

If one can find a line between two points on the perimeter of the shape which does not belong to the interior of the shape, a more optimal shape may be formed by replacing the inner section of the perimeter by this line (see figure)



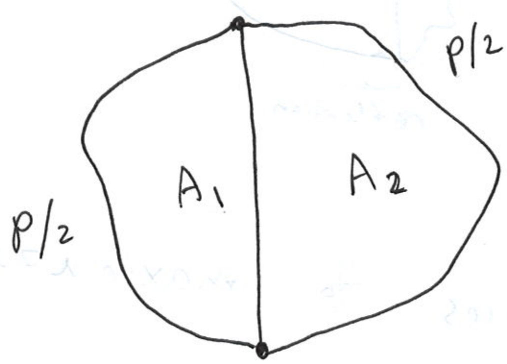
this is because a line minimizes distance between two points, so replacing the relevant section of the perimeter by this line will decrease total perimeter whilst increasing total area.

Technically, we should invoke the Jordan Curve theorem to describe the "inner" and "outer" regions of the shape, and put an orientation on the boundary. Then, if a line from point A to point B lies in the outer region, the section of the perimeter passing from A to B (notice this has an orientation and is therefore well defined) can be replaced by such a line to create a more optimal shape, proving that convexity is a necessary condition.

Proof 1.

Geometry. There is a reflective symmetry of the optimal shape in the following sense:

If one considers the line passing through two points which divides the perimeter,

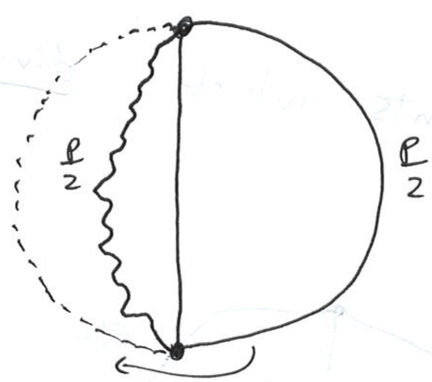


This line (by convexity) will cut the shape into two pieces of equal perimeter. If the area of the two pieces is not equal,

$$A_1 < A_2 ,$$

one can reflect the bigger piece across this line and create a more optimal shape.

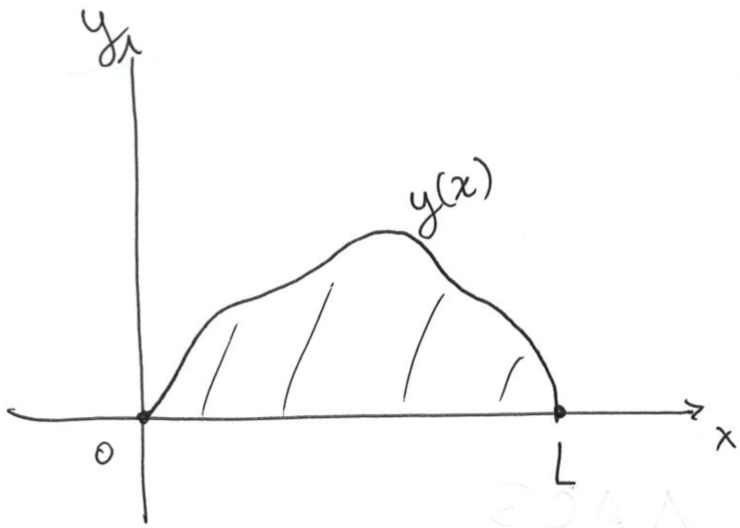
(as perimeter is preserved)



Hence it suffices to maximize the area underneath a nonnegative curve $y(x)$ of fixed arc length with boundary conditions

$$y(0) = y(L) = 0,$$

where L is <u>NOT</u> necessarily fixed.

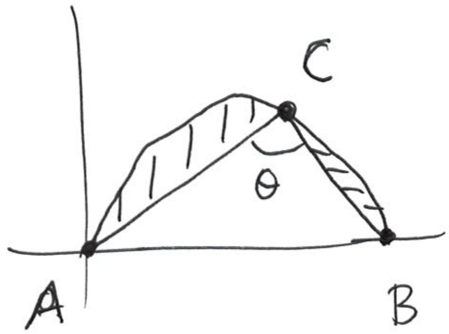


One can formulate the problem as

$$\max_{y(x)} \int_0^L y(x)\,dx \quad \Bigg| \quad \frac{P}{2} \equiv \int_0^L \sqrt{1+(y')^2}\,dx$$

and solve using Calculus of Variations (Lagrange multipliers), but we will not pursue this here.

Instead, consider the following geometric construction:



consider the triangle ΔACB

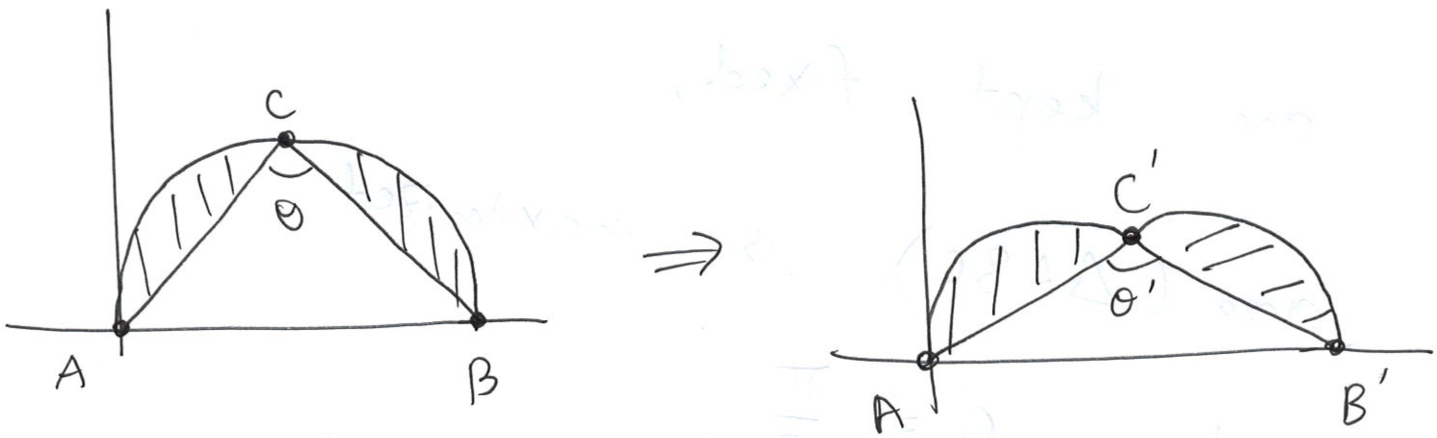
formed by the points $(0,0) = A$

$(L,0) = B$

$(x, y(x)) = C$

where $x \in (0, L)$.

This triangle lies within the shape by convexity, and cuts the half shape shown above into three pieces (the triangle and the two shaded regions)



Treat the shaded regions as solid, rigid objects, and the point C as a "hinge" which can be adjusted (by changing $\theta := \measuredangle ACB$) to maximize the area of ΔACB whilst keeping the perimeter and area of shaded regions fixed (see figure above). The area of ΔABC is given by

$$\text{area}(\Delta ABC) = |\overline{AC}||\overline{BC}| \sin(\theta)$$

Since $|\overline{AC}|$ and $|\overline{BC}|$

are kept fixed,

area (ΔABC) is maximized

when $\theta = \frac{\pi}{2}$,

and since C was chosen

arbitrary, we conclude the

optimal shape is the set

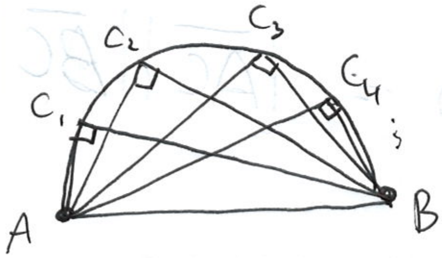
of points $\mathbb{C} = \{ (x, y(x)) \mid \angle ACB = 90^\circ \}$

which form right angles with respect

to A, B, and this is

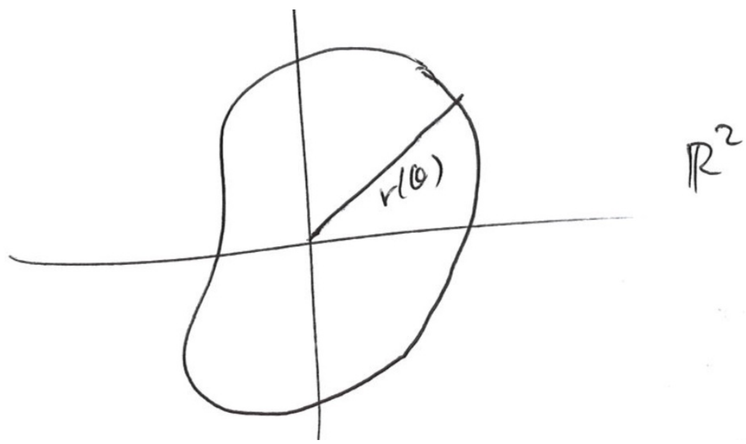
precisely the boundary of a

semi circle. $\square$



Problem 2 (Variational Method)

Consider a "shape" to be a simply connected region of $\mathbb{R}^2$, with coordinates chosen such that the origin belongs to the interior of the shape, and with a perimeter described by a function $r(\theta)$. Here we invoke the convexity of the shape to construct such a coordinate system, since otherwise $r(\theta)$ might not be well defined:



In polar coordinates, the differential arc length of the boundary is given by

$$ds^2 = r^2 d\theta^2 + dr^2$$

Hence the perimeter is

$$P \equiv \oint ds = \oint \sqrt{r^2 d\theta^2 + dr^2} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The area is given by

$$A = \int_R dA = \int_0^{2\pi} \int_0^{r(\theta)} \rho\, d\rho\, d\theta = \frac{1}{2} \int_0^{2\pi} r^2(\theta) d\theta$$

Therefore, we want $r(\theta)$ to solve the variational problem

$$\max\left\{\int r^2 d\theta\right\} \quad \Big| \quad \int \sqrt{r^2 + (r')^2} d\theta \equiv \text{const}$$

This problem is equivalent to the problem

$$\min\left\{\int \sqrt{r^2 + (r')^2} d\theta\right\} \quad \Big| \quad \int r^2 d\theta \equiv \text{const}$$

That is, the shape which maximizes area given a fixed perimeter coincides with the shape that minimizes perimeter given a fixed area. Now we solve this variational problem using calculus, specifically the method of Lagrange Multipliers.

The Lagrangian is given by

$$\mathscr{L}(r,r';\theta) = \sqrt{r^2+(r')^2} - \lambda r^2$$

Noting that Lagrangian does not depend explicitly on θ, we obtain first integral:

$$0 = \frac{d}{d\theta}\left(r'\frac{\partial \mathscr{L}}{\partial r'} - \mathscr{L}\right) = \frac{d}{d\theta}\left(\frac{(r')^2}{\sqrt{r^2+(r')^2}} - \sqrt{r^2+(r')^2} + \lambda r^2\right)$$

$$0 = \frac{d}{d\theta}\left(\frac{-r^2}{\sqrt{r^2+(r')^2}} + \lambda r^2\right)$$

Expanding the derivative, we find

$$0 = \frac{(-2rr')(r^2+(r')^2) + r^2(rr' + r'r'')}{(r^2+(r')^2)^{3/2}} + 2\lambda r r'$$

$$2\lambda = \frac{(2)(r^2+(r')^2) - r(r+r'')}{(r^2+(r')^2)^{3/2}} = \frac{2(r')^2+r^2-rr''}{(r^2+(r')^2)^{3/2}}$$

Here we identify the expression on the right as the curvature κ of a plane curve in polar coordinates, that is,

$$\kappa(\theta) = \frac{2(r')^2+r^2-rr''}{(r^2+(r')^2)^{3/2}} = 2\lambda$$

Thus our analysis shows the shape which solves the variational problem has constant curvature. We recover the circle. $\boxslash$

Problem 3

This problem is difficult and there are many possible ways to arrive at a solution. I will develop some theory that will make the problem much easier, following the textbook by J. Michael Steele, "An Introduction to Stochastic Calculus with Financial Applications", which is in my opinion a literary masterpiece.

Define the random variable X_i as the monetary outcome of the i^{th} coin flip:

$$P(X_i = 1) = 1/2, \quad P(X_i = -1) = 1/2$$

The gambler's bank account value after n coin flips may be described by the random variable S_n:

$$S_n = S_0 + \sum_{i=1}^{n} X_i$$

Where S_0 is the initial value of the bank account. Note that $S_n - S_0$ is the net profit after n coin flips.

Assume that the game stops when the gambler's bank account reaches the value $S_n = A$ or $S_n = -B$. Define the "stopping time", T, as follows:

$$T = \min\{n \geq 0 : S_n = A \text{ or } S_n = -B\}$$

(In the specific case of part (a) of this problem, $S_0 = 0$, $A = 1$ and $B = \infty$. But, following J. Michael Steele, I will solve the more general problem first.)

We are concerned about the expected value of T, which can be interpreted as how long the game will take, on average, to finish. Define a function $g(k)$ to measure the expected value of T given that the gambler starts with $S_0 = k$ dollars:

$$g(k) = \mathbb{E}[T \mid S_0 = k]$$

After the first coin flip, time will increase by 1, and the gambler's bank account value will have increased or decreased by 1 (each with probability 1/2), so we obtain the following recurrence relation:

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1 \tag{1}$$

Also, we assume the game stops when $S_n = A$ or $S_n = -B$, which gives the following boundary condition:

$$g(A) = g(-B) = 0$$

This is a linear difference equation, and in fact there is a constant second difference. To see this, define the forward difference operator Δ:

$$\Delta(g(k-1)) = g(k) - g(k-1)$$

$$\Delta^2(g(k-1)) = g(k+1) - 2g(k) + g(k-1)$$

With this notation, equation (1) becomes

$$\frac{1}{2}\Delta^2(g(k-1)) = -1$$

In the continuous case, the solution to this differential equation is a quadratic polynomial, with roots given by the boundary conditions at $A, -B$. In fact, the solution in the discrete case is identical with these boundary conditions:

$$g(k) = -(k-A)(k+B)$$

We desire the expected length of the game when $S_0 = 0$, which is namely

$$\boxed{g(0) = \mathbb{E}[T \mid S_0 = 0] = AB}$$

Now lets solve part (a) and (c) of the problem. Notice that, for all $B > 0$,

$$\mathbb{E}\left[\min\{n \geq 0 : S_n = 1\}\right] \geq \mathbb{E}\left[\min\{n \geq 0 : S_n = 1 \text{ or } S_n = -B\}\right] = B$$

Thus

$$\mathbb{E}\left[\min\{n \geq 0 : S_n = 1\}\right] = \infty$$

Part (c) is even more immediate; we obtain

$$\mathbb{E}\left[\min\{n \geq 0 : S_n = 1500 \text{ or } S_n = -1500\}\right] = (1500)^2$$

For part (b), there is no definitive best strategy. The question states that we must spend a total of \$100 on coin flips. Clearly we should not spend more than this, since we are playing a losing game. Any betting strategy will have an equal expected profit, namely $-\$4$, regardless of the strategy, since the expectation value is a linear operator and the coin has a fixed win probability. Risky strategies, ones which gamble a lot of money on a single flip, are much more likely to make net positive profit since they have a higher variance about the mean of $-\$4$. On the other hand, these strategies are also more likely to lose big.

There is another, more nuanced question. Lets restrict our attention back to \$1 coin flips, and suppose our gambler is stubborn : they refuse to leave the game until losing or winning a NET amount of \$100. The coin is now unfair, with probability of heads $0 < p < 1$, and probability of tails $q = 1 - p$. For now, only consider 1 dollar bets as before.

Recall that T is the stopping time when the gambler reaches bank account value of A or $-B$. We are now concerned with the probability that, when the game stops, the gambler's bank account reaches A dollars (instead of $-B$, which is the only other possibility). Define

$$f(k) = \mathbb{P}(S_T = A \mid S_0 = k)$$

Using similar logic, we look at what happens after one coin flip to build a recurrence relation. After one flip, there is a probability p that the gambler's bank account increases by 1 dollar, and probability q that the gambler's bank account decreases by 1 dollar.

$$f(k) = pf(k+1) + qf(k-1)$$

With boundary condition

$$f(A) = 1, \quad f(-B) = 0$$

Rearranging this equation, and using the fact that $q = 1 - p$, we find

$$0 = p(f(k+1) - f(k)) + q(f(k-1) - f(k))$$

So

$$\Delta f(k) = \frac{q}{p}\Delta f(k-1)$$

Iterating this equation, we find

$$\Delta f(k+j) = \left(\frac{q}{p}\right)^j \Delta f(k)$$

Since $f(-B) = 0$,

$$f(k) = \sum_{j=0}^{k+B-1} \Delta f(j) = \sum_{j=0}^{k+B-1} \left(\frac{q}{p}\right)^j \Delta f(-B) = \frac{1-(q/p)^{k+B}}{1-(q/p)}\Delta f(-B)$$

Since $f(A) = 1$, we find

$$1 = f(A) = \frac{1-(q/p)^{A+B}}{1-(q/p)}\Delta f(-B)$$

so

$$\Delta f(-B) = \frac{1-(q/p)}{1-(q/p)^{A+B}}$$

Finally we have solved for $f(k)$, which is

$$f(k) = \frac{1-(q/p)^{k+B}}{1-(q/p)}\left(\frac{1-(q/p)}{1-(q/p)^{A+B}}\right) = \frac{1-(q/p)^{k+B}}{1-(q/p)^{A+B}}$$

Our desired probability is therefore

$$\boxed{f(0) = \mathbb{P}(S_T = A | S_0 = 0) = \frac{1-(q/p)^B}{1-(q/p)^{A+B}}}$$

As J. Michael Steele remarks, "This formula would transform the behavior of millions of the world's gamblers, if they could only take it to heart. "

In the case of part (b), $p = 0.48$, and we want to determine that optimal betting strategy. All of the above analysis was done assuming the betting strategy of 1 dollar coin flips. According to the above formula, the probability that the gambler wins \$100 before losing \$100 is

$$\frac{1-(.52/.48)^{100}}{1-(.52/.48)^{200}} \approx \frac{3.34}{10000}$$

If all of the money is bet on a single coin flip, the probability that the gambler wins \$100 before losing \$100 is 0.48. The difference is monumental.

So for our stubborn gambler, it is best to take the most risk.