SYMPLECTIC AND ORTHOGONAL ROBINSON-SCHENSTED ALGORITHMS

PETER E. TRAPA

1. Introduction

Let $G_{\mathbb{R}}$ denote a linear reductive real Lie group with maximal compact subgroup $K_{\mathbb{R}}$. Write $\mathfrak{g}$ and $\mathfrak{k}$ for the corresponding complexified Lie algebras and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. Let $\mathfrak{B}$ denote the flag variety for $\mathfrak{g}$. The points of the cotangent bundle $T^*\mathfrak{B}$ can be thought of as pairs consisting of a Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and a covector $\xi \in \mathfrak{n}^*$. The projection $\mu : (\mathfrak{b}, \xi) \mapsto \xi$ from $T^*\mathfrak{B}$ to the nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$ is the moment map for the G action on T^*B, and is the famous Grothendieck-Springer resolution of $\mathcal{N}(\mathfrak{g}^*)$.

Now consider an orbit of the complexification K of $K_{\mathbb{R}}$ on $\mathfrak{B}$, say Q, and its conormal bundle $T^*_Q(\mathfrak{B})$. Because Q is a K orbit, the image $\mu(T^*_Q\mathfrak{B})$ is contained in $\mathfrak{p}^*$. Since μ is equivariant and proper and since $T^*_Q\mathfrak{B}$ is K invariant and irreducible[1], $\mu(T^*_Q\mathfrak{B})$ is an irreducible K invariant subvariety of $\mathcal{N}(\mathfrak{p}^*)$. Since there are only a finite number of K orbits on $\mathcal{N}(\mathfrak{p}^*)$, $\mu(T^*_Q\mathfrak{B})$ is the closure of a single K orbit on $\mathcal{N}(\mathfrak{p}^*)$.

In Proposition 3.3.1, we give a simple algorithm to compute the moment map images $\mu(T^*_Q\mathfrak{B})$ explicitly for the groups $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ and $O(p,q)$. In Proposition 3.4.1, by analyzing the intersection of $T^*_Q\mathfrak{B}$ with the fiber of μ over a generic point ξ of the image, we obtain a new parametrization of the orbits of $A_G(\xi)$ on the irreducible components of the Springer fiber $\mu^{-1}(\xi)$ in terms of domino tableaux; here $A_G(\xi)$ is the component group of the centralizer of ξ in G. We then show (Proposition 3.5.1) that this parametrization is closely related to the computations of annihilators of derived functor modules for the groups under consideration.

Previously the moment map computations for $Sp(2n, \mathbb{R})$ and $O(p,q)$ were treated by Yamamoto ([Ya1]–[Ya2]). Her algorithms are significantly different from ours. In particular, they are not well suited for the analysis of the components of the Springer fiber.

Our computations have very nice combinatorial interpretations as generalizations of the classical Robinson-Schensted algorithm. As explained in Proposition 2.6.1, which I learned from lectures of Springer and which applies to general $G_{\mathbb{R}}$, the map

$$Q \mapsto \left(\mu(T^*_Q(\mathfrak{B})), \overline{T^*_Q(\mathfrak{B})} \cap \mu^{-1}(\xi)\right)$$

is bijective. When $G_{\mathbb{R}} = GL(n, \mathbb{C})$, this bijection indeed reduces to the Robinson-Schensted algorithm ([St]). For the classical groups we consider, the domain can be parametrized in terms of involutions in a symmetric group with certain signs attached to the fixed points of the involutions. As remarked above, the first component in the image is a nilpotent K orbit on $\mathfrak{p}^*$, and in our case can be parametrized by signed tableaux. Meanwhile the second

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[1] Actually, if K is disconnected, then $T^*_Q(\mathfrak{B})$ need not be irreducible, but this doesn't affect the foregoing discussion in an essential way.

component, can be parametrized by domino tableaux. Thus the bijection has very much the same combinatorial flavor of the classical Robinson-Schensted algorithm. In fact the analogy can be made more precise from a purely combinatorial perspective, and can be seen as a generalization of a symmetry of the classical Robinson-Schensted algorithm first observed by Schützenberger. This viewpoint plays a key role in our proofs in Section 3.

2. Preliminaries

2.1. **General notation.** Given a linear real reductive group $G_{\mathbb{R}}$ with Cartan involution θ, we set $K_{\mathbb{R}} = G_{\mathbb{R}}^{\theta}$, write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}_{\mathbb{R}}$ for the corresponding Lie algebras, $\mathfrak{g}$ and $\mathfrak{k}$ for their complexifications, and write G and K for the corresponding groups. The complexified Cartan decomposition is denoted $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We write $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}^*$ (or, using the trace form, $\mathfrak{g}$), and set $\mathcal{N}(\mathfrak{p}) = \mathcal{N} \cap \mathfrak{p}$.

We write $\mathfrak{B}$ for the variety of Borel subalgebras in $\mathfrak{g}$, and choose a basepoint $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \in \mathfrak{B}$. Let W denote the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. Write $\mu : T^*\mathfrak{B} \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g}$ for the moment map of the G-action on the cotangent bundle to $\mathfrak{B}$. For a subvariety $Q \subset \mathfrak{B}$, we write $T_Q^*\mathfrak{B}$ for the conormal bundle to Q. Given $N \in \mathcal{N}$, we let $\mu^{-1}(N)$ for the fiber of μ over N; it consists of the Borel subalgebras in $\mathfrak{B}$ containing N. To emphasize this, we may also write $\mathfrak{B}^N$ in place of $\mu^{-1}(N)$.

For $N \in \mathcal{N}(\mathfrak{p})$, we write $A_G(N)$ for the component group of the centralizer of N in G, and write $A_K(N)$ for the component group of the centralizer in K. Clearly $A_K(N)$ maps to $A_G(N)$, and both groups act on $\mathrm{Irr}(\mathfrak{B}^N)$, the irreducible components of $\mathfrak{B}^N$.

2.2. **Tableaux.** We adopt the standard (English) notation for Young diagrams and standard Young tableaux of size n. We let $\mathrm{YD}(n)$ denote the set of Young diagrams of size n, and $\mathrm{SYT}(n)$ the set of standard Young tableaux of size n. Write $\mathrm{RS}(w)$ for the right (or '$Q-$' or 'counting') standard Young tableaux of size n that the Robinson-Schensted algorithm attaches to an element w of the symmetric group S_n.

A standard domino tableau of size $2n$ is a Young diagram of size $2n$ which is tiled by two-by-one and one-by-two dominos labeled in a standard configuration; that is, the tiles are labeled with distinct entries $1, \ldots, n$ so that the entries increase across rows and down columns. A Young diagram of size $2n$ which admits such a tiling is called a domino shape. A Young diagram of size $2n{+}1$ is called a domino shape if after removing its upper-left box, it admits such a standard tiling.

We let $\mathrm{SDT}_C(2n)$ (resp. $\mathrm{SDT}_D(2n)$) denote the set of standard domino tableau of size $2n$ whose shape is that of a nilpotent orbit for $Sp(2n,\mathbb{C})$ (resp. $O(2n,\mathbb{C})$; i.e. whose odd (resp. even) parts occur with even multiplicity. Finally, we define $\mathrm{SDT}_B(2n{+}1)$ to be the set of Young diagrams of size $2n+1$ and shape of the form of a nilpotent orbit for $O(2n{+}1,\mathbb{C})$ (i.e. even parts occur with even multiplicity), whose upper left box is labeled 0, and whose remaining $2n$ boxes are tiled by dominos labeled $1, \ldots n$ in a standard configuration.

An element of $\mathrm{SDT}_C(2n)$ has special shape if the number of even parts between consecutive odd parts or greater than the largest odd part is even. An element of $\mathrm{SDT}_D(2n)$ (resp. $\mathrm{SDT}_B(2n{+}1)$) has special shape if the number of odd rows between consecutive even rows is even and the number of odd rows greater than the largest even row is even (resp. odd).

A signed Young tableau of signature (p, q) is an arrangement of p plus signs and q minus signs in a Young diagram of size $p{+}q$ so that the signs alternate across rows, modulo the equivalence of interchanging rows of equal length. We denote the set of signature (p, q) signed tableau by $\mathrm{YT}_{\pm}(p, q)$.

2.3. **Evacuation.** We briefly recall Schützenberger's shape-preserving evacuation algorithm,

$$\mathbf{ev} \ : \ \mathrm{SYT}(n) \longrightarrow \mathrm{SYT}(n);$$

see [Sa, Chapter 3.11], for instance, for more details. Given $T \in \mathrm{SYT}(n)$, begin by interchanging the index 1 with the index immediately to its right or immediately below it according to which index is smaller. By successively repeating this procedure, the index 1 eventually ends up in a corner of T. Begin to build a new tableau $\mathbf{ev}(T)$ (the evacuation of T) of the same shape as T, by entering the index n in the (corner) location occupied by 1. Now remove 1 from the shuffled T, repeat the shuffling procedure, and enter $n-1$ in $\mathbf{ev}(T)$ according to the ultimate location that 2 occupies in the current rearrangement of T. Repeating this procedure defines $\mathbf{ev}(T) \in \mathrm{SYT}(n)$. As a consequence of Proposition 2.3.1 below, $\mathbf{ev}$ is an involution on $\mathrm{SYT}(n)$.

A tableau $T \in \mathrm{SYT}(n)$ is called self-evacuating if $\mathbf{ev}(T) = T$. We need to record the following property of $\mathbf{ev}$ due to Schützenberger; see [Sa, Theorem 3.11.4] for an exposition.

Proposition 2.3.1. *For $w \in S_n$,*

$$\mathrm{RS}(w_\circ w w_\circ) = \mathbf{ev}\big(\mathrm{RS}(w^{-1})\big).$$

If particular, if σ is an involution, then $\mathrm{RS}(\sigma)$ is self-evacuating if and only if $w_\circ \sigma w_\circ = \sigma$.

We now recall the bijection

$$\mathbf{dom} \ : \ \{T \in \mathrm{SYT}(n) \mid \mathbf{ev}(T) = T\} \longrightarrow \mathrm{SDT}(n)$$

defined inductively as follows. Begin by applying the evacuation procedure to 1 in T. Since $\mathbf{ev}(T) = T$, at the penultimate step (just before 1 reaches a corner of T), 1 will be adjacent to n. Hence we can remove 1 and n from the shuffled T and replace these two labels by a domino labeled by $[n/2]$, the greatest integer less than $n/2$. Now continue by evacuating 2 from what remains of the shuffled T. At the penultimate step 2 is adjacent to $n-1$, and hence defines a domino labeled $[n-2/2]$. Repeating this procedure defines an element $\mathbf{dom}(T) \in \mathrm{SDT}(n)$ (In the case that $n = 2m+1$ is odd, we change the label of the upper left hand corner of $\mathbf{dom}(T)$ from $m+1$ to 0.)

2.4. **Primitive ideals.** Consider the set $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ of primitive ideals in $\mathfrak{U}(\mathfrak{g})$ which contain the maximal ideal of $\mathfrak{Z}(\mathfrak{g})$ parametrized (via the Harish-Chandra isomorphism) by ρ. In case of simple classical $\mathfrak{g}$, we now discuss the combinatorial parametrization of $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ due to Joseph, Barbasch-Vogan, and Garfinkle.

If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, then there is a bijection from $\mathrm{SYT}(n) \to \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ obtained as follows. Given $T \in \mathrm{SYT}(n)$, let w be any element of S_n such that $\mathrm{RS}(w) = T$. The primitive ideal $I(T) \in \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ parametrized by T is the annihilator of the simple highest weight module $L(w)$ that arises as a quotient of the Verma module induced from $w w_\circ \rho - \rho$. (See [T2, Section 3], for instance, for the exact details of this parametrization.)

For future reference, we need to record a symmetry property of this parametrization.

Proposition 2.4.1. *Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, and fix $I \in \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$. Let I' denote the primitive ideal obtained from I by applying the diagram automorphism for $\mathfrak{g}$. Write T and T' for the tableaux parametrizing I and I'. Then*

$$T' = \mathbf{ev}(T),$$

the evacuation of T (Section 2.3).

Sketch. It is essentially built into Joseph's parametrization of $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ that $T = \mathrm{RS}(w)$ if and only if $T' = \mathrm{RS}((w_\circ w w_\circ)^{-1})$. So Proposition 2.3.1 gives the current proposition. □

If $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, then there is a map from $\mathrm{SDT}(2n)$ to $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ obtained as follows. First we include $W(C_n) \subset W(A_{2n+1}) = S_{2n}$ as the centralizer of the long word $w_\circ \in S_{2n}$, i.e. as the fixed points of the diagram automorphism of A_{2n-1}. Given $T \in \mathrm{SDT}(2n)$, we let $w \in W(C_n) \subset S_{2n}$ be any element whose evacuated right Robinson-Schensted tableaux coincides with T; i.e. in the notation of Section 2.3, $T = \mathbf{dom}(\mathrm{RS}(w))$ (which makes sense by Proposition 2.3.1). Then the corresponding primitive ideal is $\mathrm{Ann}(L(w))$. This map is a bijection when restricted to the subset of $\mathrm{SDT}_C(2n)$ consisting of tableau of special shape in the sense of Garfinkle citeg:i.

If $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, then there is a map from $\mathrm{SDT}(2n+1)$ to $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ obtained in the analogous way. We include $W(B_n) \subset W(A_{2n+1}) = S_{2n+1}$ as the fixed point of the diagram involution. Given $T \in \mathrm{SDT}_B(2n+1)$, let $w \in W(B_n) \subset S_{2n+1}$ be any element such that $T = \mathbf{dom}(\mathrm{RS}(w))$. Then the corresponding primitive ideal is $\mathrm{Ann}(L(w))$. This map is a bijection when restricted to tableaux in $\mathrm{SDT}_B(2n+1)$ of special shape.

Finally, if $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, there is a map from $\mathrm{SDT}(2n)$ to $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ obtained as follows. We include $W(D_n)$ in $W(B_n)$ in the obvious way, and then include $W(C_n) \subset S_{2n}$ as above. Given $T \in W(D_n) \subset S_{2n}$, let w be any element of $W'(D_n)$ such that $T = \mathbf{dom}(\mathrm{RS}(w))$. Then the corresponding primitive ideal is $\mathrm{Ann}(L(w))$. This map is a bijection when restricted to tableaux in $\mathrm{SDT}_D(2n)$ of special shape

2.5. **Representation of the hyperoctahedral group.** Let $\mathcal{Y}$ denote the Young lattice. This is the lattice poset whose unique minimal element is the empty set, whose elements consist of Young diagrams (of any size) and whose covering relations are given by the operation of adding a corner to a Young diagram. Let $\mathcal{D}$ denote the even domino lattice, defined in the analogous way; i.e. the elements are domino shapes of even size and the covering relations are the addition of domino corners. Let $\mathcal{D}'$ denote the odd domino lattice of type B; here the minimal element is a single box, and the other elements are domino shapes of odd size, and the covering relations are the addition of domino corners. It is a standard fact that as lattice posets, $\mathcal{Y} \times \mathcal{Y} \simeq \mathcal{D} \simeq \mathcal{D}'$. The latter isomorphism is trivial. For the first, see [S], for instance.

Recall that the irreducible representations of the hyperoctahedral group $W(B_n) = W(C_n)$ are parametrized by pairs of standard Young tableaux whose aggregate size is n, i.e. by elements of $\mathcal{Y} \times \mathcal{Y}$; see, for instance, [CMc, Chapter 10] for this standard fact. Note also that the dimension of such a representation parametrized by (D, D') is the number of paths from $\emptyset$ to (D, D') in $\mathcal{Y} \times \mathcal{Y}$. Using the isomorphism $\mathcal{Y} \times \mathcal{Y} \simeq \mathcal{D} \simeq \mathcal{D}'$, we obtain the following parametrization.

Proposition 2.5.1. *The irreducible representation of the hyperoctahedral group $S_n \ltimes (\mathbb{Z}/2)^n$ are parametrized by domino shapes of size $2n$ (or $2n+1$). Moreover, if we write $\pi(D)$ for the representation corresponding to a given domino shape, the dimension of $\pi(D)$ is the number of standard domino tableaux of shape $\pi(D)$.*

Remark 2.5.2. The identical argument shows that the irreducible representations of the group of r colored permutations, $G(r, n) := S_n \rtimes (\mathbb{Z}/r)^n$, are parametrized by Young diagrams of size nr that can be tiled by rim hooks of size r; for the definition of rim hook, see [Sa, Chapter 4.10]. Here we are using that the rim hook lattice is isomorphic to r copies of the Young lattice, which follows by exactly the same argument used to establish the domino case. In the context of $G(r,n)$, the corresponding dimension formula counts the number of standard rim hook tilings. Either by using the decomposition of the group algebra of $G(r,n)$ into irreducibles or an argument from [S], it follows that the number of elements in $G(r,n)$ is equal to the number of same-shape standard r-rim hook tableaux of size rn. A constructive bijection (for all r) was constructed in [SW], but when $r = 2$ it does *not* reduce to the bijection defined in the first paragraph of the introduction. This suggests that there should exist an algorithm from $G(r,n)$ to same-shape pairs of standard rim hook tableau generalizing the bijection of the introduction. It is this bijection that should have applications to the (as of yet nonexistent) theory of cells for $G(r,n)$; see [Br].

2.6. **A framework for generalized Robinson-Schensted algorithms.** Fix $G_{\mathbb{R}}$ as in Section 2.1. Recall that the set of K orbit on $\mathcal{N}(\mathfrak{p})$ is finite, and write $\{N_1, N_2, \ldots, N_k\}$ for a set of representatives of such orbits. Let A_i denote the component group of the centralizer of N_i in K. Given $Q \in K\backslash\mathfrak{B}$, let $N_Q \in \{N_1, N_2, \ldots, N_k\}$ denote the representative whose K orbit is dense in the moment map image of $T^*_Q(\mathfrak{B})$.

Proposition 2.6.1. *The map*

$$Q \mapsto \left(N_Q, \mu^{-1}(N_Q) \cap \overline{T^*_Q(\mathfrak{B})}\right)$$

is a bijection

$$K\backslash\mathfrak{B} \longrightarrow \{(N_i, C) \mid C \in A_i\backslash Irr(\mathfrak{B}^{N_i}), i = 1, \ldots k\} \tag{2.1}$$

Sketch. Using Spaltenstein's dimension formula ([Spa1]), one can check that the K saturation of (N_i, C) (viewed as a subvariety of the conormal variety $T^*_K\mathfrak{B}$) is irreducible of dimension equal to the dimension of $\mathfrak{B}$. Since the conormal variety is pure of dimension $\dim(\mathfrak{B})$ and since its irreducible components are exactly the closures of the conormal bundles to K orbits on $\mathfrak{B}$, we conclude that there is some Q such that $T^*_Q\mathfrak{B}$ is dense in $K \cdot (N_i, C)$. This gives the bijection of the proposition. More details can be found in [T1, Proposition 3.1]. (In that paper, it was attributed to Springer, but it appears to have been observed independently by a number of people.) □

Because the sets appearing in Equation (2.1) each admit a combinatorial parametrization, Proposition 2.6.1 gives rise to an interesting family of combinatorial algorithms. (The Robinson-Schensted terminology is explained by Example 2.7.1.)

2.7. **Irr**$(\mathfrak{B}^N)$ **for** $G = GL(n,\mathbb{C})$**.** Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, and fix a nilpotent element $N \in \mathfrak{g}$. Then $\mathrm{Irr}(\mathfrak{B}^N)$ is parametrized by $\mathrm{SYT}(n)$ as follows. If a flag $(F_0 \subset F_1 \subset \cdots \subset F_n)$ is an element of $\mathfrak{B}^N$, then the restriction of N to F_i is a nilpotent endomorphism, and hence specifies a Young diagram D_i of size i. We define a tableau $T(F) \in \mathrm{SYT}(n)$ by requiring that the shape of the first i boxes of $T(F)$ coincide with D_i, for all i. The assignment $F \mapsto T(F)$ is constant on an open piece of each component of $\mathfrak{B}^N$, and defines the parametrization of $\mathrm{Irr}(\mathfrak{B}^N)$ by $\mathrm{SYT}(n)$ ([St], [Spa2]).

Example 2.7.1. Consider the case of $G_{\mathbb{R}} = GL(n,\mathbb{C})$ in Proposition 2.6.1. Then G identifies with two copies of $GL(n,\mathbb{C})$, K with the diagonal $GL(n,C)$, and $\mathfrak{B}$ consists of two copies of the flag variety for $GL(n,\mathbb{C})$. The Bruhat decomposition implies that the K orbits on $\mathfrak{B}$ are parametrized by the symmetric group S_n. The N_i are parametrized by partitions of n according to the above discussion $\mathrm{Irr}(\mathfrak{B}^N)$ is parametrized by $\mathrm{SYT}(n)$. Hence Proposition 2.6.1 asserts the existence of a bijection from S_n to same-shape triples consisting of a partition of n and a pair of standard Young tableaux of the corresponding shape. (The first datum is redundant.) Steinberg ([St]) proved that this coincides with the Robinson-Schensted algorithm.

2.8. **K orbits on $\mathcal{N}(\mathfrak{p})$ for $Sp(2n,\mathbb{R})$ and $O(p,q)$.** The following result is well-known; see [CMc, Chapter 9], for instance.

Proposition 2.8.1. *Recall the notation of 2.2.*

(1) *For $G_{\mathbb{R}} = U(p,q)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by $\mathrm{YT}_{\pm}(p,q)$. (As a matter of notation, we set $\mathrm{YT}_{\pm}(SU(p,q)) = \mathrm{YT}_{\pm}(p,q)$.)*

(2) *For $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(Sp(2n,\mathbb{R})) \subset \mathrm{YT}_{\pm}(n,n)$$

of elements such that for each fixed odd part, the number of rows beginning with $+$ coincides with the number beginning with $-$.

(3) *For $G_{\mathbb{R}} = O(p,q)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(O(p,q)) \subset \mathrm{YT}_{\pm}(p,q)$$

consisting of signed tableaux such that for each fixed even part, the number of rows beginning with $+$ equals the number beginning with $-$.

2.9. **The conormal variety and Weyl group representations.** Write $T^*_K(\mathfrak{B})$ for the union (over K orbits Q on $\mathfrak{B}$) of the conormal bundles $T^*_Q(\mathfrak{B})$. Clearly $T^*_K(\mathfrak{B})$ is pure of dimension $\dim(\mathfrak{B})$ and its irreducible components are just the closures of the conormal bundles $T^*_Q(\mathfrak{B})$. In particular the fundamental classes $[\overline{T^*_Q(\mathfrak{B})}]$ of the conormal bundle closures are a basis for the top Borel-Moore homology group $\mathrm{H}_{\mathrm{top}}(T^*_K(\mathfrak{B}),\mathbb{Z})$. A standard convolution construction defines a module structure on $\mathrm{H}_{\mathrm{top}}(T^*_K(\mathfrak{B}),\mathbb{Z})$ for the convolution algebra $\mathrm{H}_{\mathrm{top}}\big(T^*_{\mathrm{diag}(G)}(\mathfrak{B}\times\mathfrak{B}),\mathbb{Z}\big)$ which, according to a theorem of Kazhdan-Lusztig, is the group algebra $\mathbb{Z}[W]$. It is an easy consequence of the definition that this action is suitably graded in the sense that for a fixed K orbit on $\mathcal{N}(\mathfrak{p})$ (say $\mathcal{O}_K$),

$$\sum_{Q \text{ s.t. } \mu(T^*_Q(\mathfrak{B}))\subset\overline{\mathcal{O}_K}} [\overline{T^*_Q(\mathfrak{B})}]$$

is W-invariant, and hence

$$\mathbf{M}(\mathcal{O}_K) := \sum_{Q \text{ s.t. } \mu(T^*_Q(\mathfrak{B}))\subset\overline{\mathcal{O}_K}} [\overline{T^*_Q(\mathfrak{B})}] \Bigg/ \sum_{Q \text{ s.t. } \mu(T^*_Q(\mathfrak{B}))\subsetneq\overline{\mathcal{O}_K}} [\overline{T^*_Q(\mathfrak{B})}].$$

is a representation of W. In particular, the orbits Q such that $\mu(T^*_Q(\mathfrak{B})) = \overline{\mathcal{O}_K}$ index a basis of a Weyl group representation.

Fix $N \in \mathcal{N}(\mathfrak{p})$, and recall the $W \times A_G(N)$ representations on $\mathrm{H}_{\mathrm{top}}(\mathfrak{B}^N)$ defined by Springer. As a matter of notation, we write $\mathrm{sp}(\mathcal{O})$ for the $A_G(N)$ invariants of this representations; here $\mathcal{O} = G\cdot N$. In any event, since $A_K(N)$ maps to $A_G(N)$, we can consider the $A_K(N)$ invariants in $\mathrm{H}_{\mathrm{top}}(\mathfrak{B}^N)$. The following result is taken from [Ro].

Theorem 2.9.1 (Rossmann). *Fix $N \in \mathcal{N}(\mathfrak{p})$ and let $\mathcal{O}_K = K \cdot N$. Then as a W representation,*

$$\mathbf{M}(\mathcal{O}_K) = \mathrm{H}_{\mathrm{top}}(\mathfrak{B}^N)^{A_K(N)}.$$

The isomorphism maps the fundamental class of closure of $T^*_Q(\mathfrak{B})$ (where $\mu(T^*_Q(\mathfrak{B})) = \overline{\mathcal{O}_K}$) to the fundamental class of $\mu^{-1}(N) \cap \overline{T^*_Q(\mathfrak{B})}$ (compare Proposition 2.6.1).

Lemma 2.9.2. *If $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ or $O(p,q)$, and $N \in \mathcal{N}(\mathfrak{p})$, then the natural map $A_K(N) \longrightarrow A_G(N)$ is surjective. In particular, the $A_K(N)$ and $A_G(N)$ orbits on $Irr(\mathfrak{B}^N)$ coincide.*

Proof. This follows from explicit centralizer calculations. We omit the details. ☐

Corollary 2.9.3. *Let $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ or $O(p,q)$, and let $\mathcal{O}_K$ be parametrized by a signed tableau S (Proposition 2.8.1). Write $\pi(S)$ for the representation of W corresponding to the shape of S by Proposition 2.5.1. Then $\mathbf{M}(\mathcal{O}_K) \simeq \pi(S)$, and*

$$\mathrm{H}_{\mathrm{top}}(T^*_K(\mathfrak{B}), \mathbb{Z}) \simeq \bigoplus_{S \in \mathrm{YT}_{\pm}(G_{\mathbb{R}})} \pi(S).$$

In particular, if we write $d(S)$ for the number of standard domino tableaux whose shape coincides with that of S. Then

$$\#\{Q \mid \mu(T^*_Q(\mathfrak{B})) = \overline{\mathcal{O}_K}\} = d(S),$$

and

$$\#K\backslash\mathfrak{B} = \sum_{S \in \mathrm{YT}_{\pm}(G_{\mathbb{R}})} d(S). \tag{2.2}$$

Proof. Proposition 2.5.1 reduces the corollary to establishing $\mathbf{M}(\mathcal{O}_K) \simeq \pi(S)$. By Theorem 2.9.1 and Lemma 2.9.2, this amounts to showing $\mathrm{sp}(\mathcal{O}) = \pi(S)$, where $\mathcal{O}$ is the G saturation of the orbit $\mathcal{O}_K$ parametrized by S. This follows from Lusztig's computation of the Springer correspondence for classical groups in terms of symbols, together with Proposition 2.5.1; cf. [Mc2, Section 2–3]. ☐

The corollary thus gives the existence a bijection from $K\backslash\mathfrak{B}$ to the same-shape subset of pairs consisting of an element of $\mathrm{YT}_{\pm}(G_{\mathbb{R}})$ and a standard domino tableaux. This will be constructed in Proposition 3.2.2 below as a Robinson-Schensted algorithm in the sense of Section 2.6.

2.10. **Involutions with signed fixed points.** Write $\Sigma(n)$ for the set of involutions in the symmetric group S_n. Let

$$\Sigma_{\pm}(n) = \{(\sigma, \epsilon) \in \Sigma \times \{+,-,0\}^n \mid \epsilon_j = 0 \text{ if and only if } \sigma(j) \neq j\},$$

which we view as the set of involutions in S_n with signed fixed points. We write $\Sigma_{\pm}[U(p,q)]$ for the subset of $\Sigma_{\pm}(p+q)$ consisting of element (σ, ϵ) such that

$$\begin{aligned} p &= \#\{j \mid \epsilon_j = +\} \; + \; (1/2)\#\{j \mid \sigma(j) \neq j\} \\ q &= \#\{j \mid \epsilon_j = -\} \; + \; (1/2)\#\{j \mid \sigma(j) \neq j\}. \end{aligned}$$

Define $\Sigma_{\pm}[Sp(2n)]$ to be the subset of elements (σ, ϵ) in $\Sigma_{\pm}[U(n,n)]$ such that

$$\begin{cases} 1. \text{ (Antisymmetry of signs)} & \epsilon_{2n+1-j} = -\epsilon_j; \\ 2. \text{ (Symmetry of involution)} & \sigma(2n+1-j) = 2n+1-\sigma(j). \end{cases}$$

Similarly define $\Sigma_\pm[O(p,q)]$ to be the subset of elements (σ,ϵ) in $\Sigma_\pm[U(p,q)]$ such that

$$\begin{cases} 1.\ \text{(Symmetry of signs)} & \epsilon_{2(p+q)+1-j} = +\epsilon_j; \\ 2.\ \text{(Symmetry of involution)} & \sigma(2(p+q)+1-j) = 2n+1-\sigma(j). \end{cases}$$

Note that the symmetric group S_n acts on $\Sigma_\pm(n)$ in the obvious way: $w\cdot(\sigma,\epsilon) = (\sigma',\epsilon')$, where $\sigma = w\sigma w^{-1}$, and $\epsilon_{w^{-1}i} = \epsilon_i$. Let $w_\circ$ denote the long word in the symmetric group. Observe that

$$\Sigma_\pm[O(p,q)] = \text{ the fixed points of } w_\circ \text{ on } \Sigma_\pm[U(p,q)]. \tag{2.3}$$

The same statement is almost true for $\Sigma_\pm[Sp(2n)]$, but we must introduce an additional twist. Define $\mathbf{s} : \Sigma_\pm[U(p,q)] \to \Sigma_\pm[U(q,p)]$ via $\mathbf{s}(\sigma,\epsilon) = (\sigma,-\epsilon)$. Then

$$\Sigma_\pm[Sp(2n)] = \text{ the fixed points of } \mathbf{s}\circ w_\circ \text{ on } \Sigma_\pm[U(p,q)]. \tag{2.4}$$

2.11. K **orbits on** $\mathfrak{B}$**.** Fix, once and for all, a signature (p,q) Hermitian form on $\mathbb{C}^{p+q}$, write $G_\mathbb{R} = U(p,q)$ for its isometry group, and fix a Cartan involution θ for $G_\mathbb{R}$. Recall the notation of Section 2.10. It is well known that $\Sigma_\pm[U(p,q)]$ parametrizes the K orbits on $\mathfrak{B}$. (Formulas are given in [Ya1], for instance.) Given $\delta\in\Sigma_\pm[U(p,q)]$, we write Q_δ for the corresponding orbit.

For a choice of nondegenerate symplectic form on $\mathbb{C}^{2n}$, let $\mathfrak{B}'_\omega$ denote the set of Borel subalgebras in $\mathfrak{sp}(\mathbb{C}^{2n},\omega)$. Then $\mathfrak{B}'_\omega$ embeds in $\mathfrak{B}$, the set of Borels in $\mathfrak{gl}(2n,\mathbb{C})$. Set $p=q=n$. Then there exists a choice of ω such that

$$Q'_\delta := Q_\delta \cap \mathfrak{B}'_\omega \neq \emptyset \iff \delta\in\Sigma_\pm[Sp(2n)].$$

Write $G'_\mathbb{R} = Sp(\mathbb{C}^{2n},\omega)\cap U(n,n) \simeq Sp(2n,\mathbb{R})$. Then Q'_δ is an orbit for K' on $\mathfrak{B}'_\omega$, where K' is the complexification of the fixed points of θ (the Cartan involution for $U(n,n)$) on $G'_\mathbb{R}$. As an example of the choices involved, fix a basis $e_1,\dots,e_{2n}$ for $\mathbb{C}^{2n}$, and define $U(n,n)$ with respect to

$$\left\langle \sum_{i=1}^{2n} a_ie_i, \sum_{i=1}^{2n} b_ie_i \right\rangle = \sum_{i=1}^{n} a_ib_i - \sum_{i=n+1}^{2n} a_ib_i,$$

then we take

$$\omega = e_1\wedge e_{2n} + e_2\wedge e_{2n-1} + \cdots + e_n\wedge e_{n+1}.$$

More details of this parametrization can be found in [Ya1].

Similarly, a choice of nondegenerate symmetric form R on $\mathbb{C}^{2(p+q)}$ induces an inclusion of $\mathfrak{B}''_R$ (the variety of Borels in $\mathfrak{so}(\mathbb{C}^{2(p+q)},R)$) into $\mathfrak{B}$, the variety of Borels in $\mathfrak{gl}(2(p+q),\mathbb{C})$. There exists a choice of R such that

$$Q''_\delta = Q_\delta\cap\mathfrak{B}''_R \neq\emptyset \iff \delta\in\Sigma_\pm[O(p,q)].$$

Write $G''_\mathbb{R} := O(\mathbb{C}^{2(p+q)},R)\cap U(2p,2q)\simeq O(p,q)$. Then Q''_δ is an orbit for K'' on $\mathfrak{B}''_\mathbb{R}$, where again K'' is the complexification of the fixed points of θ on $G''_\mathbb{R}$. Explicit details of these choices may be found in [Ya2].

When we speak of $Sp(2n,\mathbb{R})$ and $U(n,n)$ in the same context, we will always assume they are defined compatibly as above. A similar remark applies to $O(p,q)$ and $U(p,q)$. These choices are analogous to the inclusion of $W(C_n)$ into $W(A_{2n-1})$ as the centralizer of $w_\circ$ made in Section 2.4.

2.12. **A Robinson-Schensted algorithm for $U(p,q)$.** We now describe a bijection from $\Sigma_{\pm}[U(p,q)]$ to the set of same-shape pairs consisting of a standard Young tableau and a signature (p,q) signed tableau (Section 2.2).

Given $(\sigma,\epsilon) \in \Sigma_{\pm}[U(p,q)]$, form a sequence of pairs of the form

$$(i,\epsilon_i) \text{ if } \sigma(i)=i; \text{ and}$$
$$(i,\sigma(i)) \text{ if } i<\sigma(i).$$

Arrange the pairs in order by their largest entry, with the convention that a sign has numerical size zero. Write $\pi_1,\dots,\pi_r$ for the resulting ordered sequence. From such a sequence, we now describe how to build a same-shape pair of tableaux

$$\big(\mathrm{RS}_{\mathrm{u},\pm}(\delta), \mathrm{RS}_{\mathrm{u}}(\delta)\big) \in \mathrm{YT}_{\pm}(p,q) \times \mathrm{SYT}(p{+}q).$$

Each tableau is constructed by inductively adding the pairs π_j. So suppose that we have added $\pi_1,\dots,\pi_{j-1}$ to get a (smaller) same-shape pair of tableau $(T_{\pm},T)$. If $\pi_j=(k,\epsilon_k)$, then we first add the sign ϵ_k to the topmost row of (a signed tableau in the equivalence class of) $T_{\pm}$ so that the resulting tableau has signs alternating across rows. Then add the index j to T in the unique position so that the two new tableaux have the have the same shape. If $\pi_j=(k,\sigma(k))$ we first add k to T using the Robinson-Schensted bumping algorithm to get a a new tableau T', and then add a sign ϵ (either $+$ or $-$ as needed) to $T_{\pm}$ so that the result is a signed tableau $T'_{\pm}$ of the same shape as T'. We then add the pair $(\sigma(k),-\epsilon)$ (by the recipe of the first case) to the first row strictly below the row to which ϵ was added. We continue inductively to get $(\mathrm{RS}_{\mathrm{u},\pm}(\delta),\mathrm{RS}_{\mathrm{u}}) \in \mathrm{YT}_{\pm}(p,q)\times\mathrm{SYT}(p{+}q)$. (For a more formal definition, the reader is referred to [G].)

The first statement in the next theorem explains why this algorithm is a generalized Robinson-Schensted algorithm in the sense of Section 2.6. The concluding statement indicates its representation theoretic significance.

Theorem 2.12.1. *Let $G_{\mathbb{R}}=U(p,q)$, fix $\delta\in\Sigma_{\pm}[U(p,q)]$, and let Q_δ be the corresponding K orbit on $\mathfrak{B}$. In terms of the parametrizations of Proposition 2.8.1(1) and Section 2.7, the map*

$$Q_\delta \mapsto \big(N_Q, \mu^{-1}(N_Q)\cap \overline{T^*_{Q_\delta}\mathfrak{B}}\big)$$

of Proposition 2.6.1 coincides with

$$\delta \mapsto \big(\mathrm{RS}_{\mathrm{u},\pm}(\delta),\mathrm{RS}_{\mathrm{u}}(\delta)\big).$$

Let $X(Q_\delta)$ be the Harish-Chandra module with trivial infinitesimal character attached via the Beilinson-Bernstein parametrization to the trivial local system on Q_δ, and recall the parametrizations of Section 2.4 and Proposition 2.8.1(1). Then $\mathrm{RS}_{\mathrm{u}}(\delta))$ *parametrizes* $\mathrm{Ann}(X(\delta))$ *and the closure of the orbit parametrized by* $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$ *is the associated variety of $X(Q_\delta)$.*

Proof. The annihilator statement is the main result in [G]. The remainder is proved in [T1, Theorem 5.6]. □

3. Main results

3.1. **Symmetry properties of the Robinson-Schensted algorithm for $U(p,q)$.** We need to examine how the algorithm of Section 2.12 behaves under the action of $w_\circ$ and $\mathbf{s}$ introduced at the end of Section 2.10. Recall the evacuation operation $\mathbf{ev}$ introduced in Section 2.3.

Proposition 3.1.1. *Fix* $\delta \in \Sigma_{\pm}[U(p,q)]$ *and write* $(\mathrm{RS}_{\mathrm{u},\pm}(\delta), \mathrm{RS}_{\mathrm{u}}(\delta)) = (S,T)$. *Then*

$$\mathrm{RS}_{\mathrm{u},\pm}(w_\circ \cdot \delta) = S_{\mathrm{rev}} \qquad \mathrm{RS}_{\mathrm{u}}(w_\circ \cdot \delta)) = \mathbf{ev}(T);$$

and

$$\mathrm{RS}_{\mathrm{u},\pm}(\mathbf{s}\delta) = -S \qquad \mathrm{RS}_{\mathrm{u}}(\mathbf{s}\delta)) = T;$$

where S_{rev} *is obtained from* S *by reversing each row of* S*, and* $-S$ *is the tableau obtained by inverting all signs in* S.

Proof. The only part of the proposition which is not obvious from the definitions is the assertion that $\mathrm{RS}_{\mathrm{u}}(w_\circ \cdot \delta)) = \mathbf{ev}(T)$. One can probably prove this directly without too much difficulty, but we opt for a slightly more abstract argument. Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. It is clear that the $\mathfrak{U}(\mathfrak{g})$-modules $X(\delta)$ and $X(w_\circ \cdot \delta)$ differ by the diagram automorphism of $\mathfrak{gl}(n,\mathbb{C})$ (say τ). So their annihilators $\mathrm{Ann}(X(\delta))$ and $\mathrm{Ann}(X(w_\circ \cdot \delta))$ also differ by τ. Proposition 2.4.1 implies that the tableaux parametrizing $\mathrm{Ann}(X(\delta))$ and $\mathrm{Ann}(X(w_\circ \cdot \delta))$ differ by evacuation. But the last assertion in Theorem 2.12.1 implies that these tableaux are $\mathrm{RS}_{\mathrm{u}}(\delta)$ and $\mathrm{RS}_{\mathrm{u}}(w_\circ \cdot \delta)$, so the current proposition follows. □

3.2. **A Robinson-Schensted algorithm for** $Sp(2n,\mathbb{R})$ **and** $O(p,q)$**.** We begin by noting a simple corollary to Proposition 3.1.1.

Corollary 3.2.1. *Recall the inclusions* $\Sigma_{\pm}[O(p,q)], \Sigma_{\pm}[Sp(2n)] \subset \Sigma_{\pm}[U(p,q)]$.

(1) *If* $\delta \in \Sigma_{\pm}[Sp(2n)]$*, then* $\mathrm{RS}_{\mathrm{u},\pm}(\delta) \in \mathrm{YT}_{\pm}(Sp(2n))$.
(2) *If* $\delta \in \Sigma_{\pm}[O(p,q)]$*, then* $\mathrm{RS}_{\mathrm{u},\pm}(\delta) \in \mathrm{YT}_{\pm}(O(p,q))$.

Proof. Consider the statement for $Sp(2n,\mathbb{R})$, and fix $\delta \in \Sigma_{\pm}[Sp(2n)]$. From Proposition 3.1.1 and Equation (2.4), one concludes that if we reverse each row of $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$ and subsequently invert all signs in the resulting tableau, we must again obtain $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$. This can be achieved if and only if the number of odd rows of $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$ of a fixed length beginning with a plus sign coincides with the number of such rows beginning with a minus sign. Thus $\mathrm{RS}_{\mathrm{u},\pm}(\delta) \in \mathrm{YT}_{\pm}(Sp(2n))$. A similar argument establishes the corollary for $O(p,q)$. □

Proposition 3.2.2. *There are bijective maps*

$$(\mathrm{RS}_{\mathrm{sp},\pm}, \mathrm{RS}_{\mathrm{sp}}) \ : \ \Sigma_{\pm}[Sp(2n)] \to \ \textit{same-shape subset of } \mathrm{YT}_{\pm}(Sp(2n)) \times \mathrm{SDT}(2n)$$
$$(\mathrm{RS}_{\mathrm{o},\pm}, \mathrm{RS}_{\mathrm{o}}) \ : \ \Sigma_{\pm}[O(p,q)] \to \ \textit{same-shape subset of } \mathrm{YT}_{\pm}(O(p,q)) \times \mathrm{SDT}(p+q),$$

obtained as follows. The maps $\mathrm{RS}_{\mathrm{sp},\pm}$ *and* $\mathrm{RS}_{\mathrm{o},\pm}$ *are the restriction of the* $\mathrm{RS}_{\mathrm{u},\pm}$*, the signed tableau part of the Robinson-Schensted algorithm for* $U(p,q)$ *(Section 2.12). The maps* $\mathrm{RS}_{\mathrm{sp},\pm}$ *and* $\mathrm{RS}_{\mathrm{o},\pm}$ *are obtained by composing* RS_{u} *with the domino evacuation algorithm* **dom** *of Section 2.3.*

Proof. Equations (2.3) and (2.4) and Proposition 3.1.1 (together with Corollary 3.2.1) imply that the maps described in the Proposition are indeed well-defined. The maps $(\mathrm{RS}_{\mathrm{sp},\pm}, \mathrm{RS}_{\mathrm{sp}})$ and $(\mathrm{RS}_{\mathrm{o},\pm}, \mathrm{RS}_{\mathrm{o}})$ are clearly injective, since $(\mathrm{RS}_{\mathrm{u},\pm}, \mathrm{RS}_{\mathrm{u}})$ and **dom** are injective. On the other hand, $\Sigma_{\pm}[Sp(2n)]$ parametrizes K orbits on the flag variety for $Sp(2n,\mathbb{R})$ (Section 2.11), so Equation (2.2) in Corollary 2.9.3 implies $(\mathrm{RS}_{\mathrm{sp},\pm}, \mathrm{RS}_{\mathrm{sp}})$ is surjective (and hence bijective). A similar argument implies $(\mathrm{RS}_{\mathrm{o},\pm}, \mathrm{RS}_{\mathrm{o}})$ is bijective. □

Remark 3.2.3. Without resorting to Section 2.9, it is easy to give a purely combinatorial proof that the maps in Proposition 3.2.2 are surjective (and hence bijective). For instance, suppose (S,T) is an element of the same-shape subset of $\mathrm{YT}_{\pm}(Sp(2n)) \times \mathrm{SDT}(2n)$. We are to find $\delta \in \Sigma_\pm[Sp(2n)]$ such that $(\mathrm{RS}_{\mathrm{u},\pm}(\delta), \mathbf{dom} \circ \mathrm{RS}_{\mathrm{u}}(\delta)) = (S, T)$. By Theorem 2.12.1, there exists such a $\delta \in \Sigma_\pm[U(p,q)]$, and we need only show that $\mathbf{s} \circ w_\circ(\delta) = \delta$. Suppose not, i.e. suppose $\eta := \mathbf{s} \circ w_\circ(\delta) \neq \delta$. But since $(\mathrm{RS}_{\mathrm{u},\pm}(\delta), \mathrm{RS}_{\mathrm{u}}(\delta)) = (S, \mathbf{dom}^{-1}(T))$ and $\mathbf{dom}^{-1}(T)$ is self-evacuating, Proposition 3.1.1 implies $(\mathrm{RS}_{\mathrm{u},\pm}(\eta), \mathrm{RS}_{\mathrm{u}}(\eta)) = (S, \mathbf{dom}^{-1}(T))$. Since $(\mathrm{RS}_{\mathrm{u},\pm}, \mathrm{RS}_{\mathrm{u}})$ is injective (Theorem 2.12.1) we conclude $\eta = \delta$, contradiction. An identical argument works for $O(p,q)$. This gives a combinatorial proof of Proposition 3.2.2. We have included the argument relying on Section 2.9 to highlight the connection with Weyl group representations.

3.3. **Moment map images of conormal bundles for $Sp(2n,\mathbb{R})$ and $O(p,q)$.** The next result shows that the algorithms in Proposition 3.2.2 fit the generalized Robinson-Schensted framework of Section 2.6.

Proposition 3.3.1. *Fix $G_\mathbb{R} = Sp(2n,\mathbb{R})$ or $O(p,q)$. Recall the parametrizations of $K\backslash\mathfrak{B}$ (Section 2.11) and the algorithm $\mathrm{RS}_{\mathrm{u},\pm}$ of Section 2.12. For $\delta \in \Sigma_\pm(G_\mathbb{R})$, the orbit parametrized by $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$ (Section 2.8) is dense in the moment map image $\mu(T^*_{Q_\delta}(\mathfrak{B}))$.*

Proof. Embed $G_\mathbb{R}$ compatibly into $G'_\mathbb{R} = U(n,n)$ or $U(p,q)$ as in Section 2.11. (Here we are inverting the notational role of $G_\mathbb{R}$ and $G'_\mathbb{R}$, but this should cause little confusion.) For $\delta \in \Sigma_\pm(G_\mathbb{R}) \subset \Sigma_\pm[U(p,q)]$, let Q_δ denote the corresponding K orbit on the flag variety $\mathfrak{B}$ for $G_\mathbb{R}$, and let Q'_δ denote the corresponding orbit on $\mathfrak{B}'$ for $G'_\mathbb{R}$. Given a signed tableau $S \in \mathrm{YT}_\pm(G_\mathbb{R}) \subset \mathrm{YT}_\pm(U(p,q))$, write $\mathcal{O}_S$ for the corresponding K orbit on $\mathcal{N}(\mathfrak{p})$ for $G_\mathbb{R}$, and adopt the analogous notation for $\mathcal{O}'_S$. Write $\mu(\delta)$ for the dense orbit in $\mu(T^*_{Q_\delta}(\mathfrak{B}))$, and write $\mu'(\delta)$ for the dense orbit in $\mu'(T^*_{Q'_\delta}(\mathfrak{B}'))$. We also let $\mu(\delta)$ denote the corresponding element of $\Sigma_\pm(G_\mathbb{R})$.

We first establish that $\mu(\delta) \subset \overline{\mathcal{O}_{\mathrm{RS}_{\mathrm{u},\pm}(\delta)}}$, by using the equivariance of the moment map, and the known moment map image computation for $U(p,q)$ (Theorem 2.12.1). It is clear that

$$K' \cdot T^*_{Q_\delta}(\mathfrak{B}) \subset T^*_{Q'_\delta}(\mathfrak{B}').$$

The equivariance of the moment map, together with the fact that $\mathcal{O}'_{\mathrm{RS}_{\mathrm{u},\pm}\delta} = \mu'(\delta)$ (Theorem 2.12.1), we conclude that

$$K' \cdot \mu(\delta) \subset \overline{\mathcal{O}'_{\mathrm{RS}_{\mathrm{u},\pm}(\delta)}}.$$

By intersecting with $\mathcal{N}(\mathfrak{p})$ (and noting that the parametrizations of Proposition 2.8.1 are suitably compatible), we conclude that $\mu(\delta) \subset \overline{\mathcal{O}_{\mathrm{RS}_{\mathrm{u},\pm}(\delta)}}$, as claimed.

Now suppose that there is some δ for which $\mu(\delta) \neq \mathrm{RS}_{\mathrm{u},\pm}(\delta)$, i.e. for which

$$\mu(\delta) \subsetneq \overline{\mathcal{O}_{\mathrm{RS}_{\mathrm{u},\pm}(\delta)}}.$$

We may assume that δ is chosen so that $\mathcal{O}_{\mathrm{RS}_{\mathrm{u},\pm}(\delta)}$ has the minimal possible dimension (say d) among all δ for which $\mu(\delta) \neq \mathrm{RS}_{\mathrm{u},\pm}(\delta)$. We now simply count the number of elements in

$$A(\delta) := \{\eta \in \Sigma_\pm[G_\mathbb{R}] \mid \mu(\eta) = \mu(\delta)\}.$$

Because d was assumed to be minimal,

$$\mathrm{RS}^{-1}_{\mathrm{u},\pm}(\mu(\delta)) \subset A(\delta).$$

By hypothesis $\delta \in A(\delta)$ but $\delta \notin \mathrm{RS}^{-1}_{\mathrm{u},\pm}(\mu(\delta))$. So we conclude that

$$\#A(\delta) > \#\mathrm{RS}^{-1}_{\mathrm{u},\pm}(\mu(\delta)). \tag{3.1}$$

Corollary 2.9.3 says that the number of elements in $A(\delta)$ is the number of standard domino tableau of shape equal to that of $\mathrm{RS}_{\mathrm{u},\pm}(\delta)$. Proposition 3.2.2 say that $\mathrm{RS}_{\mathrm{u},\pm}^{-1}(\mu(\delta))$ has the same number of elements. So Equation (3.1) gives a contradiction, and the proposition is proved. □

3.4. **A domino tableau parametrization of $A_G(N)$ orbits on $\mathbf{Irr}(\mathfrak{B}^N)$.** Let $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$ or $O(p,q)$. Fix a K orbit $\mathcal{O}_K$ on $\mathcal{N}(\mathfrak{p})$, fix $N \in \mathcal{O}_K$, and let S be the signed tableau parametrizing $\mathcal{O}_K$ (Proposition 2.8.1). We now describe a bijection

$$\{T \in \mathrm{SDT}(n) \text{ whose shape is that of } N\} \longrightarrow A_G(N) \text{ orbits on } \mathrm{Irr}(\mathfrak{B}^N) \tag{3.2}$$

obtained as follows. Given a domino tableau S whose shape is that of N, using Proposition 3.2.2 we obtain an element $\delta \in \Sigma_{\pm}[G_{\mathbb{R}}]$ by requiring

$$\big(\mathrm{RS}_{\mathrm{u},\pm}(\delta), \mathbf{dom} \circ \mathrm{RS}_{\mathrm{u}}(\delta)\big) = (S,T).$$

Let $Q_\delta \in K\backslash\mathfrak{B}$ be the K orbit corresponding to δ (Section 2.11). Recall that the orbits of $A_K(N)$ and $A_G(N)$ on $\mathrm{Irr}(\mathfrak{B}^N)$ coincide (Lemma 2.9.2). So Propositions 2.6.1 and 3.3.1 imply that

$$\mu^{-1}(N) \cap \overline{T^*_{Q_\delta}(\mathfrak{B})}$$

is an element of the right-hand side of Equation (3.2). This defines the map in Equation (3.2). Tracing through each step, one sees that this map is bijective.

We thus we conclude that the $A_G(N)$ orbits on $\mathrm{Irr}(\mathfrak{B}^N)$ are parametrized by standard domino tableau of shape equal to the Jordan form of N. It is important to note that the definition of this parametrization involved a choice of $\mathcal{O}_K \in \mathrm{Irr}[(G\cdot N)\cap \mathfrak{p}]$, and it is not immediately clear that different choices lead to the same parametrization.

Proposition 3.4.1. *Let $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$ or $O(p,q)$, and fix $N \in \mathcal{N}(\mathfrak{p})$. The bijection*

$$\{T \in \mathrm{SDT}(n) \text{ whose shape is that of } N\} \longrightarrow A_G(N) \text{ orbits on } Irr(\mathfrak{B}^N)$$

defined in Equation (3.2) is independent of the choice of $\mathcal{O}_K$.

Proof. We give the argument for $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$. (The case of $O(p,q)$ is identical except in notation.) Fix $N_1, N_2 \in \mathcal{N}(\mathfrak{p})$ with $N_2 \in G\cdot N_1$ but $N_2 \notin K\cdot N_1$. Let S^i denote the signed tableau parametrizing $K\cdot N_i$. Fix a domino tableau T whose shape coincides with that of N_1 (or N_2). Let

$$Q_i = (\mathrm{RS}_{\mathrm{sp}} \times \mathrm{RS}_{\mathrm{sp},\pm})^{-1}(T, S^i); \tag{3.3}$$

here (and below), we are identifying elements of $\Sigma_{\pm}[Sp(2n)]$ with the K orbit on $\mathfrak{B}$ that they parametrized (Section 2.11). The proposition amounts to showing that

$$\overline{T^*_{Q_1}(\mathfrak{B})} \cap \mu^{-1}(N_1) = \overline{T^*_{Q_2}(\mathfrak{B})} \cap \mu^{-1}(N_2). \tag{3.4}$$

Recall the inclusion $G_{\mathbb{R}} \subset G'_{\mathbb{R}} = U(n,n)$ (Section 2.11 — we have inverted the notational role of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$), and let Q'_i denote the K' saturation of Q_i in $\mathfrak{B}'$. Suppose Equation (3.4) fails. Then the corresponding statement fails for $G'_{\mathbb{R}}$,

$$\overline{T^*_{Q'_1}(\mathfrak{B}')} \cap \mu'^{-1}(N_1) \neq \overline{T^*_{Q'_2}(\mathfrak{B}')} \cap \mu'^{-1}(N_2).$$

By Theorem 2.12.1, this implies that $\mathrm{RS}_{\mathrm{u}}(Q'_1) \neq \mathrm{RS}_{\mathrm{u}}(Q'_2)$. Since $\mathrm{RS}_{\mathrm{sp}}(Q_1) = \mathbf{dom}(\mathrm{RS}_{\mathrm{u}}(Q'_2))$, we conclude that $\mathrm{RS}_{\mathrm{sp}}(Q_1) \neq \mathrm{RS}_{\mathrm{sp}}(Q_2)$. But this contradicts Equation (3.3). □

Remark 3.4.2. Recently McGovern ([Mc1]) and Pietraho ([P]) gave two independent parametrizations of the $A_G(N)$ orbits on $\mathrm{Irr}(\mathfrak{B}^N)$ by domino tableau. It is expected (but still not known) that their parametrizations coincide and, moreover, that they coincide with the one in Proposition 3.4.1.

3.5. **Connection with annihilators.** For $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$ or $O(p,q)$, the maps $\mathrm{RS}_{\mathrm{sp}}$ and RS_{o} attach a domino tableau to each K orbit Q on $\mathfrak{B}$. According to [G1]–[G4], such a tableaux parametrizes a primitive ideal in the enveloping algebra of $\mathfrak{g}$. By analogy with Theorem 2.12.1, it is natural to ask whether this primitive ideal has anything to do with Harish-Chandra modules for $G_{\mathbb{R}}$ supported on the closure of Q. By contrast with the $U(p,q)$ case, this question is complicated enormously by the existence of nontrivial K equivariant local systems on Q, as well as the presence of nonspecial shapes. Nonetheless the Robinson-Schensted algorithms of Section 3.2 compute annihilators and associated varieties of derived functor modules.

Proposition 3.5.1. *Fix $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$ or $O(p,q)$ and write* RS *for* $\mathrm{RS}_{\mathrm{sp}}$ *or* RS_{o} *(Section 3.2). Let $\mathfrak{q}$ be a θ-stable parabolic of $\mathfrak{g}$ containing a fixed θ-stable Borel $\mathfrak{b}$, and let Q' denote the K orbit of $\mathfrak{q}$ on G/P (where P is the parabolic subgroup of G corresponding to $\mathfrak{q}$). Let Q be the dense K orbit in the preimage of Q' under the projection of $\mathfrak{B} = G/B \to G/P$, and write $Q = Q_\delta$ for $\delta \in \Sigma_\pm(G_{\mathbb{R}})$ (Section 2.11). Write $A_{\mathfrak{q}}$ for the Harish-Chandra module for $G_{\mathbb{R}}$ with trivial infinitesimal character attached via the Beilinson-Bernstein parametrization to the trivial local system on Q. Then* $\mathrm{RS}(\delta)$ *has special shape and (in the parametrization of Section 2.4),*

$$\mathrm{Ann}(A_{\mathfrak{q}}) = \mathrm{RS}(\delta).$$

Moreover the closure of the K orbit parametrized by $\mathrm{RS}_\pm(\delta)$ *is the associated variety of* $A_{\mathfrak{q}}$.

Proof. The Harish-Chandra modules $A_{\mathfrak{q}}$ are derived functor modules induced from the trivial representation of the Levi factor $\mathfrak{l}$ of $\mathfrak{q}$; see [T3] for more details. Since $\mathrm{AV}(A_{\mathfrak{q}}) = \mu(T^*_Q(\mathfrak{B}))$ (see the introduction of [T3], for instance), Proposition 3.3.1 implies that $\mathrm{RS}_\pm(\delta)$ parametrizes the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}})$. So to establish Proposition 3.5.1 we need only treat the annihilator statement. Since we have computed the annihilators of $A_{\mathfrak{q}}$ modules in [T3, Section 8], and we can compute δ directly from $\mathfrak{q}$ directly (cf. [VZ]), this amounts to a combinatorial check. A better, less combinatorial approach would be to proceed by an induction on the number of simple components type A in the Levi factor $\mathfrak{l}$, in a manner exactly analogous to the computation of annihilators given in Section 8 of [T3]. In either case, the assertion is not terribly difficult, and we omit the details. □

Proposition 3.5.1 and Theorem 2.12.1 suggest the following more general questions:

(∗) Suppose X is a Harish-Chandra module for $G_{\mathbb{R}}$ with trivial infinitesimal character attached to the trivial local system on a K orbit Q_δ. Suppose further $\mathrm{RS}_\pm(\delta)$ has special shape. Then does $\mathrm{RS}_\pm(\delta)$ compute a component of the associated variety of X? Does $\mathrm{RS}(\delta)$ compute the annihilator of X?

Unfortunately this is false, as the next example indicates.

Example 3.5.2. Let $G_{\mathbb{R}} = Sp(8,\mathbb{R})$. Consider the orbit Q_δ parametrized by $\delta = (\sigma,\epsilon)$ with $\sigma = (36)$ (the transposition interchanging 3 and 6) and $\epsilon = (+,+,0,-,+,0,-,-)$. One computes $\mathrm{RS}_{\pm}(\delta)$ as

+	−	+	−
+	−		
+	−		

,

and $\mathrm{RS}_{\mathrm{sp}}(\delta)$ as

1		3
2	4	

.

Let X be the Harish-Chandra module attached to the trivial local system on Q_δ. (More precisely, X is cohomologically induced from a one-dimensional representation of $U(2,0)$ tensored with a nonunitary highest weight module for $Sp(4,\mathbb{R})$.) Then one can check that X is in the same Harish-Chandra cell as the $A_{\mathfrak{q}}$ module induced from the trivial representation on a levi factor of the form $\mathfrak{u}(2,0) \oplus \mathfrak{u}(0,2)$ and, moreover, that $\mathrm{AV}(X)$ is

+	−	+	−
+	−	+	−

,

while $\mathrm{Ann}(X)$ is

1	3
2	4

.

So $\mathrm{RS}(\delta)$ (which has special shape) does not compute $\mathrm{Ann}(X)$.

It is of considerable interest to note that Q_δ in the example admits a unique nontrivial K equivariant local system, and the annihilator and associated variety of the Harish-Chandra module attached to it *are* indeed given by $\mathrm{RS}_{\mathrm{sp}}(\delta)$ and $\mathrm{RS}_{\pm}(\delta)$. This immediately suggests a way to modify $(*)$ by allowing X to be attached to nontrivial local systems. Even though we have no counterexample to this modification, it still seems ambitious. Instead it seems plausible that if we further modify $(*)$ by placing further restrictions on $\mathrm{RS}_{\pm}(\delta)$, then the question admits a positive answer. We hope to return to this elsewhere.

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Department of Mathematics, University of Utah, Salt Lake City, UT 84112

E-mail address: `ptrapa@math.utah.edu`