ANNIHILATORS AND ASSOCIATED VARIETIES OF $A_{\mathfrak{q}}(\lambda)$ MODULES FOR $U(p,q)$

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Abstract. Vogan has conjectured that the cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ in the weakly fair range exhaust all unitary representations of $U(p,q)$ with certain kinds of real integral infinitesimal character. To prove a statement like this, it is essential to identify these modules among the set of all irreducible Harish-Chandra modules. Barbasch and Vogan have parametrized this latter set in terms of their annihilators and asymptotic supports (or, equivalently, associated varieties). In this paper, we identify the weakly fair $A_{\mathfrak{q}}(\lambda)$ in this parametrization by combining known results about their asymptotic supports together with an explicit computation of their annihilators. In particular, this determines all vanishing and coincidences among the $A_{\mathfrak{q}}(\lambda)$ in the weakly fair range, and gives the Langlands parameters of these modules.

1. Introduction

The cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ have long been known to play a distinguished role in the unitary dual of a real reductive Lie group G (see [VZ], for instance). More recently, Salamanca [Sa2] has shown that they exhaust the unitary representations of G whose infinitesimal character is real, strongly regular, and integral. The starting point of Salamanca's proof is identifying where the $A_{\mathfrak{q}}(\lambda)$ modules (with regular infinitesimal character) fit into the admissible dual of G, and the identification depends crucially on the hypothesis of regular infinitesimal character.

At singular infinitesimal character, the situation is much less well-understood, even for a relatively uncomplicated group like $U(p,q)$. The $A_{\mathfrak{q}}(\lambda)$ modules still provide a long list of unitary representations: Vogan's unitarizability theorem applies to certain $A_{\mathfrak{q}}(\lambda)$ — the weakly fair ones of Definition 3.4 — and for $U(p,q)$ they are all irreducible or zero. Vogan has conjectured that the list is complete.

Conjecture 1.1 (Vogan)**.** *The cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ in the weakly fair range exhaust the unitary Harish-Chandra modules for $U(p,q)$ whose infinitesimal character is a weight-translate of ρ.*

In this paper, we lay the foundation for a proof of the conjecture by identifying the weakly fair $A_{\mathfrak{q}}(\lambda)$ modules in the admissible dual of G. The strategy is to reduce to the (well-understood) regular case by writing a singular $A_{\mathfrak{q}}(\lambda)$ module as the image of a complicated wall-crossing translation functor applied to an $A_{\mathfrak{q}}(\lambda)$ module with regular infinitesimal character. A serious technical obstacle is that the translation functors are essentially impossible to compute in terms of various standard parametrizations of the admissible dual (like the Langlands classification, for instance). Said differently, the weakly fair $A_{\mathfrak{q}}(\lambda)$ modules do not fit nicely in the Langlands classification.

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We are thus forced to find a parametrization of the admissible dual which does behave well (or at least tractably) with respect to the relevant translation functors. The parametrization we use is due to Barbasch and Vogan [BV4], and is based on the structure of Kazhdan-Lusztig-Vogan cells for $U(p,q)$. Roughly speaking, the Barbasch-Vogan parametrization identifies Harish-Chandra modules by their annihilators and associated varieties, and so our task is to compute these invariants for the weakly fair $A_{\mathfrak{q}}(\lambda)$ modules. This is the content of our main result, Theorem 7.9: given λ in the mediocre range for $\mathfrak{q}$ (a range which properly includes the weakly fair range), the theorem gives an algorithm to determine the annihilator and associated variety of $A_{\mathfrak{q}}(\lambda)$. The techniques used in proving Theorem 7.9 are roughly a reduction to the case of maximal $\mathfrak{q}$, and should be widely applicable to other classical groups.

Theorem 7.9 determines all coincidences and vanishing among the weakly fair $A_{\mathfrak{q}}(\lambda)$ modules, and provides an algorithm to compute their Langlands parameters. Since many readers may be interested in this latter computation, we sketch explicit details in Remark 7.11.

The paper is organized as follows. After fixing notation, we discuss some (mostly) well-known results about the $A_{\mathfrak{q}}(\lambda)$ modules in Section 3. Of particular interest to experts are Theorem 3.1(b)(iv) which gives a larger range of irreducibility for the $A_{\mathfrak{q}}(\lambda)$ modules than is typically considered (the so-called mediocre range), and Proposition 3.8 which provides a description of the unitarily small representations of $U(p,q)$ whose infinitesimal character is a weight translate of ρ. In Section 4, we recall the classification of primitive ideals in $\mathfrak{gl}(n,\mathbb{C})$ and prove a weak statement describing their behavior under cohomological induction. In Section 5, we first recall a few deep facts about asymptotic supports and associated varieties. We describe these invariants abstractly for the modules $A_{\mathfrak{q}}(\lambda)$, and then make that description explicit for $U(p,q)$. In Section 6, we precisely state the Barbasch-Vogan parametrization and give an elegant identification of the parameters of the $A_{\mathfrak{q}}(\lambda)$ modules in the good range. While Theorem 6.4 and Corollary 6.12 only treat the good range, they are interesting in their own right. Since their statements are quite clean and the proofs are relatively lightweight, many readers may find these results easiest to digest. At any rate, they are prerequisite to tackling the main results which follow.

In Section 7, we extend the algorithms of Section 6 to the mediocre range and state our main results (Theorem 7.9 and Corollary 7.12). We prove Theorem 7.9 in Section 8 by carefully understanding the effect of certain wall crossing translation functors from regular to singular infinitesimal character. In Section 9, we prove that any mediocre $A_{\mathfrak{q}}(\lambda)$ is isomorphic to a weakly fair one. (As explained in Remark 3.7, this is a small piece of Conjecture 1.1.)

Many special cases of the main results presented here are treated in Kobayashi's work [K]. Additional special cases were covered by Friedman [F].

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2. Notation and Structure Theory

Let $G = U(p,q)$ be the group of complex linear transformations of $\mathbb{C}^{p+q}$ preserving a Hermitian form defined by a matrix with p pluses and q minuses on the diagonal. (We will be a little more precise about the arrangement of the signs below.) Let $K \cong U(p) \times U(q)$ be the fixed points of the Cartan involution of inverse conjugate transpose. Let θ be the differentiated involution and let $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$ be the corresponding decomposition. Write $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}$ for the corresponding complexifications; for example $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, with $n = p + q$.

Fix the diagonal torus $T \subset K$ with Lie algebra $\mathfrak{t}_o$, and set $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_o$. Write $\Delta(\mathfrak{g}, \mathfrak{t})$ for the roots of $\mathfrak{t}$ in $\mathfrak{g}$ and make the standard choice of positive roots,

$$\Delta^+ = \{e_i - e_j \mid i < j\}.$$

Write ρ for the half-sum of the positive roots. Let $\alpha_i = e_i - e_{i+1} \in \Delta(\mathfrak{g}, \mathfrak{t})$ denote the ith simple root, and Σ denote the collection of all simple roots. With these choices, a weight $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}_{\mathbb{R}}^*$ is dominant if $\nu_1 \geq \cdots \geq \nu_n$.

Let $\mathfrak{b}$ be the Borel subalgebra corresponding to Δ^+. Write $W \simeq S_n$ for the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$, and let w_o denote the long element in W. For a dominant $\nu \in \mathfrak{t}^*$ and $w \in W$ define Verma modules by

$$M_{\mathfrak{b}}(w\nu) = \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{ww_o\nu - \rho}),$$

and denote their unique irreducible quotients by $L_{\mathfrak{b}}(w\nu)$. The definition is arranged so that $L_{\mathfrak{b}}(\nu) = M_{\mathfrak{b}}(\nu)$ and $L_{\mathfrak{b}}(w_o\nu)$ is finite-dimensional (if ν is integral and regular).

We will need a very explicit description of (representatives of K-conjugacy classes of) θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$; this is standard and well-known (see, for example, [V8, Example 4.5] for omitted details). Let $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ be an ordered sequence of pair of positive integers (not both zero). Set $p = \sum_i p_i$, $q = \sum_i q_i$, and $n_i = p_i + q_i$. Define $U(p,q)$ with respect to the Hermitian form defined by a diagonal matrix consisting of p_1 pluses, then q_1 minuses, then p_2 pluses, and so on. Let $\mathfrak{l}$ denote the block diagonal subalgebra

$$\mathfrak{gl}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_r, \mathbb{C}),$$

let $\mathfrak{u}$ denote the strict block upper-triangular subalgebra, and write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\rho(\mathfrak{u})$ for the half-sum of the roots in $\mathfrak{u}$. Then $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. As the ordered sequences of pairs range over all

$$\{(p_1, q_1), \ldots (p_r, q_r)\}, \qquad \sum p_i = p, \ \sum q_i = q,$$

the $\mathfrak{q}$ constructed in this way exhaust the K conjugacy classes of θ-stable parabolic subalgebras for $\mathfrak{g}$.

2.1. Notation for θ-stable parabolics.

Whenever we speak of (the K-conjugacy class of) a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ attached to a sequence $\{(p_1, q_1), \ldots, (p_r, q_r)\}$, we shall always mean the one described above. In the coordinates given above, any unitary one-dimensional $(\mathfrak{l}, L \cap K)$-module, restricted to T, has differential

$$\lambda = (\overbrace{\lambda_1, \ldots, \lambda_1}^{n_1 = p_1 + q_1}, \ldots\ldots, \overbrace{\lambda_r, \ldots, \lambda_r}^{n_r = p_r + q_r}) \in \mathfrak{t}_{\mathbb{R}}^*,$$

with each $\lambda_i \in \mathbb{Z}$.

2.2. **Translation functors, τ-invariants, and primitive ideals.** Let F be a finite-dimensional irreducible representation of $\mathfrak{g}$ with extremal weight μ. For any weight $\gamma \in \mathfrak{t}^*$, let P_γ denote the projection, defined on the category of $Z(\mathfrak{g})$-finite $U(\mathfrak{g})$ modules, onto generalized infinitesimal character γ. Define the translation functor

$$\psi_\nu^{\nu+\mu}(X) = P_{\nu+\mu} \circ (F \otimes \cdot\,) \circ P_\nu(X).$$

Certain translation functors will arise frequently, and we give them special names. Suppose $\nu \in \mathfrak{t}^*_{\mathbb{R}}$ is dominant, integral, and regular, and for a simple root α, let μ_α be an extremal weight of a finite-dimensional representation so that $\nu_\alpha = \nu + \mu_\alpha$ is still dominant but lies exactly on the α wall: $\langle \nu_\alpha, \beta \rangle = 0$ if and only if $\beta = \pm\alpha$. We denote the corresponding translation functor $\psi_\nu^{\nu_\alpha}$ by ψ_α. Given an irreducible $U(\mathfrak{g})$ module X with infinitesimal character ν we define its τ-invariant

$$\tau(X) = \{\alpha \in \Sigma \mid \psi_\alpha(X) = 0\}.$$

(Neither μ_α nor ψ_α is well-defined, but the τ-invariant definition is.) Next let

$$\mu_i = e_{i+1} + \cdots + e_n,$$

and consider the finite-dimensional representation of $\mathfrak{g}$ with highest weight $w_o\mu_i$. We will write T_i for $\psi_\nu^{\nu+\mu_i}$, suppressing the dependence on ν (now allowed to be arbitrary). We let T_i^k denote the k-fold composition of T_i.

A two-sided ideal in the enveloping algebra $U(\mathfrak{g})$ is called a primitive ideal if it is the annihilator of a simple $U(\mathfrak{g})$ module. A primitive ideal is said to have infinitesimal character ν if it contains the maximal ideal in $Z(\mathfrak{g})$ corresponding to ν. Denote the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character ν by $\mathrm{Prim}(U(\mathfrak{g}))_\nu$. If ν is dominant, regular, and integral, we define the τ-invariant of $I = \mathrm{Ann}(X) \in \mathrm{Prim}(U(\mathfrak{g}))_\nu$ to be the subset of simple roots consisting of those α for which $\psi_\alpha(X)$ is zero.

2.3. **Tableau notation.** Given a partition $n = n_1 + \cdots + n_k$ with the n_i decreasing, we may attach a left justified arrangement of n boxes with n_i boxes in the ith row. Call such an arrangement a Young diagram of size n. If $\nu = (\nu_1, \ldots, \nu_n)$ is an n-tuple of real numbers, a ν-quasitableau is defined to be *any* arrangement of $\nu_1, \ldots, \nu_n$ in a Young diagram of size n. The underlying diagram of a quasitableau is called its shape. If a ν-quasitableau satisfies the condition that the entries weakly increase across rows and strictly increase down columns, it is said to be a ν-tableau. A ν-tableau whose entries strictly increase across rows is called standard. If $\nu = (1, 2, \ldots, n)$, then a standard ν-tableau is called a Young tableau. Replacing 'increasing' by 'decreasing' in the definition of a ν-tableau defines a ν-antitableau.

For technical reasons, we will need to switch between two sets of data: the data of a Young tableau together with a decreasing n-tuple $\nu = (\nu_1 \geq \cdots \geq \nu_n)$; and the data of a ν-antitableau. Given a decreasing n-tuple ν and a Young tableau S_{Y}, we get a ν-antitableau by changing the ith entry of S_{Y} to ν_i. For the converse construction, we need to adopt the convention that given two occurrences of an identical entry in a ν-antitableau, one is said to be larger than the other if it occurs strictly to the left of the other. Then given a ν-antitableau S_{A} we first order $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, and construct a Young tableau S_{Y} from S_{A} as follows. Locate the the largest occurrence (in the sense of the convention just mentioned) of ν_1 in S_{A} and relabel it '1'. If $\nu_2 = \nu_1$, then locate the next largest occurrence of ν_1 in S_{A} and relabel it '2'; if $\nu_2 < \nu_1$, locate the largest occurrence of ν_2 and relabel it '2'.

Continuing in this way, we obtain a Young tableau S_{Y}. For example,

$$S_{\mathrm{A}} = \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 2 & \\ \cline{1-2}\end{array}\ , \qquad S_{\mathrm{Y}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}\ .$$

We call S_{Y} the underlying tableau of S_{A}.

A skew diagram is any diagram obtained by removing a smaller Young diagram from a larger one that contains it. A skew column is a skew tableau whose shape consists of at most one box per row and whose entries strictly increase when moving down in the diagram. A skew column is called difference-one if its consecutive entries (when moving down the column) *decrease* by exactly one when moving down the column.

A signed Young tableau of signature (p,q) is an equivalence class of Young diagrams whose boxes are filled with p pluses and q minuses so that the signs alternate across rows; two signed Young diagrams are equivalent if they can be made to coincide by interchanging rows of equal length. (Note that the equivalence relation preserves shapes.) A skew column of a signed tableau is any arrangement of pluses and minuses in a skew diagram consisting of at most one box per row.

3. The modules $A_{\mathfrak{q}}(\lambda)$

In this section, we recall the definition and properties of the modules $A_{\mathfrak{q}}(\lambda)$. Most of the material in this section is standard, but part of Theorem 3.1b(iv) is new (see Remark 3.3).

We adopt the notation of [KV] for our cohomological induction functors, and return for the moment to the setting of an arbitrary reductive $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the complexification of a maximally compact θ-stable Cartan subalgebra $\mathfrak{h}_o$. Choose a θ-stable system of positive roots $\Delta^+(\mathfrak{g},\mathfrak{h})$ and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ containing $\mathfrak{h}$ with $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{g},\mathfrak{h})$. A one-dimensional unitary $(\mathfrak{l}, L\cap K)$-module $\mathbb{C}_\lambda$ is determined by $\lambda \in \mathfrak{h}^*$, its differential restricted to $\mathfrak{h}$. Define $\mathbb{C}_\lambda^{\#} = \mathbb{C}_\lambda \otimes_{\mathbb{C}} \bigwedge^{top} \mathfrak{u}$ viewed as a $(\bar{\mathfrak{q}}, L\cap K)$ module and form

$$\mathcal{L}_j(\mathbb{C}_\lambda) = (\Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j(\mathrm{ind}^{\mathfrak{g},L\cap K}_{\bar{\mathfrak{q}},L\cap K}(\mathbb{C}_\lambda^{\#}));$$

here Π_j is the derived Bernstein functor. For $S = \dim(\mathfrak{u}\cap\mathfrak{k})$, write $A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$.

Here are the main properties of these modules:

Theorem 3.1. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic and let $\mathbb{C}_\lambda$ be a one-dimensional unitary $(\mathfrak{l}, L\cap K)$ module; set $S = dim(\mathfrak{u}\cap\mathfrak{k})$.*

(a) *$\mathcal{L}_j(\mathbb{C}_\lambda)$ has infinitesimal character $\lambda+\rho$.*

(b) *Suppose $\mathrm{ind}^{\mathfrak{g}}_{\bar{\mathfrak{q}}}(\mathbb{C}^{\#}_{\lambda+t\rho(\mathfrak{u})})$ is irreducible for all $t \geq 0$; then:*

(i) *$\mathcal{L}_j(\mathbb{C}_\lambda) = 0$ for $j \neq S$.*

(ii) *$A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$ is a unitarizable $(\mathfrak{g},K)$ module.*

(iii) *If the infinitesimal character $\lambda+\rho$ is regular, then $A_{\mathfrak{q}}(\lambda)$ is nonzero and irreducible.*

(iv) *Suppose further that $G = U(p,q)$; then $A_{\mathfrak{q}}(\lambda)$ is either irreducible unitary or zero.*

Remark 3.2. Part (a) says that (for G linear and $\mathrm{rank}(G) = \mathrm{rank}(K)$) the infinitesimal character of an $A_{\mathfrak{q}}(\lambda)$ module is always a translate of ρ by a weight of a finite dimensional representation of $\mathfrak{g}$. This explains the infinitesimal character condition in Conjecture 1.1.

Remark 3.3. Assertion (iv) is a special feature of the $U(p,q)$ setting; in general such an $A_{\mathfrak{q}}(\lambda)$ need not be irreducible or zero. In the case of general G, Chapter 8 of [KV] provides sufficient conditions from which to conclude (iv). More precisely, under

Hypothesis 1:

$$A_{\mathfrak{q}}(\lambda') \text{ is irreducible; and}$$
$$\psi^{\lambda+\rho}_{\lambda'+\rho}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(\mathbb{C}^{\#}_{\lambda'})) = \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(\mathbb{C}^{\#}_{\lambda}); \text{ and}$$

Hypothesis 2:

$$\text{The Kostant problem for } \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(\mathbb{C}^{\#}_{\lambda}) \text{ has a positive solution;}$$

one can conclude $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero. (For a definition of the Kostant problem, see [Jo] for example.) The second hypothesis is subtle in general, but it certainly holds if the closure of the (complex) Richardson orbit $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{l}}(\mathcal{O}_{zero})$ is normal and equivariantly simply connected. Of course this is always the case for $\mathfrak{gl}(n,\mathbb{C})$, and hence we obtain (iv) when $G = U(p,q)$ and λ in the weakly fair range. For general G, the relevant orbit closures can fail to be normal and simply connected and we see, at least morally, why the irreducibility result can fail in general.

Under the assumptions of Theorem 3.1b(iv), Hypothesis 1 can be verified by taking $\lambda'+\rho$ is dominant and regular (so that Theorem 3.1(b)(iii) implies $A_{\mathfrak{q}}(\lambda')$ is irreducible) and applying Lemma 3.13. When λ is in the weakly fair range of Definition 3.4, Lemma 3.13 holds for general G, as a consequence of [KV, Lemma 8.39]. Outside the weakly fair range, the lemma (and hence Theorem 3.1b(iv)) are apparently new, although the proof is surprisingly simple.

Perhaps more importantly for us is that the work of Sections 6-8 allows us to deduce Theorem 3.1b(iv) from Theorem 3.1b(iii), without referring to Chapter 8 in [KV]. (To be fair, our proof is extremely complicated and far from conceptual.) However, it is reasonable to expect that the arguments of Sections 6-8 can be used to detect reducibility of singular $A_{\mathfrak{q}}(\lambda)$ modules for classical groups other than $U(p,q)$.

We set aside the definition of certain ranges of positivity for λ and $\mathfrak{q}$.

Definition 3.4. A one-dimensional unitary $(\mathfrak{l}, L\cap K)$-module λ said to be in the mediocre range for $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ if $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(\mathbb{C}^{\#}_{\lambda+t\rho(\mathfrak{u})})$ is irreducible for all $t \geq 0$. We say that λ is in the (weakly) good range if $\lambda+\rho$ is (weakly) dominant; and say that λ is in the (weakly) fair range if $\lambda+\rho(\mathfrak{u})$ is (weakly) dominant.

A module $A_{\mathfrak{q}}(\lambda)$ is said to be good, or in the good range, if λ is in the good range for $\mathfrak{q}$. Similar terminology applies for weakly good, fair, weakly fair, and mediocre.

The fair range is easily seen to contain the good range, and it's not too hard to see that the mediocre range contains the fair range (see [KV, Theorem 5.105]). The next lemma makes these ranges explicit in the $U(p,q)$ setting (from which the containments become obvious).

Lemma 3.5. *Recall Notation 2.1, and let* $\mathfrak{q}$ *correspond to* $\{(p_1,q_1),\ldots,(p_r,q_r)\}$, *and set* $n_i = p_i+q_i$. *Fix a one-dimensional unitary* $(\mathfrak{l}, L\cap K)$*-module*

$$\lambda = (\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1=p_1+q_1},\ldots\ldots,\overbrace{\lambda_r,\ldots,\lambda_r}^{n_r=p_r+q_r}).$$

(a) λ *is in the good range for* $\mathfrak{q}$ *if and only if*

$$\lambda_i - \lambda_{i+1} > -1, \qquad \textit{for all } i.$$

(b) *λ is in the fair range for $\mathfrak{q}$ if and only if*

$$\lambda_i - \lambda_{i+1} > -\frac{n_i + n_{i+1}}{2}, \qquad \textit{for all } i.$$

(c) *λ is in the mediocre range for $\mathfrak{q}$ if and only if*

$$\lambda_i - \lambda_j \geq -\max(n_i, n_j) - \sum_{i<k<j} n_k, \qquad \textit{for all } i < j.$$

Remark 3.6. The weakly good and fair ranges are characterized by relaxing the strict inequalities in (a) and (b) to weak ones. Parts (a) and (b) follow directly from the definitions. Part (c) is deeper; it is proved in Satz 4 and Corollar 4 of [Ja]. Finally note that the condition in (c) isn't transitive, so we need to consider all pairs $i < j$.

Remark 3.7. Since the fair range is properly contained in the mediocre range, Theorem 3.1 suggests that we are perhaps excluding some unitary representations by restricting our attention to the weakly fair range. Conjecture 1.1 says that this should not be the case and, in fact, in Theorem 9.1, we prove that any mediocre $A_{\mathfrak{q}}(\lambda)$ is isomorphic to a weakly fair one; so we obtain no new unitary representations inside the mediocre range (but outside the fair range).

Before turning to more detailed matters about the $A_{\mathfrak{q}}(\lambda)$ modules below, we discuss which ones have their infinitesimal character in a central translate of $\overline{W\rho}$, the convex hull of the Weyl group orbit of ρ. (The irreducible unitary ones with this property are unitarily small in the sense of [SaV].) For general λ, the condition is complicated, but in the weakly fair range, the complications magically disappear.

Proposition 3.8. *Retain the notations of Lemma 3.5.*

(a) *Choose $\sigma \in S_r$ so that $\lambda_{\sigma(1)} \geq \cdots \geq \lambda_{\sigma(r)}$. Then $\lambda \in \overline{W\rho}$ (modulo the center of $\mathfrak{g}$) if and only if*

$$\lambda_{\sigma(i)} - \lambda_{\sigma(i+1)} \leq \frac{(n_{\sigma(i)} + n_{\sigma(i+1)})}{2}, \qquad \textit{for all } i.$$

(b) *If λ is in the weakly fair range for $\mathfrak{q}$, $\lambda + \rho \in \overline{W\rho}$ (modulo the center of $\mathfrak{g}$) if and only if*

$$\langle \lambda, \alpha \rangle \leq 0, \ \textit{for all } \alpha \in \Delta^+;$$

or, explicitly, if and only if $\lambda_i - \lambda_{i+1} \leq 0$ for all i.

Said differently, given the Salamanca-Vogan conjecture and Conjecture 1.1, the $A_{\mathfrak{q}}(\lambda)$ modules with

$$-\langle \rho(\mathfrak{u}), \alpha \rangle \leq \langle \lambda, \alpha \rangle \leq 0$$

exhaust the unitarily small representations of $U(p,q)$ whose infinitesimal character is a weight translate of ρ.

Sketch. For part (a), we can clearly assume that λ is dominant and σ is the identity. We can also assume (by modifying λ by a central element) that the sum of the entries of λ is zero; i.e. that λ lives in the dual of the semisimple piece of the diagonal Cartan subalgebra. (We no longer are assuming that the entries of λ are integers, of course.) In order for this kind of λ to live in $\overline{W\rho}$, it must be inside each codimension-one face of $\overline{W\rho}$ which contains the point ρ. We can characterize such faces as follows. Given a simple reflection s_i, let $S(i) \simeq S_{i-1} \times S_{n-i}$ be the subgroup of S_n generated by the simple reflections other than s_i. (That is, $S(i)$ is the

Weyl group of the levi factor of a maximal parabolic subgroup.) Then the codimension-one faces containing ρ are precisely the convex hulls $\overline{S(i)\rho}$. The condition that λ lie inside the ith such face is exactly the ith condition given in part (a), thus completing the sketch of the first part. (The reader is encouraged to draw the rank two picture.)

For part (b), apply (a) to $\lambda+\rho$. □

Returning to more immediate questions, the good $A_{\mathfrak{q}}(\lambda)$ with infinitesimal character ρ are (almost) parametrized by the set of θ-stable parabolics, but there is some repetition. In our $U(p,q)$ setting, for instance, a simple induction in stages argument shows that coincidences arise from adjacent compact factors (of the same signature) in $\mathfrak{l}_o$.

Lemma 3.9. *Suppose that $\mathfrak{q}'$ corresponds to $\{(p'_1,q'_1),\ldots,(p'_{r+1},q'_{r+1})\}$ (as in Notation 2.1) and that for some $i \leq r$, $q'_i = q'_{i+1} = 0$. Let $\mathfrak{q}$ correspond to the sequence $\{(p_1,q_1),\ldots,(p_r,q_r)\}$ obtained by combining the ith and $(i+1)$st entries:*

$$(p_j,q_j) = \begin{cases} (p'_j,q'_j) & \text{if } j < i, \\ (p'_i + p'_{i+1}, 0) & \text{if } j = i, \\ (p'_{j-1}, q'_{j-1}) & \text{if } j > i. \end{cases}$$

Then $A_{\mathfrak{q}}(\mathbb{C}_{triv}) \simeq A_{\mathfrak{q}'}(\mathbb{C}_{triv})$. The analogous statement holds if $p'_i = p'_{i+1} = 0$.

These are the only coincidences that can arise, however.

Proposition 3.10. *The good $A_{\mathfrak{q}}(\lambda)$ for $U(p,q)$ with infinitesimal character ρ are parametrized by ordered sequences of pairs of integers*

$$\{(p_1,q_1),\ldots,(p_r,q_r)\} \text{ with } \sum p_i = p,\ \sum q_i = q,$$

so that no adjacent pairs are of the form $(p_i,0),(p_{i+1},0)$ or $(0,q_i),(0,q_{i+1})$. The correspondence takes a sequence to $A_{\mathfrak{q}}(\mathbb{C}_{triv})$ where $\mathfrak{q}$ is defined as in Notation 2.1.

Example 3.11. In the case of $U(p,1)$, the parameters appearing in the proposition are all of the form

$$\{(i,0),(p-i-j,1),(j,0)\}, \quad 0 \leq i,j \leq p\ , i+j \leq p;$$

here if the pair $(0,0)$ appears we ignore it. For future reference, we denote the above set of pairs by $[i,j]$ and the corresponding $A_{\mathfrak{q}}(\mathbb{C}_{triv})$ as $X[i,j]$.

We now record a few results, specific to the $U(p,q)$ setting, describing the effect of translation functors on the $A_{\mathfrak{q}}(\lambda)$.

Lemma 3.12. *Let $\mathfrak{q}$ correspond to an ordered sequence $\{(p_1,q_1),\ldots,(p_r,q_r)\}$, let λ be in the good range for $\mathfrak{q}$, and set $n_i = p_i + q_i$. Consider a simple root $\alpha = e_k - e_{k+1}$ and write (uniquely)*

$$k = \sum_{i \leq j} n_j + l, \text{ with } 0 \leq l < n_{j+1}.$$

Then $\alpha \in \tau(A_{\mathfrak{q}}(\lambda))$ (Notation 2.2) if and only if one of the following conditions holds

(a) *$l \geq 1$; or*

(b) *If $l = 0$, the consecutive entries $(p_j,q_j),(p_{j+1},q_{j+1})$ are of the form $(p_j,0),(p_{j+1},0)$ or of the form $(0,q_j),(0,q_{j+1})$.*

In particular, if the sequence $\{(p_1,q_1),\ldots,(p_r,q_r)\}$ is of the kind described in Proposition 3.10, case (b) can never occur, and $\tau(A_{\mathfrak{q}}(\lambda))$ consists of the simple roots of $\mathfrak{t}$ in $\mathfrak{l}$.

Pf. One can certainly prove the lemma by understanding the Langlands parameters of the good $A_{\mathfrak{q}}(\lambda)$ (as is done in [VZ]) and then applying Vogan's τ invariant calculation of [V3]. This requires some fairly serious bookkeeping, so we sketch an alternative proof.

The ideas given below together with induction in stages reduce the Lemma to the case when $\mathfrak{q}$ is maximal and of the form of Proposition 3.10. So let $\mathfrak{q}$ be associated to the sequence $\{(p_1, q_1), (p_2, q_2)\}$ with neither both p's nor both q's zero. The roots of the form specified by condition (a) in the lemma are exactly the $\alpha_k = e_k - e_{k+1}$ with $k \neq p_1 + q_1$. Now the translation functor ψ_α commutes with the derived Bernstein functor, so the composition factors of $\psi_\alpha(A_{\mathfrak{q}}(\lambda))$ are of the form $\Pi_S(Z)$ where Z is a composition factor of $\psi_\alpha(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda))$. By a Mackey isomorphism, this latter module admits a generalized Verma flag which can be explicitly examined to conclude that if condition (a) is satisfied, $\psi_\alpha(A_{\mathfrak{q}}(\lambda))$ is zero.

To finish the sketch we need to show that for $k = p_1 + q_1$, $\alpha_k \notin \tau(A_{\mathfrak{q}}(\lambda))$. If this were the case, the τ invariant would consist of all simple roots. Lemma 4.2 then implies that the shape of the annihilator of $A_{\mathfrak{q}}(\lambda)$ is a single column, which (by the 'same-shape' result of [BV1]) in turn implies that the asymptotic support of $A_{\mathfrak{q}}(\lambda)$ is zero. Since we have assumed that $\mathfrak{q}$ is associated to $\{(p_1, q_1), (p_2, q_2)\}$, Proposition 5.4 and Lemma 5.6 imply that either $p_1 = p_2 = 0$ or $q_1 = q_2 = 0$, which contradicts our original assumption on the p's and q's. □

The next lemma will be the basis of moving from good to worse ranges. The first assertion is Lemma 8.1 below; the second follows from the first using the ideas of the preceding proof.

Lemma 3.13. *Let λ' be in the good range for $\mathfrak{q}$, let $A \subset \Sigma - \tau(A_{\mathfrak{q}}(\lambda'))$, and recall Notation 2.2. Suppose*

$$\lambda = \lambda' + \sum_{i \in A} k_i \mu_i \qquad \text{(with each } k_i \geq 0\text{)}$$

is in the mediocre range for $\mathfrak{q}$, and set $\nu = \lambda' + \rho$. Let T be any translation functor of the form

$$T = \prod_{i \in A} T_i^{k_i},$$

with the factors of the product being taken in any order. Then

$$T(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_{\lambda'}^{\#})) = \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_{\lambda}^{\#}),$$

and $T(A_{\mathfrak{q}}(\lambda')) = A_{\mathfrak{q}}(\lambda)$.

4. Primitive ideals in $\mathfrak{gl}(n, \mathbb{C})$

We begin with a convenient choice of Joseph's parametrization of primitive ideals in $\mathfrak{gl}(n, \mathbb{C})$.

Theorem 4.1. *For a dominant integral $\nu = (\nu_1 \geq \cdots \geq \nu_n) \in \mathfrak{t}^*$, the set Prim$(U(\mathfrak{g}))_\nu$ is in bijection with the set of ν-antitableau (Notations 2.2, 2.3).*

We now describe how we want the parametrization of the theorem to work. Duflo's theorem asserts that the map from (the involutions of) W to $\mathrm{Prim}(U(\mathfrak{g}))_\nu$ sending w to $\mathrm{Ann}_{U(\mathfrak{g})}(L(w\nu))$ is surjective. Assume now that ν is non-singular, dominant, and integral; we treat the singular case in a moment. Joseph proved that

$$\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{b}}(w\nu)) = \mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{b}}(w'\nu))$$

if and only if $RS(w) = RS(w')$; here $RS(w)$ denotes the the 'counting' tableau of the Robinson-Schensted algorithm (see [Sag]). We obtain a ν-antitableau by changing the ith entry to ν_i, thus describing the parametrization of the theorem in the regular case. (At first glance the 'antitableau' parametrization appears like a ridiculous complication. It does, however, have the advantage of making the statements of our main theorems much cleaner.)

The singular case follows from the translation principle as discussed after Theorem 4.4 below. In order to state that theorem, we need to first consider τ-invariants on the level of tableaux. Jantzen first showed that, for ν regular and integral, $\tau(L_{\mathfrak{b}}(w\nu))$ coincides with the combinatorial definition of $\tau(w)$ coming from the Bruhat order. Combined with an easy observation about the Robinson-Schensted algorithm, this implies that one can read off the τ invariant of a primitive ideal (with regular integral infinitesimal character) from its tableau:

Lemma 4.2. *Let ν be dominant, integral, and regular, and fix $I \in Prim(U(\mathfrak{g}))_\nu$. Then $\alpha = e_i - e_{i+1}$ is in the τ-invariant of I if and only if ν_{i+1} is strictly below ν_i in the tableau corresponding to I (by the procedure of the previous paragraph).*

The tableau condition comes up sufficiently often that we set it aside in a definition.

Definition 4.3. Let ν be dominant, regular, and integral. The simple root $\alpha = e_i - e_{i+1}$ is said to be in the τ-invariant of a ν-standard tableau S if and only if ν_{i+1} is strictly below ν_i in S; or, equivalently, if and only if $i+1$ is strictly below i in the underlying tableau of S.

Now we isolate the relevant version of the translation principle. (See [KV, Chapter 7] and the references given there.)

Theorem 4.4. *Let ν be regular, integral, and dominant, and recall Notation 2.2. Suppose there is a finite-dimensional representation with extremal weight μ so that $\nu' = \nu + \mu$ is again dominant (but potentially singular). Let $A = \{\alpha \in \Sigma \mid \langle \nu', \alpha \rangle = 0\}$.*

(a) *The translation functor $\psi_\nu^{\nu'}$ establishes a bijection between irreducible $U(\mathfrak{g})$ modules with infinitesimal character ν whose τ invariants are contained in the complement of A and irreducible $U(\mathfrak{g})$ modules with infinitesimal character ν'.*

(b) *$\psi_\nu^{\nu'}$ is well-defined on the level of primitive ideals and defines a bijection between primitive ideals with infinitesimal character ν whose τ-invariants are contained in the complement of A and primitive ideals with infinitesimal character ν'.*

To complete the description of the parameterization of Theorem 4.1, consider a primitive ideal I' of dominant (but potentially singular) infinitesimal character ν'. Let ν be as in Theorem 4.4; then there is a unique primitive ideal I of infinitesimal character ν with $\psi_\nu^{\nu'}(I) = I'$. We have already described a standard ν-antitableau parametrizing I. Under Theorem 4.1, I' is parameterized by the unique standard ν'-antitableau whose underlying tableau coincides with that of I. Notice that in terms of the parameterization of Theorem 4.1, the tableau of $I' = \psi_\nu^{\nu'}(I)$ is obtained by changing the coordinates of the antitableau corresponding to I from ν to ν'. We shall see in Lemma 8.12 that much more complicated translation functors can be described in this way.

To be absolutely explicit, we summarize how to go from a ν-antitableau S_{A} to the highest weight module whose annihilator it parametrizes. Take $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, and construct the underlying Young tableau S_{Y} using the procedure described in Notation 2.3. Consider the set of elements of S_n whose Robinson-Schensted counting tableau is S_{Y}. For a given one of these elements, say w, S_{A} parametrizes the annihilator of the highest weight module $L_{\mathfrak{b}}(w\nu)$.

In Section 6, we will need some weak information about how annihilators behave under cohomological induction. The next lemma is that kind of statement.

Lemma 4.5. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a maximal θ-stable parabolic for $U(p,q)$ with $\mathfrak{l}_o = \mathfrak{u}(p_1,q_1) \oplus \mathfrak{u}(p_2,q_2)$; set $n_i = p_i + q_i$. Let $X' \otimes X''$ be an irreducible $(\mathfrak{l}, L \cap K)$-module with infinitesimal character ν satisfying*

$$\langle \nu + \rho(\mathfrak{u}), \alpha \rangle > 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

Set

$$X = \mathcal{L}_S(X' \otimes X'').$$

Then X is irreducible and the first n_1 boxes of the underlying tableau of $\mathrm{Ann}(X)$ *coincide with the underlying tableau of* $\mathrm{Ann}(X')$ *(Notation 2.3, Theorem 4.1).*

Sketch. By [KV, Theorem 8.2], X is irreducible. The results of [V1] imply that the first n_1 boxes of the underlying tableau of $\mathrm{Ann}(X)$ are characterized by applying sequences of wall-crossing translation functors to X. The walls in question correspond to the first $n_1 - 1$ simple roots of $\mathfrak{g}$. Using the ideas of the proof of Lemma 3.12, it follows that the wall crossing information is identical for X and X'. The lemma follows. □

We isolate the precise statement that we will need in a corollary, which follows by induction using an easy induction in stages argument taking $X' = A_{\mathfrak{q}'}(\lambda')$ and X'' an appropriate one-dimensional representation.

Corollary 4.6. *Let $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ be the θ-stable parabolic corresponding to the ordered set $\{(p_1,q_1),\ldots,(p_r,q_r)\}$. Let $r \geq s$ and let $\mathfrak{q}' \subset \mathfrak{gl}(n',\mathbb{C})$ be the θ-stable parabolic corresponding to $\{(p_1,q_1),\ldots,(p_s,q_s)\}$. Suppose $\mathbb{C}_\lambda$ is a one-dimensional $(\mathfrak{l}, L \cap K)$-module in the good range for $\mathfrak{q}$, and let $\mathbb{C}_{\lambda'}$ denote the $(\mathfrak{l}', L' \cap K')$ module obtained by restriction. Then the first n' boxes of the underlying tableau of* $\mathrm{Ann}(A_{\mathfrak{q}}(\lambda))$ *coincide with the underlying tableau of* $\mathrm{Ann}(A_{\mathfrak{q}'}(\lambda'))$.

5. Asymptotic supports and associated varieties for the $A_{\mathfrak{q}}(\lambda)$

For the moment we return to the general setting of an arbitrary reductive group G. Given an irreducible Harish-Chandra module X, one is led to the study of singularities of its distribution characters at the identity. The relevant notion is due to Barbasch and Vogan ([BV1]); roughly speaking, the distribution character of X has an asymptotic expansion whose leading term is a real linear combination of (Fourier transforms of canonical measures on) real nilpotent orbits, all of the same dimension. This linear combination, denoted $\mathcal{AS}(X)$, is called the asymptotic cycle of X; the closure of the union of the orbits appearing in $\mathcal{AS}(X)$ is called the asymptotic support of X and is denoted $\mathrm{AS}(X)$.

Nilpotent orbits also arise naturally through Vogan's construction of the associated variety of X ([V2], [V7]). Using a good filtration on X, one forms the associated graded object which turns out to be a finitely generated module over $S(\mathfrak{g}/\mathfrak{k})$ and therefore corresponds to an algebraic cycle in $(\mathfrak{g}/\mathfrak{k})^*$. This cycle is called the associated variety of X and is denoted $\mathcal{AV}(X)$; it is an *integral* linear combination of nilpotent $K_{\mathbb{C}}$ orbits on $\mathfrak{p}$. The union of terms appearing in $\mathcal{AV}(X)$ is denoted $\mathrm{AV}(X)$.

Although the asymptotic support is a purely analytic invariant, and the associated variety is a purely algebraic one, the next result indicates the formal possibility that the two may be related.

Proposition 5.1 (The Kostant-Sekiguchi bijection; see [CMc]). *The set of nilpotent coadjoint orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}^*$ is in bijection with the set of real nilpotent coadjoint orbits of G on $\mathfrak{g}_o^*$.*

Let Φ denote the bijection of the theorem (taking G orbits to $K_{\mathbb{C}}$ ones), and extend Φ in the obvious way to linear combinations and unions of orbits. Barbasch and Vogan conjectured the following result; a proof has been announced by Schmid and Vilonen [ScVi].

Theorem 5.2. $\Phi(\mathcal{AS}(X)) = \mathcal{AV}(X)$.

In particular, note that the coefficients appearing in $\mathcal{AS}(X)$ are all integers, and that $\Phi(\mathrm{AS}(X)) = \mathrm{AV}(X)$.

A natural question to ask is how asymptotic supports behave under cohomological induction. In particular, we can ask for the asymptotic support of an $A_{\mathfrak{q}}(\lambda)$. Since the asymptotic support of a finite-dimensional representation is zero, one expects $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$ to be (somehow) induced from the zero orbit. A precise statement appears in Proposition 5.4, but first we need to define a notion of induction for real orbits.

Definition 5.3. Let $\mathcal{O}_{\mathfrak{l}}$ be a real nilpotent coadjoint orbit for L in $\mathfrak{l}_o^*$. Suppose L is a Levi subgroup of G and $\mathfrak{q}$ is a θ-stable parabolic of $\mathfrak{g}$ with $\mathfrak{l}_o = \mathfrak{q} \cap \bar{\mathfrak{q}}$. Let $\mathcal{O}_{\mathfrak{l},\theta} = \Phi(\mathcal{O}_{\mathfrak{l}})$ be the corresponding $(K \cap L)_{\mathbb{C}}$ orbit in $\mathfrak{p}^* \cap \mathfrak{l}^*$ (Proposition 5.1). Then $K_{\mathbb{C}} \cdot (\mathcal{O}_{\mathfrak{l},\theta} + (\mathfrak{u}^* \cap \mathfrak{p}^*))$ has a unique open $K_{\mathbb{C}}$ orbit $\mathcal{O}_{\mathfrak{g},\theta}$. We define

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \Phi^{-1}(\mathcal{O}_{\mathfrak{g},\theta}).$$

(Note that the induced orbit depends on $\mathfrak{q}$ and not just $\mathfrak{l}$.)

The following proposition was conjectured in [BV4]; for $U(p,q)$, Barbasch and Vogan knew a proof based on explicit computation. The general statement given below is well-known to experts.

Proposition 5.4. *Let λ be in the good range for $\mathfrak{q}$. Then $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$ is the closure of the Richardson orbit $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$.*

Pf. Let $X = A_{\mathfrak{q}}(\lambda)$, let $Q \subset G_{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{q}$, and consider the closed $K_{\mathbb{C}}$ orbit $\mathcal{O}$ of the identity coset of the partial flag variety $G_{\mathbb{C}}/Q$. Let $\Delta(X)$ denote the (partial flag) $\mathcal{D}$-module localization of X at dominant regular infinitesimal character $\lambda + \rho$. Then [BoBr, Corollary 1.9] implies that $\mathrm{AV}(X)$ is the image under the moment map of the support of the characteristic cycle of $\Delta(X)$. Since λ is dominant, it is not difficult to see that the support of the characteristic cycle is the conormal bundle of $\mathcal{O}$ (see [Ch, Lemma 1.4], for instance). The image of the conormal bundle is the $K_{\mathbb{C}}$ saturation of $\mathfrak{u}^* \cap \mathfrak{p}^*$ which is just the closure of the orbit $\Phi^{-1}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero}))$ appearing in Definition 5.3. Since $\mathrm{AV}(X) = \Phi(\mathrm{AS}(X))$ (Theorem 5.2), the proposition is proved.

Now we return to the $\mathfrak{u}(p,q)$ setting to record some explicit results.

Lemma 5.5 (see [CMc], 9.3.3). *Nilpotent orbits in $\mathfrak{u}(p,q)$ are parametrized by signed Young diagrams of signature (p,q) (Notation 2.3).*

We conclude this section with a lemma that gives the results of certain orbit inductions on the level of tableaux. In its statement, an empty row is to be interpreted as ending with both plus and minus signs.

Lemma 5.6. *Let $\mathcal{O}_1$ be a nilpotent orbit in $\mathfrak{u}(p_1,q_1)$ corresponding to the signed tableau T_1. Let $\mathcal{O}_{zero}$ be the zero orbit in $\mathfrak{u}(p_2,q_2)$. Let $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{gl}(p+q,\mathbb{C})$ be associated to the sequence $\{(p_1,q_1),(p_2,q_2)\}$. Then the signed tableaux of signature (p,q) corresponding to the induced orbit*

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_1 \times \mathcal{O}_{zero})$$

is obtained by adding r pluses and s minuses, from top to bottom, to the row-ends of T_1 so that

(a) *at most one sign is added to each row-end; and*
(b) *the signs of the resulting diagram must alternate across rows.*

(The resulting diagram may not necessarily have rows of decreasing length, but one can choose a tableau equivalent to T_1 so that the result does have rows of decreasing length.)

Sketch. One may prove Lemma 5.6 as follows. First note that $\mathcal{O}_1$ is itself Richardson so we can write $\mathcal{O}_1 = \mathrm{ind}_{\mathfrak{q}_1'}^{\mathfrak{g}_1}(\mathcal{O}_{zero})$; then an appropriate induction in stages argument shows that the lemma computes $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}}(\mathcal{O}_{zero})$ for some $\mathfrak{q}'$ (possibly) different than $\mathfrak{q}$. Hence the lemma reduces to the computation of Richardson orbits. As remarked at the end of the proof of Proposition 5.4, this computation amounts to composing the Kostant-Sekiguchi bijection with the computation of the moment map image of a certain conormal bundle. A. Yamamoto [Ya] has given an algorithm to perform this latter computation in terms of signed tableau. Tracking through these steps gives the algorithm of the lemma. □

Remark 5.7. At best, this sketch again requires substantial bookkeeping. In particular, one needs to understand the $K_{\mathbb{C}}$ orbits on $G_{\mathbb{C}}/B$ which parametrize the $A_{\mathfrak{q}}(\lambda)$ modules. This by itself is rather involved — it follows from Vogan and Zuckerman's description of the $A_{\mathfrak{q}}(\lambda)$ Langlands parameters [VZ] and an application of the Matsuki correspondence. As an alternative, it is a relatively straightforward exercise to compute $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$ directly from the definitions. We leave the details to the interested reader.

Remark 5.8. Garfinkle ([G]) has given a map taking Langlands parameters of Harish-Chandra modules for $U(p,q)$ with trivial infinitesimal character to signed tableaux. A number of people have conjectured that her algorithm in fact computes associated varieties, and McGovern ([Mc2]) has independently checked that this is indeed the case. In any event, we have given enough details above to give an explicit proof: the algorithm of Lemma 5.6 coincides with Garfinkle's algorithm for $A_{\mathfrak{q}}(\lambda)$ modules, and the general case is reduced to this by the Harish-Chandra cell structure described in the beginning of Section 6.

Example 5.9. We continue the example of $U(p,1)$ initiated in 3.11. If $p \geq 2$, there are four nilpotent orbits in $\mathfrak{u}(p,1)$. They are parametrized by the signed tableaux

$$\begin{array}{|c|}\hline - \\ \hline + \\ \hline \vdots \\ \hline + \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline + & - \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline + & - & + \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \quad .$$

and we will abbreviate these tableaux by their top row. We can apply the algorithm of Lemma 5.6 to the compute the Richardson orbits corresponding to the asymptotic support of the good $A_{\mathfrak{q}}(\lambda)$ with infinitesimal character ρ parametrized in Example 3.11. Using the notation established there we get

$$\mathrm{AS}(X[0,0]) = \boxed{-}\ ;$$

$$\mathrm{AS}(X[i,0]) = \boxed{+|-}\ , \quad i \neq 0;$$
$$\mathrm{AS}(X[0,j]) = \boxed{-|+}\ , \quad j \neq 0;$$
$$\mathrm{AS}(X[i,j]) = \boxed{+|-|+}\ , \quad i,j \neq 0.$$

This concludes the example.

We will need the following technical lemma for applications below.

Lemma 5.10. *Let $\mathfrak{q}$ be attached to the sequence $\{(p_1,q_1),\ldots,(p_r,q_r)\}$. For $j \leq r$, let $\mathfrak{q}(j)$ be the θ-stable parabolic of $\mathfrak{g}(j) = \mathfrak{gl}(\sum_{i\leq j}(p_i+q_i),\mathbb{C})$ attached to the subsequence $\{(p_1,q_1),\ldots,(p_j,q_j)\}$. Let S be the (equivalence class of) signature (p,q) tableau corresponding to $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$. For any representative $\hat{S}$ of S, write $\hat{S} = \coprod \hat{S}_i$ for the partition of $\hat{S}$ into disjoint skew columns (Notation 2.3) obtained by requiring*

$$\mathrm{shape}(\coprod_{i\leq j} \hat{S}_i) = \mathrm{shape}(\mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero})).$$

Then for any $\hat{S}$, we have the following conclusion:

$$\coprod_{i\leq j} \hat{S}_i = \mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero}).$$

In particular, the number of plus (or minus) signs in each $\hat{S}_j$ is independent of the choice of $\hat{S}$.

Sketch. The first assertion is not as obvious as it may seem. The main point is that in the lemma $\hat{S}$ is fixed, yet at each stage the algorithm of Lemma 5.6 potentially requires rows to be interchanged (cf. the parenthetic comment concluding the statement of Lemma 5.6). The reason this introduces no complications is as follows. Write $\hat{T}^j$ for any representative of $\mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero})$. Suppose $\hat{T}^j$ has several rows of length m, with at least one length m row ending $+$ and at least one ending in $-$. Then the corresponding rows of $\hat{T}^{j-1}$ either *all* have length $m-1$ or *all* have length m. Given this observation, the first assertion follows. The final assertion is clear. $\square$

6. The Barbasch-Vogan parametrization and the good range.

In [V6] and [BV4], the definition of Kazhdan-Lusztig cells is adapted to the real case giving an equivalence relation on the set of Harish-Chandra modules with infinitesimal character ρ. Equivalence classes contain modules with the same asymptotic support and, for $U(p,q)$ each class contains a canonically defined $A_{\mathfrak{q}}(\lambda)$. Thus, by Proposition 5.4, the asymptotic support of *any* Harish-Chandra module for $U(p,q)$ with trivial infinitesimal character is irreducible. In fact, Barbasch and Vogan proved that cells are completely characterized by the signed tableau corresponding to the asymptotic support of any element in the cell. Moreover, all such tableaux arise in this way.

As in the complex case, the elements of a Harish-Chandra cell parametrize an integral basis for a subquotient of the coherent continuation representation. (The subquotient is minimal with respect to the property of being spanned by irreducible characters.) By a counting argument, Barbasch and Vogan showed that all subquotients in question are irreducible, a phenomenon which McGovern [Mc1] has subsequently shown is a consequence of the fact that all irreducible representations of S_n are special. In any case, Barbasch and Vogan proved that the subquotient corresponding to the cell parametrized by a signed tableau $T_{\pm}$ is simply the irreducible representation of S_n given (in Young's notation) as shape$(T_{\pm})$, and as a consequence deduce the following theorem for infinitesimal character $\nu = \rho$. The general case follows from a translation principle (cf. Theorem 4.4).

Theorem 6.1 ([BV4]). *Suppose that $\nu \in \rho + \mathbb{Z}^n$ is a weight lattice translate of the infinitesimal character of the trivial representation. The map assigning an irreducible Harish-Chandra module for $U(p,q)$ with infinitesimal character ν to the pair consisting of its annihilator and its asymptotic support is an injection. On the level of tableaux (Section 4 and 5), the map assigns a ν-antitableau and a signature (p,q) signed tableau (of the same shape) to each irreducible module of infinitesimal character ν, and any such pair arises in this way.*

Remark 6.2. When $\nu = \rho$, the map described in the theorem is formally analogous to the Robinson-Schensted algorithm arising in the computation of Kazhdan-Lusztig cells for $\mathfrak{sl}(n,\mathbb{C})$. This analogy can be made precise in terms of the geometry of the generalized Steinberg variety; see [Tr] for details.

The main goal of this paper is to identify the parameters of the weakly fair $A_{\mathfrak{q}}(\lambda)$, and we need to start by identifying the good $A_{\mathfrak{q}}(\lambda)$. For a fixed regular integral infinitesimal character ν, Garfinkle described an algorithm taking Langlands parameters to pairs of tableaux and proved that the algorithm computes annihilators by a very detailed and complicated combinatorial calculation with the generalized τ-invariant [G]. Moreover, [VZ] explicitly gives the Langlands parameters of the good $A_{\mathfrak{q}}(\lambda)$, so combining these results one obtains a tableau characterization of the good $A_{\mathfrak{q}}(\lambda)$. This is entirely tractable, but we choose to avoid these relatively ponderous references and side-step the issue of Langlands parameters by a simple application of the results of Sections 4 and 5. We take that up now.

The idea is to build up the tableaux parameters of an $A_{\mathfrak{q}}(\lambda)$ step-by-step from the simple factors of $\mathfrak{l}_o$. The inductive proof is quite simple but the notation for the general case is a little overwhelming. We indicate the inductive procedure in the following example, and leave it to the reader to formulate the general proof of the theorem which follows.

Example 6.3. Let $\mathfrak{q} \subset \mathfrak{gl}(8,\mathbb{C})$ correspond to $\{(2,2),(1,3)\}$, and let

$$\lambda = (\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2)$$

be in the good range for $\mathfrak{q}$. Let $\lambda^{(1)} = (\lambda_1, \dots, \lambda_1)$ be the indicated character of $\mathfrak{gl}(4,\mathbb{C})$, and similarly for $\lambda^{(2)}$. Set $\nu = \lambda + \rho$ and $\mu = \lambda^{(1)} + \rho(\mathfrak{gl}(4,\mathbb{C}))$. We compute the tableau parameters $(S, S_{\pm})$ giving the annihilator and associated variety of $A_{\mathfrak{q}}(\lambda)$. To illustrate the induction, we write

$$A_{\mathfrak{q}}(\lambda) = \mathcal{L}(\mathbb{C}_{\lambda^{(1)}} \otimes \mathbb{C}_{\lambda^{(2)}}) = \mathcal{L}(A_{\mathfrak{q}^{(1)}}(\lambda^{(1)}) \otimes \mathbb{C}_{\lambda^{(2)}});$$

here $\mathfrak{q}^{(1)} = \mathfrak{gl}(4,\mathbb{C})$. Now $A_{\mathfrak{q}^{(1)}}(\lambda^{(1)}) = \mathbb{C}_{\lambda^{(1)}}$, so its tableau parameters are

$$\begin{array}{|c|}\hline \mu_1 \\ \hline \mu_2 \\ \hline \mu_3 \\ \hline \mu_4 \\ \hline \end{array} \quad , \quad \begin{array}{|c|}\hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline \end{array} \quad .$$

Proposition 5.4 abstractly computes $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$, and Lemma 5.6 does so explicitly. Using the algorithm of the lemma, we obtain

$$S_{\pm} = \mathrm{AS}(A_{\mathfrak{q}}(\lambda)) \quad = \quad \begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline - & + \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

Now S must have the same shape as $S_{\pm}$, and Corollary 4.6 tells us the first four coordinates of S; so far, then, we know that S looks like

$$S = \begin{array}{|c|c|} \hline \nu_1 & \\ \hline \nu_2 & \\ \hline \nu_3 & \\ \hline \nu_4 & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

Finally τ-invariant considerations (Lemma 3.12 and Lemma 4.2) imply that the remaining coordinates $\nu_5, \dots, \nu_8$ must be sequentially entered moving strictly down S; there is only one way to do this:

$$S = \begin{array}{|c|c|} \hline \nu_1 & \nu_5 \\ \hline \nu_2 & \nu_6 \\ \hline \nu_3 & \nu_7 \\ \hline \nu_4 & \multicolumn{1}{c}{} \\ \cline{1-1} \nu_8 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

This completes the inductive computation of $(S, S_{\pm})$ for $A_{\mathfrak{q}}(\lambda)$.

Theorem 6.4. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ corresponds to $\{(p_1, q_1), \dots, (p_r, q_r)\}$ (Notation 2.1), let $\mathbb{C}_\lambda$ be a one-dimensional $(l, L \cap K)$-module in the good range for $\mathfrak{q}$ (Definition 3.4), and let*

$$\nu = (\nu_1^{(1)}, \dots, \nu_{p_1+q_1}^{(1)}, \dots\dots, \nu_1^{(r)}, \dots, \nu_{p_r+q_r}^{(r)}) = \lambda + \rho.$$

The tableau parameters (Theorem 6.1) of $A_{\mathfrak{q}}(\lambda)$ are obtained inductively as follows. Start with the empty pair of tableaux and assume that the $(s-1)$ step has been completed giving a pair $(S^{(s-1)}, S_{\pm}^{(s-1)})$. $S_{\pm}^{(s)}$ is obtained by adding p_s pluses and q_s minuses to $S_{\pm}^{(s-1)}$ according to the algorithm of Lemma 5.6; $S^{(s)}$ is the tableau of the same shape of $S_{\pm}^{(s)}$ obtained by adding the coordinates $\nu_1^{(s)}, \dots, \nu_{p_s+q_s}^{(s)}$ sequentially from top to bottom in the remaining unspecified boxes.

Remark 6.5. Because of the good range condition on λ, the algorithm of Theorem 6.4 automatically produces a ν-antitableau S. But even if λ isn't good, the algorithm still produces a ν-quasitableau. Theorem 7.9 describes how to straighten this quasitableau into a ν-antitableau which corresponds to the annihilator of $A_{\mathfrak{q}}(\lambda)$.

Remark 6.6. The proof of Theorem 6.4 generalizes far beyond $U(p, q)$ and, in the case of a classical real Lie groups $G_{\mathbb{R}}$, essentially reduces the computation of annihilators of $A_{\mathfrak{q}}(\lambda)$ modules to the relatively elementary issue of computing real Richardson orbits. When the cell structure of $G_{\mathbb{R}}$ is particularly simple (for instance, if each cell contains an $A_{\mathfrak{q}}(\lambda)$ module), then the argument is powerful enough to compute annihilators of *any* Harish-Chandra module for $G_{\mathbb{R}}$. Precise details will appear elsewhere.

Example 6.7. Consider again $U(p,1)$ and recall Examples 3.11 and 5.9. Recall the $A_{\mathfrak{q}}(\lambda)$ modules $X[i,j]$ of infinitesimal character $\rho = (\rho_1, \ldots, \rho_n)$. Using Theorem 6.4, we can compute annihilators of these modules.

$$\mathrm{Ann}(X[0,0]) = \begin{array}{|c|}\hline \rho_1 \\ \hline \vdots \\ \hline \rho_n \\ \hline \end{array}$$

$$\mathrm{Ann}(X[i,0]) = \begin{array}{|c|c|}\hline \rho_1 & \rho_{k_i} \\ \hline \vdots \\ \cline{1-1} \vdots \\ \cline{1-1} \end{array}, \qquad k_i = i+1; i \neq 0$$

$$\mathrm{Ann}(X[0,j]) = \begin{array}{|c|c|}\hline \rho_1 & \rho_{l_j} \\ \hline \vdots \\ \cline{1-1} \vdots \\ \cline{1-1} \end{array}, \qquad l_j = n+1-j; j \geq 1$$

$$\mathrm{Ann}(X[i,j]) = \begin{array}{|c|c|c|}\hline \rho_1 & \rho_{k_i} & \rho_{l_j} \\ \hline \vdots \\ \cline{1-1} \vdots \\ \cline{1-1} \end{array}, \qquad k_i = i+1; l_j = n+1-j; i,j \neq 0.$$

An easy count of Langlands parameters shows that the modules $X[i,j]$ exhaust the irreducible Harish-Chandra modules with infinitesimal character ρ. By Theorem 3.1, the $X[i,j]$ are all unitary, and so we have verified Conjecture 1.1 explicitly for infinitesimal character ρ. (This case was handled originally by Baldoni-Barbasch [BaBa].) The interested reader can explicitly prove Theorem 6.1 in this case by using the description of the coherent contінutation representation given in [C].

In terms of the program described in the introduction, the more important kind of result is determining when a pair of tableaux actually parametrizes an $A_{\mathfrak{q}}(\lambda)$. Such a statement follows by formally examining the algorithm of Theorem 6.4. (Corollary 6.8 is restated a little more cleanly in Corollary 6.12).

Corollary 6.8. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*$ be dominant, integral, and regular. Consider a pair $(S_\pm, S)$ consisting of a signature (p,q) signed tableau and a ν-antitableau. Partition S into disjoint union of difference-one skew columns (Notation 2.3) $S_1, \ldots S_m$ ordered by their maximal entry, and let $\hat{S}_{\pm,j}$ denote the corresponding skew columns of a representative $\hat{S}_\pm$ of $S_\pm$. Set $S^k = \coprod_{i \leq k} S_i$ and similarly define $\hat{S}^k_\pm$. Assume that each S^k is itself a tableau. Then $(S_\pm, S)$ parametrizes a good $A_{\mathfrak{q}}(\lambda)$ for $U(p,q)$ if and only if there is a representative $\hat{S}_\pm$ of $S_\pm$ such that the following condition holds for all j: if a row is skipped in the arrangement of S_j in S, then all rows in $\hat{S}^j_\pm$ below (and including) the first skipped row and above (and including) the last row of S_j end in the same sign.*

Moreover, $\mathfrak{q}$ and λ can be read off from the S_j as follows: $\mathfrak{q}$ corresponds to the ordered sequence of pairs of integers obtained from the number of plus and minus signs in $\hat{S}_{\pm,j}$; and $\lambda = \nu - \rho$. (The data of $\mathfrak{q}$ and λ is independent of the choice of representative $S_\pm$.)

Sketch. Suppose $(S, S_\pm) = (\mathrm{Ann}(A_{\mathfrak{q}}(\lambda)), \mathrm{AV}(A_{\mathfrak{q}}(\lambda)))$, for some λ in the good range for $\mathfrak{q}$. We are to find a partition of S and a representative $\hat{S}_\pm$ of $S_\pm$ satisfying the requirements of the corollary. Theorem 6.4 gives a partition of $S = \coprod S_i$ into disjoint difference-one skew columns, and Lemmas 5.6 and 5.10 imply that the corresponding partition of $\hat{S}_\pm$ (for any choice of representative $\hat{S}_\pm$) satisfies the required conditions.

Conversely, if such a partition and representative of $(S, S_\pm)$ are given, Theorem 6.4 clearly implies $(S, S_\pm) = (\mathrm{Ann}(A_{\mathfrak{q}}(\lambda)), \mathrm{AV}(A_{\mathfrak{q}}(\lambda)))$, where $\mathfrak{q}$ and λ are defined as in the second paragraph of the corollary. The final parenthetical assertion follows from the concluding assertion in the statement of Lemma 5.10. □

Remark 6.9. The partition in the corollary may not be unique; the failures of uniqueness correspond exactly to the adjacent pairs condition in Lemma 3.9.

Now we introduce a little more notation designed to rewrite the statement of Corollary 6.8 in a more compact form which generalizes.

Definition 6.10. Let $S_1, \ldots, S_r$ be a set of disjoint difference-one skew columns of a ν-quasitableau S, and suppose $S = \coprod S_i$. Then we say that the S_i form a partition of S into difference-one skew columns if $S^j = \coprod_{a \leq j} S_i$ is a quasitableau for each $j = 1, \ldots, r$.

If $\hat{S}_\pm$ is a representative of a signed tableau $S_\pm$ of the same shape of S, any partition of S into skew columns induces a partition $\hat{S}_\pm = \coprod \hat{S}_{\pm,i}$ of $\hat{S}_\pm$ into skew columns. Let (p_i, q_i) denote the number of plus and minus signs in $\hat{S}_{\pm,i}$, and let $\mathfrak{q}^j$ be the θ-stable parabolic corresponding to the ordered sequence $\{(p_1, q_1), \ldots, (p_j, q_j)\}$ (as in Notation 2.3) of the appropriate $\mathfrak{g}^j = \mathfrak{gl}(n^j, \mathbb{C})$. We say that the partition $S = \coprod S_i$ is consistent with (the representative) $\hat{S}_\pm$ if

$$\hat{S}^j_\pm = \coprod_{i \leq j} \hat{S}_{\pm,i} = \mathrm{ind}_{\mathfrak{q}^j}^{\mathfrak{g}^j}(\mathcal{O}_{zero}), \qquad \text{for all } j.$$

Using Lemma 5.6, we obtain an explicit condition for $S = \coprod S_i$ to be consistent with $\hat{S}_\pm$: if a row is skipped in the arrangement of S_j in S, then all rows in $\hat{S}^j_\pm$ below (and including) the first skipped row and above (and including) the last row of S_j must end in the same sign.

Suppose we are given a partition of a ν-antitableau S into difference-one skew columns, $S = \coprod S_j$, consistent with $\hat{S}_\pm = \coprod \hat{S}_{\pm,i}$. To this data, we may attach a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ to the sequence $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ as above; and we obtain a unitary one-dimensional representation, $\mathbb{C}_\lambda$, of $\mathfrak{l}$ as follows. Set $[\nu]$ equal to the n-tuple of numbers obtained by concatenating the entries appearing in the skew columns $S_1, \ldots, S_r$, and view $[\nu]$ as a functional on $\mathfrak{t}$; then set $\lambda = [\nu] - \rho$. Note that $\mathfrak{q}$ and λ constructed in this way are independent of the choice of representative $\hat{S}_\pm$. (Clearly λ is independent of the choice; the last sentence of Lemma 5.10 implies that $\mathfrak{q}$ is too.) We say that $\mathfrak{q}$ and λ are associated to the partition of $S = \coprod S_i$ consistent with $S_\pm$.

Finally, we translate the ranges of positivity of Definition 3.4 to the level of tableaux. (The point is that if λ is in the, say, mediocre range for $\mathfrak{q}$, and λ and $\mathfrak{q}$ are associated to some partition $S = \coprod S_i$, we want to define the columns S_i to be mediocre.) If $i < j$, two difference-one skew columns, S_i and S_j in a partition of S are said to be in mediocre position if either of the following conditions is satisfied: the smallest entry in S_i is greater than or equal to the smallest entry in S_j; or the largest entry in S_i is greater than or equal to the largest entry of S_j. The skew columns are said to be in (weakly) fair position if the average of the entries in S_i is (weakly) greater than the average of the entries in S_j. Similarly, S_i and S_j are said to be in (weakly) good position if the smallest entry in the S_i is (weakly) larger than the largest entry in S_j. Finally, we say that S_i and S_j are in nice position if *both* the smallest entry in S_i is greater than or equal to the smallest entry in S_j, and the largest entry in S_i is greater than or equal to the largest entry of S_j. (We have not encountered the nice condition before, but it will be important in the combinatorics of Section 8.) The entire partition is called mediocre, fair, good, or nice if all pairs of its skew columns are in the specified position.

Remark 6.11. We have the following implications on the ranges defined above:

$$\begin{array}{ccccc} \text{good} & \Longrightarrow & \text{fair} & \Longrightarrow & \text{mediocre} \\ & \searrow & \Uparrow & \nearrow & \\ & & \text{nice} & & \end{array}$$

With the above definitions, Corollary 6.8 becomes:

Corollary 6.12. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*$ be dominant, integral, and regular. Let X be an irreducible Harish Chandra module for $U(p,q)$ of infinitesimal character ν and consider $(S, S_\pm) = (\mathrm{Ann}(X), \mathrm{AS}(X))$ (Theorem 6.1). Then $X \cong A_{\mathfrak{q}}(\lambda)$ if and only if there is a partition of S into difference-one skew columns consistent with a representative of $S_\pm$ so that $\mathfrak{q}$ and λ are associated to this partition (Definition 6.10).*

We will generalize this in Corollary 7.12 below.

7. Statement of main theorems

As mentioned in Remark 6.5, the algorithm of Theorem 6.4 has an obvious analog outside the good range. But when $A_{\mathfrak{q}}(\lambda)$ is no longer good, there is nothing to guarantee that the quasitableau produced is in fact an antitableau. Sometimes it is, and in these cases, the algorithm of Theorem 6.4 does produce the annihilator of $A_{\mathfrak{q}}(\lambda)$ (this has to be proved, of course). But sometimes the quasitableau is not an antitableau, and we need a way to convert it into the one parametrizing the corresponding annihilator. In order to do this, we must move outside the class of $A_{\mathfrak{q}}(\lambda)$ modules to a larger class of representations that still retains most of the nice translation properties of the $A_{\mathfrak{q}}(\lambda)$'s. Combinatorially, this procedure introduces an equivalence relation on the the set of partitions of ν-quasitableaux into difference-one skew columns. Then given X, we can conclude that X is isomorphic to a weakly fair $A_{\mathfrak{q}}(\lambda)$ if and only if there exists a suitably consistent representative in the equivalence class of some nice partition of $\mathrm{Ann}(X)$. Moreover, the equivalence relation will keep track of all coincidences and vanishing among the weakly fair (and, in fact, mediocre) $A_{\mathfrak{q}}(\lambda)$.

Now we make these matters more precise, and begin to describe the equivalence relation. As a first step we need to define a rough measure of the size and singularity of two adjacent columns in a partition of S into difference-one skew columns. (The manner in which 'size' is to be interpreted is discussed in the remark following the definition.)

Definition 7.1. Given two adjacent columns $C = S_j$ and $D = S_{j+1}$ of a partition of S into difference-one skew columns (Definition 6.10), we first define an integer depending only on the shape of C and D in the following way. Label the entries of C and D (moving sequentially down each skew column) as $c_1, \ldots, c_k$, and $d_1, \ldots, d_l$. For $1 \le m \le \min(k,l)$ define a condition

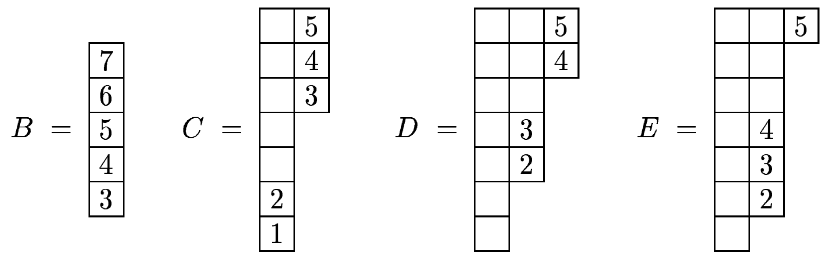
$$\text{condition } m: \quad c_{k-m+i} \text{ is strictly left of } d_i \text{ in } S \quad \text{for } 1 \le i \le m.$$

Define the overlap of C and D, denoted overlap(C,D) to be the largest $m \leq \min(k,l)$ so that condition m holds. (If condition m never holds, define overlap$(C,D) = 0$.)

The singularity of C and D, denoted sing(C,D), is defined to be the number of pairs of identical entries among the c_i and d_j. This is an integer which depends on the entries of C and D (and is independent of shape of C and D).

Remark 7.2. When $S = C \coprod D$ consists of only two skew columns, then it is easy to see that the overlap is simply the number of rows of length two in S, and hence is precisely related to the size of the variety associated to the primitive ideal corresponding to S. In the general case, an analogous interpretation of overlap(C,D) exists, but since we do not need the precise statement, we leave it to the reader to formulate.

Example 7.3. Consider



Then overlap$(B,C) =$ sing$(B,C) = 3$; overlap$(C,D) =$ sing$(C,D) = 4$; and overlap$(C,E) = 3 <$ sing$(C,E) = 4$.

With these definitions in hand, we can now define the equivalence relation. The reader is encouraged to read Example 7.6 concurrently with the definition.

Definition 7.4. We define an equivalence relation on the set consisting of mediocre partitions of ν-quasitableau into difference-one skew columns (Notation 2.3, Definition 6.10) together with the formal quasitableau 0. Suppose $S = \coprod S_i$ is a mediocre partition of a ν-quasitableau into difference-one skew columns. The equivalence relation will be generated by replacing adjacent skew columns S_i, S_{i+1} in weakly mediocre position with S_i', S_{i+1}' in nice position (Definition 6.10); here $R = S_i \coprod S_{i+1}$ and $R' = S_i' \coprod S_{i+1}'$ have identical shape and the entries of R' are a permutation of those of R. The resulting equivalence is given by

$$S = \coprod_j S_j \sim S' = \coprod_{j<i} S_j \coprod S_i' \coprod S_{i+1}' \coprod_{j>i+1} S_j.$$

We now describe $R' = S_i' \coprod S_{i+1}'$. Since we will constantly refer to this procedure, we set it aside.

Procedure 7.5. Suppose that S_i and S_{i+1} have lengths r and s respectively, and recall Definition 7.1.

(a) If

$$\text{overlap}(S_i, S_{i+1}) > \text{sing}(S_i, S_{i+1}),$$

or

$$\text{overlap}(S_i, S_{i+1}) = \text{sing}(S_i, S_{i+1}) < \min(r,s),$$

Then $R' = R$.

(b) If
$$\text{overlap}(S_i, S_{i+1}) < \text{sing}(S_i, S_{i+1}),$$
then S is defined to be equivalent to 0, the (formal) zero tableau.

(c) Assume
$$\text{overlap}(S_i, S_{i+1}) = \text{sing}(S_i, S_{i+1}) = \min(r, s).$$
We begin by describing a rearrangement, R' of the coordinates of $R = S_i \coprod S_{i+1}$. Let $a_1, \dots, a_r$ denote the sequential entries of S_i and $b_i, \dots, b_s$ likewise for S_{i+1}. Assume that the b's are a subset of the a's (the opposite case is described below), and write them as
$$\begin{array}{l} a_1, \dots, a_{l+1}, \dots, a_{l+s}, \dots, a_r, \qquad a_{l+i} = b_i \\ \qquad\qquad b_1, \dots, b_s. \end{array}$$
If $b_s = a_r$, then define $R' = R$. Otherwise let $S_{i+1}(-1)$ denote the column obtained from S_{i+1} by subtracting one from each entry. Set $R(-1) = S_i \coprod S_{i+1}(-1)$. By induction (the case $b_s = a_r$ is the base case), we can assume that $[R(-1)]'$ (i.e. the procedure applied to $R(-1)$) has been defined. We will construct R' by changing one member of each of the duplicate labels $b_i - 1 (1 \leq i \leq s)$ in $[R(-1)]'$ to b_i. Begin by considering the unique entry a_{l+1} in $[R(-1)]'$. There is at most one box labeled $b_1 - 1$ strictly to the right of a_{l+1} in $[R(-1)]'$. If such a box exists, then add one to its entry. If no such box exists, then add one to the entry in the left-most box labeled $b_1 - 1$ in $[R(-1)]'$. In either case call the resulting skew-tableau $[R(-1)]'_1$. Now construct $[R(-1)]'_2$ by the same procedure applied to $[R(-1)]'_1$, but instead considering the entries a_{l+2} and $b_2 - 1$. Continue in this way, and define $R' = [R(-1)]'_s$.

(d) Keep the assumption on overlap and singularity as in the previous case, but suppose that the a's are a subset of the b's. Write them as
$$\begin{array}{l} \qquad\qquad a_1, \dots, a_r \\ b_1, \dots, b_{l+1}, \dots, b_{l+r}, \dots, b_s \qquad a_i = b_{l+i}. \end{array}$$
Informally, we compose the algorithm of the previous case with an automorphism of $\mathfrak{g}$ coming from the Dynkin diagram. If $l = 0$, define $R' = R$. Otherwise, set $R(+1) = S_i(+1) \coprod S_{i+1}$, where $S_i(+1)$ is obtained by adding 1 to each entry of S_i. Consider the unique entry b_{l+r} in $[R(1)]'$ (the skew tableau obtained by the inductive hypothesis applied to $R(1)$). Then at most one box $a_r + 1$ is strictly left of b_{l+r} in $[R(1)]'$. If such a box exists, change its label to a_r. If no such box exists, change the label of the right-most occurrence of $a_r + 1$ to a_r. Let $[R(1)]'_1$ denote the resulting skew tableau. Continue as in (c), and define $R' = [R(1)]'_s$.

To finish the definition, we must give a partition $R' = S'_i \coprod S'_{i+1}$ into difference-one skew columns in nice position. We construct S'_{i+1} as follows. Its last entry consists of the the smallest entry, say c, in R'. (Recall if two entries appear in a skew tableau, the smaller one is the one that occurs strictly right of the other). Its next to last entry consist of the smallest occurrence of $c - 1$ which is strictly above and weakly to the right of the last entry of S'_{i+1}. We continue in this way until we reach the last duplicate entry of R' (i.e. the entry b_1 in case (c) or a_1 in case (d)). This defines S'_{i+1} and S'_i is defined to be what remains. It is not hard to see that the resulting $R' = S'_i \coprod S'_{i+1}$ is actually a nice partition of R' into difference-one skew columns.

Example 7.6. Recall the difference-one skew columns C, D, E of Example 7.3. Consider the partitions

$$S \;=\; \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 3 & \\ 3 & 2 & \\ 2 & & \\ 1 & & \end{array} \;=\; B \coprod C \coprod D \;\; ; \text{ and}$$

$$T \;=\; \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & \\ 5 & 3 & \\ 4 & 4 & \\ 3 & 3 & \\ 2 & 2 & \\ 1 & & \end{array} \;=\; B \coprod C \coprod E \;.$$

(a) Referring to Example 7.3, we know $\text{overlap}(B, C) = \text{sing}(B, C) < \min(5, 5)$. Hence Procedure 7.5(a) applies to $R = B \coprod C$ to give $R' = R$. Moreover, the partition described in the last paragraph of Definition 7.4 is $R' = B \coprod C$.

(b) Again referring to Example 7.3, we see that Procedure 7.5(b) applies to $R = C \coprod E$ to give zero. Hence T is equivalent to the zero tableau.

(c) Finally consider $R = C \coprod D$. Applying Procedure 7.5(c) and the definition of the partition of R', we obtain

$$R' = C' \coprod D' \;=\; \begin{array}{cc} \square & 5 \\ \square & 4 \\ \square & \\ \square & \\ \square & \\ 3 & \\ 2 & \end{array} \;\coprod\; \begin{array}{ccc} \square & \square & 5 \\ \square & \square & 4 \\ \square & 3 & \\ \square & 2 & \\ \square & 1 & \\ \square & & \\ \square & & \end{array} \;.$$

We thus obtain

$$B \coprod C \coprod D = \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 3 & \\ 3 & 2 & \\ 2 & & \\ 1 & & \end{array} \;\sim\; \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 2 & \\ 3 & 1 & \\ 3 & & \\ 2 & & \end{array} = B \coprod C' \coprod D' \,.$$

Now applying Procedure 7.5(a) to $B \coprod C'$ shows that $S \sim 0$, since $\text{overlap}(B, C') = 2 < \text{sing}(B, C') = 3$.

The next lemma will be used frequently. Its proof amounts to the fact that the mediocre condition was defined not only for adjacent pairs of columns, but all pairs (see Definition 6.10 and Remark 3.6).

Lemma 7.7. *Let $S = \coprod S_j$ be a mediocre partition of a ν-quasitableau into difference-one skew columns. If $S \sim S' = \coprod S'_j$, then the partition of S' is again mediocre.*

The following very technical lemma will be useful below. (The reader is encouraged to skip it, and refer back when necessary.)

Lemma 7.8. *Let $T = T_1 \coprod T_2 \coprod T_3$ be a partition of a ν-antitableau into difference-one skew columns. Assume T_1 and T_2 are in nice position, T_2 and T_3 are in mediocre position, and that T_1 contains the largest entry of T while T_3 contains the smallest. Suppose that Procedure 7.5 applied to $T_2 \coprod T_3$ gives $T'_2 \coprod T'_3$. Then $T_1 \coprod T'_2 \coprod T'_3$ is nice.*

Sketch. By hypothesis T'_2 and T'_3 are in nice position, and since T_1 contains the largest entry of T by hypothesis, we need only verify that the smallest entry of T_1 is greater than or equal to the smallest entry of T'_2. Write t_i for the smallest entry of T_i, and similarly for t'_i. The nice hypothesis on T_1 and T_2 implies $t_1 \geq t_2$, so it is enough to verify the following claim: t_2 occurs in T'_2. Since t_3 is the smallest entry in T (by hypothesis), we know $t_2 \geq t_3$. So either t_2 is larger than every entry in T_3, or t_2 occurs in T_3. In the former case, Procedure 7.5(a) applies to give $T'_2 = T_2$, and we conclude t_2 occurs in T'_2 as claimed. In the latter case, t_2 occurs twice in $T'_2 \coprod T'_3$. We conclude t_2 must occur in T'_2, thus proving the claim and hence the lemma. □

We can now state our main results.

Theorem 7.9. *Let $\mathfrak{q}$ be a θ-stable parabolic and $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L \cap K)$-module in the mediocre range for $\mathfrak{q}$. Let $\nu = \lambda + \rho = (\nu_1, \ldots, \nu_n)$, and construct a ν-quasitableau S, together with a partition into difference-one skew columns $S = \coprod S_i$, as in Theorem 6.4. Then there is a algorithm (described below) to locate a distinguished $S' = \coprod S'_i$ equivalent (in the sense of Definition 7.4) to $S = \coprod S_i$ such that either: $S' = 0$; or S' is actually a ν-antitableau and $\coprod S'_i$ is a nice partition into difference-one skew columns (Definition 6.10) with*

$$\operatorname{overlap}(S'_i, S'_{i+1}) \geq \operatorname{sing}(S'_i, S'_{i+1}) \quad \textit{for all } i \textit{ (Definition 7.1).}$$

The module $A_{\mathfrak{q}}(\lambda)$ is nonzero if and only if the latter case holds and in this case,

$$\operatorname{Ann}(A_{\mathfrak{q}}(\lambda)) = S'.$$

We describe the algorithm of the theorem. Let $\mathfrak{q}$ be associated to the sequence of pairs of positive integers $\{(p_1, q_1), \ldots, (p_r, q_r)\}$, let λ be in the mediocre range for $\mathfrak{q}$, and let $\nu = \lambda + \rho$. The algorithm is defined inductively in terms of r; we consider the cases $r \leq 3$, leaving the general statement to the reader. When $r = 1$, $A_{\mathfrak{q}}(\lambda) = \mathbb{C}_\lambda$, and the theorem is trivial. If $r = 2$, Theorem 6.4 gives a mediocre partition $S = S_1 \coprod S_2$ of a ν-quasitableau into difference-one skew columns. Using Procedure 7.5, we obtain $S \sim S' = S'_1 \coprod S'_2$ or $S \sim 0$. In the former case S' is actually a ν-antitableau and $S' = S'_1 \coprod S'_2$ is a nice partition of S' into difference-one skew columns whose singularity does not exceed their overlap, as required in the theorem.

Next suppose $r = 3$. Again Theorem 6.4 gives a mediocre partition $S = S_1 \coprod S_2 \coprod S_3$ of a ν-quasitableau into difference-one skew columns. By the $r = 2$ case and Lemma 7.7, we may assume S_1 and S_2 are in nice position. Applying Procedure 7.5 to $R = S_2 \coprod S_3$, we obtain either $S \sim 0$ or $S \sim S' = S_1 \coprod S'_2 \coprod S'_3$, the partition being mediocre by Lemma 7.7. In the latter case, S'_2 and S'_3 are in nice position and their singularity does not exceed their overlap; but S_1 and S'_2 are only mediocre. So apply Procedure 7.5 to $R = S_1 \coprod S'_2$ which gives $S \sim S'' = S_1'' \coprod S_2'' \coprod S_3'$. Again if S'' is nonzero, then S_1'' and S_2'' are in nice position (with the correct overlap condition), but S_2'' and S_3' are only mediocre, so we can again apply Procedure 7.5 as before, and continue in this way.

We claim that the see-saw algorithm must eventually produce either zero or a nice partition (with the correct overlap conditions) of a ν-antitableau equivalent to S. To see this notice that when two mediocre columns $T_i \coprod T_{i+1}$ are exchanged for nice ones $T_i' \coprod T_{i+1}'$ by Procedure 7.5, the largest entry of $T_i \coprod T_{i+1}$ always resides in T_i' and the smallest entry in T_{i+1}'. (This follows immediately from the definition.) So after two see-saws, either $S \sim 0$ or $S \sim T_1 \coprod T_2 \coprod T_3$ with the largest entry of S contained in T_1 and the smallest entry in T_3. We can assume T_1 and T_2 are in nice position, and then Lemma 7.8 implies that S is equivalent to a nice parition (or zero) as claimed. The algorithm of the theorem in the $r = 3$ case is complete.

It is clear how the algorithm works for $r > 3$. (Convergence follows by induction and the $r = 3$ case.)

Remark 7.10. When λ is in the weakly good range for $\mathfrak{q}$, Theorem 4.4 and Lemma 3.13 imply that the annihilator of $A_{\mathfrak{q}}(\lambda)$ can be computed from Theorem 6.4 and τ-invariant considerations. Theorem 7.9 reduces to such considerations since the overlap requirement on the nice partition is nothing but a τ-invariant condition in this case.

Remark 7.11. We discuss how to read off the Langlands parameters of a nonzero weakly fair $A_{\mathfrak{q}}(\lambda)$, say X. Theorem 7.9 computes $\mathrm{Ann}(X)$ and Proposition 5.4 gives $\mathrm{AV}(X)$. Using the translation principle (Theorem 4.4), we can find X' with infinitesimal character ρ (say) with $\psi(X') = X$. Moreover, that theorem allows us to easily read off $\mathrm{Ann}(X')$, and of course we know that $\mathrm{AV}(X) = \mathrm{AV}(X')$. Since ψ doesn't cross any walls, its effect on Langlands parameters is easy to understand. So writing down the Langlands parameters of $X = A_{\mathfrak{q}}(\lambda)$ amounts to writing down those of X'. Given X' of trivial infinitesimal character and $(\mathrm{Ann}(X'), \mathrm{AV}(X'))$, Garfinkle ([G]) gives an algorithm to write down the $\mathbb{Z}/2$-datum of X'. Then it is a simple matter to relate this to the usual cuspidal Langlands parameters (see [V4]). Among other things, the paper [Tr] gives the explicit relationship between $\mathbb{Z}/2$-data and $K_{\mathbb{C}}$ orbits on the flag variety, which is relevant to the Beilinson-Berstein classification.

Formally inverting the statement of Theorem 7.9, we obtain:

Corollary 7.12. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*_{\mathbb{R}}$ be dominant and integral. Suppose $(S, S_\pm)$ is a pair consisting of a ν-antitableau and a signature (p, q) tableau of the same shape (Notation 2.3). Then $(S, S_\pm) = (\mathrm{Ann}(X), \mathrm{AV}(X))$ for a mediocre $X \cong A_{\mathfrak{q}}(\lambda)$ if and only if the following condition holds: there exists a nice partition $S = \coprod S_i$ into difference-one skew columns with*

$$\mathrm{overlap}(S_i, S_{i+1}) \geq \mathrm{sing}(S_i, S_{i+1}) \quad \textit{for all } i$$

such that the partition $S = \coprod S_i$ is equivalent (in the sense of Definition 7.4) to a mediocre partition $S' = \coprod S_i'$ so that the partition of S' is consistent with $S_\pm$ and so that $\mathfrak{q}$ and λ are associated to this partition of S' (Definition 6.10).

Remark 7.13. When ν is dominant and regular Theorem 7.9 and Corollary 7.12 reduce to Theorem 6.4 and Corollary 6.12.

Example 7.14. Consider the following pair of tableau for $SU(p,p-1)$, $p \geq 3$,

1	1	1	·	1
0	−1	−1	·	−1
−1				

,

+	−	+	·	·
−	+	−	·	·
+				

.

By Corollary 7.12, these tableaux do not parametrize a mediocre $A_{\mathfrak{q}}(\lambda)$, and so Conjecture 1.1 predicts that the corresponding Harish-Chandra module X is nonunitary. Using Garfinkle's algorithm [G] and the Vogan-Knapp minimal K type formula ([K]), one can again verify that X is a spherical representation. An unpublished result of Barbasch asserts that any unitary spherical representation is a weakly fair $A_{\mathfrak{q}}(\lambda)$; so we conclude that X is nonunitary, as predicted. Note that for $p > 3$, the Dirac operator inequality (a traditional means to detect nonunitarity) is necessarily inconclusive.

8. Proof of Theorem 7.9

In this section, we prove Theorem 7.9. To compute the annihilators of the mediocre $A_{\mathfrak{q}}(\lambda)$, we will use the following strategy. Given such an $A_{\mathfrak{q}}(\lambda)$, we can pull apart the overlaps of λ to obtain a good λ', and then use Lemma 3.13 to move from the good $A_{\mathfrak{q}}(\lambda')$ (where we have complete information about its annihilator and asymptotic support) to our module of interest, $A_{\mathfrak{q}}(\lambda)$. On the surface, the program seems hopeless. The translation functor, T, defined in the lemma is complicated; it is a sequence of multiple wall crossing functors, so it appears as though we need very detailed information about the coherent continuation representation in order to understand T. But Lemma 3.13 says that $T(A_{\mathfrak{q}}(\lambda')) = A_{\mathfrak{q}}(\lambda)$ which implies that whatever intermediate complications involved in computing $T(A_q(\lambda'))$ must all disappear in the final answer.

We are going to compute T by inductively applying the T_i's (of Notation 2.2) to $A_{\mathfrak{q}}(\lambda')$, and we first need to describe the effect of the translations T_i on generalized Verma modules inside the mediocre range.

Lemma 8.1. *Let $\mathfrak{q} = l \oplus \mathfrak{u}$ be the block upper triangular parabolic subgroup of $\mathfrak{gl}(n,\mathbb{C})$ with $\mathfrak{l} = \oplus_{j=1}^{k} \mathfrak{gl}(n_j,\mathbb{C})$. Let i be of the form $i = \sum_{j \leq l} n_j$, and let*

$$\mu_i = e_{i+1} + \cdots + e_n.$$

Suppose that $\mathbb{C}_\eta$ is a character of $\mathfrak{l}$ with η and $\eta + \mu_i$ in the mediocre range for $\mathfrak{q}$ (Definition 3.4). Set $\nu = \eta + \rho$ and $\nu' = \eta + \mu_i + \rho$, and consider the translation functor $T_i = \psi_\nu^{\nu'}$ as in Notation 2.2. Let $M(\eta)$ denote the (normalized) generalized Verma module

$$M(\eta) = \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(C_\eta \otimes \bigwedge^{top}(\mathfrak{u})).$$

Then

$$T_i(M(\eta)) = M(\eta + \mu_i).$$

Pf. The proof is an induction on k, the number of factors of $\mathfrak{l}$. We describe the base case when $\mathfrak{l} = \mathfrak{gl}(n_1,\mathbb{C}) \oplus \mathfrak{gl}(n_2,\mathbb{C})$, giving enough details so that the reader can complete the induction. In this case, $i = n_1$, and without loss of generality (by, say, Theorem 4.4) we may take

$$\eta = (\overbrace{0,\ldots,0}^{n_1},\overbrace{t,\ldots,t}^{n_2}),$$

the mediocre hypothesis on $\eta + \mu_i$ implying $t < \max(n_1, n_2)$. As outlined in the proof of Lemma 4.2, $T_{n_1}(M(\eta))$ admits a filtration with generalized Verma quotients $M(\kappa)$ characterized by

(a) κ is of the form $\eta+\lambda$ where λ is a highest weight (for $\mathfrak{t}\cap\mathfrak{l}$ in $\mathfrak{l}$) of an irreducible constituent of the finite-dimensional $\mathfrak{g}$ module F^{μ_i} of extremal weight μ_i restricted to $\mathfrak{l}$; and

(b) $\kappa+\rho \in W(\eta+\mu_i+\rho)$.

Concretely, F^{μ_i} is nothing but $\bigwedge^{n_2}(\mathbb{C}^n)$, so the restriction in (a) is easy to compute: the highest weights λ are exactly

$$\lambda_{l,m} = (\overbrace{1,\ldots,1,\underbrace{0,\ldots,0}_{l}}^{n_1},\overbrace{1,\ldots,1,\underbrace{0,\ldots,0}_{m}}^{n_2}), \qquad \text{with } l+m = n_1.$$

Now, using the mediocre assumption on t, one can check directly that $\lambda = \lambda_{0,n_2} = \mu_i$ is the unique highest weight satisfying the requirements (a) and (b) above (hence giving the conclusion that $T(M(\eta)) = M(\eta+\mu_i)$, and completing the $k = 2$ case). Since this is the crux of the matter, we give the argument in detail.

Write

$$\eta + \rho = (\rho_1, \ldots, \rho_{n_1}, t + \rho_{n_1+1}, \ldots, t + \rho_n), \qquad \rho_j = \frac{n - 2j + 1}{2}.$$

First assume that $n_1 \geq n_2$; together with the mediocre assumption on $t + \mu_i$, this implies that

$$\rho_1 > t + \rho_{n_1+1}.$$

Hence the value $\rho_1 + 1$ does not appear as a coordinate of $\eta+\mu_i+\rho$. Thus if $\eta+\lambda_{l,m}+\rho \in W(\eta+\mu_i+\rho)$, we must have that the first coordinate of $\eta + \lambda_{l,m} + \rho$ is ρ_1+1. Hence the first coordinate of $\lambda_{l,m}$ is 0. This implies $\lambda_{l,m} = \lambda_{0,n_2}$, as claimed.

On the other hand, if $n_1 < n_2$, then the mediocre hypothesis on $\eta+\mu_i$ implies that

$$\rho_{n_1} > t + \rho_n.$$

Thus the value $t+\rho_n$ does not appear as an entry of $\eta+\mu_i+\rho$. So if $\eta+\lambda_{l,m}+\rho \in W(\eta+\mu_i+\rho)$, the last coordinate of $\lambda_{l,m}$ must be 1. Hence $\lambda_{l,m} = \lambda_{0,n_2}$, as claimed.

The general case follows by using the $k = 2$ arguments on adjacent Levi factors of $\mathfrak{l}$ and proceeding inductively. □

Corollary 8.2. *Retain the notations of the previous lemma, and for $1 \leq l \leq m \leq n$, let*

$$r = \sum_{j\leq l} n_j, \qquad s = \sum_{j\leq m} n_j$$

Suppose η, $\eta+\mu_r$, $\eta+\mu_s$, and $\eta+\mu_r+\mu_s$ are in the mediocre range for $\mathfrak{q}$. Then

$$T_rT_s(M(\eta)) = M(\eta+\mu_r+\mu_s) = T_sT_r(M(\eta))$$

The previous lemma and corollary complete the proof of Lemma 3.13. We will, however, need a strengthened version of Corollary 8.2.

Lemma 8.3. *Retain the notations and assumptions of the previous lemma and corollary, and let F^r and F^s denote the irreducible representations of $\mathfrak{g}$ with extremal weights μ_r and μ_s. Then the translation functors $T_r(T_s(M(\eta))) = T_s(T_r(M(\eta)))$ can be computed as*

$$P(M(\eta) \otimes F^r \otimes F^s),$$

where P denotes the projection on infinitesimal character $\eta+\rho+\mu_r+\mu_s$ (as in Notation 2.2).

Pf. All of the ideas of the general setting are captured in the case when $\mathfrak{l}$ is the sum of three blocks. So assume $\mathfrak{l} = \oplus_{i=1}^{3}\mathfrak{gl}(n_1,\mathbb{C})$, with $r = n_1$ and $s = n_1+n_2$. Write $T_{rs}(M(\eta))$ instead of $P(M(\eta)\otimes F^r \otimes F^s)$. As in the proof of Lemma 8.1, $T_{rs}(M(\eta))$ admits a filtration with generalized Verma quotients $M(\kappa)$ characterized by

(a) κ is of the form $\eta+\lambda$ where λ is a highest weight (for $\mathfrak{t}\cap\mathfrak{l}$ in $\mathfrak{l}$) of an irreducible constituent of $F^r\otimes F^s$ restricted to $\mathfrak{l}$; and

(b) $\kappa+\rho \in W(\eta+\mu_r+\mu_s+\rho)$.

Using the fact that $F^r = \bigwedge^{n_2+n_3}(\mathbb{C}^n)$ and $F^s = \bigwedge^{n_2}(\mathbb{C}^n)$, it is not difficult to see that

$$F^r\otimes F^s = \bigoplus F[2^l 1^m],$$

where the sum is over all pairs l and m with $2l+m = 2n_2+n_3$ and $0\le l\le n_2$; here $F[2^k1^l]$ is the finite dimensional representation of $\mathfrak{gl}(n,\mathbb{C})$ with highest weight

$$2(e_1+\cdots e_k)+(e_{k+1}+\cdots e_{k+l}).$$

We thus see that the $\mathfrak{l}$ highest weights of $F^r\otimes F^s$ restricted to $\mathfrak{l}$ are all of the form

$$\lambda = (\overbrace{\underbrace{2,\ldots,2}_{l_1},\underbrace{1,\ldots,1}_{m_1},0,\ldots,0}^{n_1} \mid \overbrace{\underbrace{2,\ldots,2}_{l_2},\underbrace{1,\ldots,1}_{m_2},0,\ldots,0}^{n_2} \mid \overbrace{\underbrace{2,\ldots,2}_{l_3},\underbrace{1,\ldots,1}_{m_3},0,\ldots,0}^{n_3}),$$

with

$$2(l_1+l_2+l_3)+(m_1+m_2+m_3) = 2n_2+n_3.$$

Arguing as in Lemma 8.1 (and using the mediocre hypothesis crucially), we conclude that the only λ of the above form which satisfies the requirements of (a) and (b) above is

$$\lambda = (\overbrace{0,\ldots,0}^{n_1},\overbrace{1,\ldots,1}^{n_2},\overbrace{2,\ldots,2}^{n_3}).$$

Hence the lemma amounts to proving that the $\mathfrak{l}$ representation F with highest weight λ occurs exactly once in the restriction of $F^r\otimes F^s$ to $\mathfrak{l}$. To see this, note that λ is an extremel weight for $F^r\otimes F^s$, and hence F occurs at most once. Clearly λ is extremel for $\mathfrak{l}$, and so we conclude F occurs exactly once. □

We then obtain the following corollary which is absolutely essential in what follows.

Corollary 8.4. *Retain the notations and assumptions of Corollary 8.2, and suppose that Y is an irreducible Harish-Chandra module with*

$$\mathrm{Ann}(M(\eta))\subset \mathrm{Ann}(Y).$$

Then

$$T_rT_s(Y) = T_sT_r(Y).$$

Now we describe how to compute the $T_i(Y)$ in terms of the coherent continuation representation. The computation in part (b) can be envisioned as first passing to regular infinitesimal character, then crossing a sequence of walls, and finally pushing to a different sequence of walls.

Lemma 8.5. *Let Y be an irreducible Harish-Chandra module with dominant infinitesimal character ν. Let ν' denote the dominant weight conjugate to $\nu + \mu_i$, write W^ν for the stabilizer in W of ν ,and similarly for $W^{\nu'}$. Let W_o denote any choice of representatives for the cosets $W^\nu/(W^\nu \cap W^{\nu'})$. Finally let Θ be a coherent family with $\Theta(\nu) = Y$. Then*

$$T_i(Y) = \sum_{w \in W_o} \Theta(w\nu').$$

In particular, if Y_{reg} is an irreducible Harish-Chandra module of (dominant) regular infinitesimal character ν_{reg} with $\psi^{\nu}_{\nu_{reg}}(Y_{reg}) = Y$ (as in Theorem 4.4(a)), then

$$T_i(Y) = \sum_{w \in W_o} \psi^{\nu'}_{\nu_{reg}}(w^{-1} \cdot \Theta(Y_{reg})).$$

Pf. Let F^i denote the finite dimensional representation with extremal weight $\nu'-\nu$, i.e. $F^i = \bigwedge^{n-i} \mathbb{C}^n$. From the definition of a coherent family, we have

$$T_i(Y) = \sum \Theta(\nu + \gamma),$$

where γ is a weight of F^i and $\nu+\gamma = w\nu'$ for some $w \in W$. Hence we are to determine when $w\nu' - \nu$ is a weight of F^i. Obviously this is the case if $w \in W^{\nu'}$, and it is easy to see that the same is true if $w \in W^\nu$. On the other hand, one can check directly that if $w \notin W^\nu W^{\nu'}$, then $w\nu' - \nu$ cannot be a weight of F^i. Finally notice that $\{w\nu' \mid w \in W_o\}$ coincides with $\{w\nu' \mid w \in W^\nu W^{\nu'}\}$. The first assertion of the lemma follows. The second assertion is clear. □

Since ν' is typically very singular, many terms of the form $\psi^{\nu'}_{\nu_{reg}}(w^{-1} \cdot \Theta(Y_{reg}))$ will vanish in the expression for $T_i(Y)$. In fact, in practice we will only need to compute the action of a single w^{-1} on $\Theta(Y_{reg})$, so the computation becomes tractable. In any event, Lemma 8.5 suggests that we need to know something about the coherent continuation representation, and the next lemma provides that kind of information.

Lemma 8.6. *Let α and β be consecutive adjacent simple roots spanning a subroot system $A_2 \subset A_{n-1}$. Let X be an irreducible Harish-Chandra module with nonsingular integral infinitesimal character, and suppose $\beta \in \tau(X)$ while $\alpha \notin \tau(X)$. Then $s_\alpha\Theta(X)$ contains a unique irreducible constituent X' such that $\beta \notin \tau(X)$ and $\alpha \in \tau(X)$. Moreover, in this setting, we have the following conclusions:*

(a) *The tableau S' parametrizing $\mathrm{Ann}(X')$ is explicitly computable as a 'hook exchange' of the tableau S parametrizing $\mathrm{Ann}(X)$. More precisely, write $\beta = e_{k-1} - e_k$ and $\alpha = e_k - e_{k+1}$. Assume X has (dominant) infinitesimal character $\nu = (\nu_1, \ldots, \nu_n)$. The τ invariant assumptions on X imply (cf. the comments preceding Definition 4.3) that the coordinates $\nu_{k-1}, \nu_k, \nu_{k+1}$ are arranged in one of two relative configurations:*

$$\begin{array}{ccc} & & \boxed{\nu_{k-1}} \\ & \boxed{\nu_{k+1}} & \\ \boxed{\nu_k} & & \end{array} \qquad or \qquad \begin{array}{ccc} & & \boxed{\nu_{k+1}} \\ & \boxed{\nu_{k-1}} & \\ \boxed{\nu_k} & & \end{array} \quad .$$

Then S' coincides with S except that in the first case, the coordinates ν_{k-1} and ν_k are interchanged; and in the second case, the coordinates ν_k and ν_{k+1} are interchanged.

(b) *In particular, if γ is a simple root orthogonal to α and β, then $\gamma \in \tau(X)$ if and only if $\gamma \in \tau(X')$.*

Pf. The first statement is Theorem 3.10(b) in [V3]. Part (a) is explained very carefully in the statement of [V1, Theorem 3.2]. Part (b) is elementary (though it obviously follows from part (a) and Lemma 4.2). □

As the equivalence relation of Section 7 suggests, we are going to essentially reduce to the case of two columns; this is the setting of the next lemma.

Lemma 8.7. *Let X be an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ. Suppose $S = \mathrm{Ann}(X)$ has a partition $S = S_1 \coprod S_2$ into difference-one skew columns of size n_1 and n_2 in good position (Definition 6.10). Let $i = n_1$ and $T = T_i$ (Notation 2.2 or as in Lemma 8.1). Set*

$$\nu(k) = \nu + k\mu_i$$

and consider the $\nu(k)$-quasitableau $S(k) = S_1 \coprod S_2(k)$ where $S_2(k)$ denotes the skew column obtained by adding k to each entry of S_2. Suppose that the partition of $S(k)$ is mediocre. Then $T^k(X)$ is nonzero if and only if $\mathrm{overlap}(S_1, S_2) \geq \mathrm{sing}(S_1, S_2(k))$. *In this case, $T^k(X)$ is irreducible and the annihilator $\mathrm{Ann}(T^k(X))$ is obtained from $S(k)$ by Procedure 7.5.*

Pf. By the translation principle (Theorem 4.4), it is enough to treat the case when $\nu = \rho$. (In this case the condition that $S(k) = S_1 \coprod S_2(k)$ be mediocre is equivalent to requiring $k \leq \max(n_1, n_2)$.) The proof of the lemma follows from a complicated induction on k. The case $k = 1$ is essentially treated by Theorem 4.1 (see especially the comments following Theorem 4.4). In a little more detail, if $\mathrm{overlap}(S_1, S_2) = 0$, then from Definition 7.1 and Lemma 4.2, we conclude that the simple root $e_{n_1} - e_{n_1+1}$ is in the τ-invariant of X. Hence $T(X)$ is zero, and this is exactly what Procedure 7.5(b) gives. On the other hand, if $\mathrm{overlap}(S_1, S_2) > 0$, then $e_{n_1} - en_1 + 1 \notin \tau(X)$, and the paragraph following Theorem 4.4 implies that $\mathrm{Ann}(T(X)) = S_1 \coprod S_2(1)$; this agrees with Procedure 7.5(a).

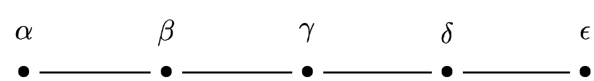
We will describe the $k = 2$ case and sketch how to reduce the $k = 3$ case to the $k = 2$ one. The formidable details of the general induction are left to the reader. For future reference, we write $\widetilde{\rho}(k)$ for the dominant weight in the Weyl group orbit of $\rho(k)$ and we let $A(k)$ denote the set of simple roots on which $\widetilde{\rho}(k)$ is singular. Then $\mathrm{sing}(S_1, S_2(k)) = \#A(k)$.

We describe the $k = 2$ case now. We apply Lemma 8.5 to compute $T^2(X)$; to apply the lemma, we take $Y = T(X)$, $Y_{reg} = X$, and $W_o = \{e, s_\gamma\}$ where s_γ is the reflection in the simple root $\gamma = e_{n_1} - e_{n_1+1}$. Hence

$$T(Y) = \psi_\rho^{\widetilde{\rho}(2)}(s_\gamma \cdot \Theta(X) + \Theta(X)) = \psi_\rho^{\widetilde{\rho}(2)}(s_\gamma \cdot \Theta(X)), \tag{1}$$

with the last equality following because $A(2) \cap \tau(X)$ is nonempty by hypothesis. Hence we are interested in locating constituents X' of $s_\gamma \cdot \Theta(X)$ so that $\tau(X') \cap A(2)$ is empty.

Label the simple roots near γ as follows:



(Of course, some of these vertices need not exist on the Dynkin diagram, so ignore them if they don't.) There are several possibilities for $A(2)$; either $A(2) = \{\beta\}$, $\{\delta\}$, or $\{\beta, \delta\}$.

Assume $A(2) = \{\beta\}$ (the $A(2) = \{\delta\}$ case being identical by symmetry, i.e. by composing with an outer A_n automorphism coming from the Dynkin diagram). Then Lemma 8.6(a) implies that there is a unique constituent X' of $s_\gamma \cdot \Theta(X)$ with $\beta \notin \tau(X')$, i.e. with $\tau(X') \cap$

$A(2)$ empty; moreover the underlying tableau of $\mathrm{Ann}(X')$ is obtained by a hook exchange also described in Lemma 8.6(d). The remarks following Theorem 4.4 and Equation (1) imply that the underlying tableau of $\mathrm{Ann}(T^2(X))$ coincides with that of $\mathrm{Ann}(X')$, and hence we have computed the annihilator of X'. On the other hand, in this case necessarily $n_1 = 1$, so Procedure 7.5(c) applies and gives non-zero $\rho(2)$-tableau. A direct check shows that this tableau is indeed $\mathrm{Ann}(X')$.

Now assume $A(2) = \{\beta, \delta\}$. By the above, we know that there is a unique constituent X' of $s_\gamma \cdot \Theta(X)$ with $\beta \notin \tau(X')$, and we know its underlying tableau. There are two possibilities here: either $\delta \in \tau(X')$, in which case $T^2(X)$ is zero (by Equation (1) and τ-invariant considerations); or $\delta \notin \tau(X')$, in which case the underlying tableau of $\mathrm{Ann}(X')$ coincides with that of $\mathrm{Ann}(T^2(X))$. We can distinguish between these two case by explicitly examining the hook exchange giving $\mathrm{Ann}(X')$.

There are two possibilities for the *relative* positions of the coordinates $n_1 - 1, n_1, n_1 + 1$, and $n_1 + 2$ in the underlying tableau of X; either

$$\begin{array}{|c|c|}\hline n_1 - 1 & n_1 + 1 \\ \hline n_1 & n_1 + 2 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline n_1 - 1 & n_1 + 1 \\ \hline n_1 \\ \cline{1-1} n_1 + 2 \\ \cline{1-1} \end{array} \quad .$$

In the first case $\mathrm{overlap}(S_1, S_2) \geq 2$, and in the second $\mathrm{overlap}(S_1, S_2) = 1$. The hook exchange of Lemma 8.6(d) interchanges the coordinates n_1 and n_1+1. Thus, by Lemma 4.2, $\delta \notin \tau(X)$ if and only if we are in the first case. Hence $T^2(X) \neq 0$ if and only if $\mathrm{overlap}(S_1, S_2) \geq 2 = k$, and in this case one may verify that the hook-exchanged underlying tableau of $\mathrm{Ann}(X')$ coincides with the one given in Procedure 7.5. The $k = 2$ case is thus completed.

Now consider the $k = 3$ case. There are five possibilities for $A(3)$:

$$A(3) = \{\alpha\}, \{\epsilon\}, \{\alpha, \gamma\}, \{\gamma, \epsilon\}, \text{ or } \{\alpha, \gamma, \epsilon\}.$$

The first case is handled exactly as the the first case treated in the $k = 2$ case. If $A(3) = \{\alpha\}$, then since $k = 3 \leq \max(n_1, n_2)$, necessarily $A(2) = \{\beta\}$. Using Lemma 8.5 to compute $T(Y)$ with $Y = T^2(X)$, we can take $Y_{reg} = X'$ as defined in the $k = 2$ case above, and $W_o = \{e, s_\beta\}$. By τ-invariant considerations,

$$T(Y) = \psi_\rho^{\widetilde{\rho}(3)}(s_\beta \cdot \Theta(X)),$$

and so we are to locate constituents of $s_\beta \cdot \Theta(X)$ that do not contain α in their τ-invariants. Again using Lemma 8.6, there is a unique such constituent of $s_\beta \cdot \Theta(X')$ with $\alpha \notin \tau(X'')$; the underlying tableau of X'' is computed by a hook exchange from that of X', and one can check explicitly that the underlying tableau of X'' is the one described by Procedure 7.5. The case $A(3) = \{\epsilon\}$ is handled in exactly the same way (or by symmetry).

Next note that by symmetry, the case of $A(3) = \{\alpha, \gamma\}$ is identical to the case of $A(3) = \{\gamma, \epsilon\}$, so assume now that $A(3) = \{\alpha, \gamma\}$ or $\{\alpha, \gamma, \epsilon\}$. In both of these cases, necessarily we have $A(2) = \{\beta, \delta\}$. Hence, using Lemma 8.5 to compute $T(Y), Y = T^2(X)$, we can take $Y_{reg} = X'$ (as defined in the $k = 2$ case above) and $W_o = \{e, s_\beta, s_\delta, s_\delta s_\beta\}$. We are thus interested in constituents X'' of $w \cdot \Theta(X')$, $w \in W_o$, with $\tau(X'') \cap A(3)$ empty.

Now in either case at hand, $\alpha \in A(3)$. Hence if a constituent Z of $w \cdot \Theta(X')$ is to survive the translation $\psi_\rho^{\widetilde{\rho}(3)}$, we must have $\alpha \notin \tau(Z)$. Since $\alpha \in \tau(X')$ and α is orthogonal to s_δ, Lemma 8.6(b) implies that either $w = s_\beta$ or $s_\delta s_\beta$. In any event, Lemma 8.6 implies that there is a unique constituent Z of $s_\beta \Theta(X)$ with $\alpha \notin \tau(Z)$; a hook exchange on $\mathrm{Ann}(X')$ computes the underlying tableau of Z. Either by explicitly examining the underlying tableau of Z or by an elementary calculation with the coherent continuation representation (along the lines of the proofs of Lemma 8.6(b),(c) in [V3]) one sees that $\gamma \notin \tau(Z)$. Since $\gamma \in A(3)$, we conclude that $\psi_\rho^{\widetilde{\rho}(3)}(s_\beta \cdot \Theta(X)) = 0$.

Thus it remains to find constituents X'' of $s_\delta s_\beta \cdot \Theta(X')$, with $\tau(X'') \cap A(3)$ empty. By the above, all such constituents arise in $s_\delta \cdot \Theta(Z)$. In fact, we are exactly in the setting of the $k = 2$ case with Z taking the place of $T(X)$ and the root δ taking the place of γ. This is essentially the inductive step.

In the case that $A(3) = \{\alpha, \gamma\}$, Lemma 8.6(a) says that there is always a constituent X'' of $s_\delta(Z)$ with $\tau(X'') \cap A(3)$ empty; its underlying tableau is computable in terms of a hook exchange on the underlying tableau of Z (i.e. two hook exchanges on the underlying tableau of X'). The underlying tableau of X'' coincides with the underlying tableau of $T^3(X)$ and one may verify directly that this is the underlying tableau of the $\rho(3)$-tableau produced by Procedure 7.5.

In the case that $A(3) = \{\alpha, \gamma, \epsilon\}$, then $T^3(X) = 0$ if and only if the X'' described in the previous paragraph has $\epsilon \in \tau(X'')$. A direct inspection of tableaux reveals that this is equivalent to requiring $\mathrm{overlap}(S_1, S_2) = 2$. In the case that $\mathrm{overlap}(S_1, S_2) \geq 3$, $\epsilon \notin \tau(X'')$ and $T^3(X) \neq 0$. Its underlying tableau is that of $\mathrm{Ann}(X'')$ and hence may be computed as in the previous paragraph and can be seen to coincide with Procedure 7.5. □

Remark 8.8. Consider a particular example of the lemma. Let $\mathfrak{q}$ be a maximal parabolic, and let $X = A_{\mathfrak{q}}(\mathbb{C}_{triv})$. The root γ is the unique simple root not contained in $\mathfrak{l}$, and the assumption on k implies that $\lambda = k\mu_i$ is in the mediocre range for $\mathfrak{q}$. The lemma gives a sharp condition on k guaranteeing that $T^k(X)$ is nonzero irreducible, and it computes its annihilator. By Lemma 3.13, $A_{\mathfrak{q}}(\lambda) = T^k(X)$, and we thus obtain a special case of Theorem 7.9. Closer inspection reveals that we have proved more: we have, in fact, deduced a special case of Theorem 3.1b(iv) using only irreducibility in the good range (Theorem 3.1b(iii)).

We need to extend this two column case to the case of adjacent skew columns in a partition of S. The arguments given above carry over to this case, so long as the adjacent columns do not interact with the rest of the tableau.

Definition 8.9. Suppose S is a ν-antitableau of size n and $S = \coprod S_i$ is a partition into skew columns. The adjacent columns S_j and S_{j+1} are said to be isolated if:

(a) the entries of S_i, $i < j$, are strictly greater than the entries of $S_j \coprod S_{j+1}$; and
(b) the entries of S_i, $i > j+1$, are strictly smaller than the entries of $S_j \coprod S_{j+1}$.

Here is the more general two column result.

Proposition 8.10. *Let X be an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ, and let S be the ν-tableau corresponding to* $\mathrm{Ann}(X)$ *(Theorem 4.1). Suppose S has a partition into skew columns $S = \coprod S_i$ and that S_j and S_{j+1} are difference-one and in good position (Definition 6.10). Let the column S_i have length n_i and set*

$$\gamma = e_t - e_{t+1}, \qquad t = \sum_{i \leq j} n_i.$$

For any integer k, let $S_{j+1}(k)$ be the skew column obtained by adding k to every entry of S_{j+1}. Assume that S_j and $S_{j+1}(k)$ are isolated (in the sense of Definition 8.9) in $\coprod_{i\leq j} S_i \coprod_{i\geq j+1} S_i(k)$. Let

$$\nu(k) = \nu + k\mu_l,$$

set $R(k) = S_j \coprod S_{j+1}(k)$ be the indicated skew $\nu(k)$-quasitableau, and assume S_j and $S_{j+1}(k)$ are in mediocre position. Then $T_\gamma^k(X)$ is nonzero if and only if

$$\operatorname{overlap}(S_j, S_{j+1}) \geq \operatorname{sing}(S_j, S_{j+1}(k)).$$

In this case, $T_\gamma^k(X)$ is irreducible and the tableau $S(k)$ which parametrizes $\operatorname{Ann}(T_\gamma^k(X))$ is given by

$$S(k) = \coprod_{i\leq j-1} S_i \coprod R'(k) \coprod_{i\geq j+2} S_i(k),$$

where $R'(k)$ is skew tableau obtained from $R(k)$ using Procedure 7.5.

Sketch. Again the translation principle reduces the lemma to the case of $\nu = \rho$. The arguments of Lemma 8.7 extend to this setting, but not immediately so, since the geometry of the adjacent columns S_j and S_{j+1} is more complicated. To prove the proposition one needs to understand the computation of $\operatorname{Ann}(T_\gamma^k(X))$ of the previous lemma in terms of explicit hook exchanges. These are precisely the hook exchanges that appear in the computation of $\operatorname{Ann}(T_\gamma^k(X))$ in the more general setting of the proposition. Since hook exchanges only depend on the relative position of the entries in the tableau, and since the relative positions are essentially the same in both the two column and adjacent column setting, the proof goes through. We leave the details to the reader. □

We have thus completed a description of the two column case. To pass to the general case, we need to prove a statement describing how to combine more than two columns and, in order to do so, we need to gain control over the formula in Lemma 8.5. In the case of Lemma 8.7, we were able to do this using Lemma 8.6. Here is the generalization that we need.

Lemma 8.11. *Let $\alpha_1 = e_1 - e_2, \ldots, \alpha_l = e_l - e_{l+1}$ span a root subsystem $A_l \subset A_{n-1}$, and write $W(l)$ for the corresponding Weyl subgroup. Let X be an irreducible Harish-Chandra module with nonsingular integral infinitesimal character, and suppose*

$$\alpha_1, \ldots, \alpha_{l-1} \notin \tau(X), \text{ but } \alpha_l \in \tau(X).$$

Then there is a unique constituent X' of $\sum_{w\in W(l)} w \cdot \Theta(X)$ such that

$$\alpha_2, \ldots, \alpha_l \notin \tau(X), \text{ but } \alpha_1 \in \tau(X).$$

Moreover, X' is actually a constituent of $s_{\alpha_1} \cdots s_{\alpha_{l-1}} \cdot \Theta(X)$, and:

(a) *The underlying tableau of X' can be explicitly computed by iterating hook exchanges through the coordinates*

$$e_{l-2-i}, e_{l-1-i}, e_{l-i}, \qquad i = 0, \ldots, l-3,$$

on the underlying tableau of X.

(b) *If γ is orthogonal to $\alpha_1, \ldots, \alpha_l$, then $\gamma \in \tau(X)$ if and only if $\gamma \in \tau(X')$.*

Pf. The statement follows by induction on l, the base case $l=2$ being treated by Lemma 8.6. The induction is complicated to write down, but all the ideas are contained in the proof of the $l=2$ case. We refer the reader to the details of Theorem 3.10(b) in [V3]. □

Now we can prove a statement about 'nice' multi-column translations. On level of tableaux, these translations are easy to compute: one simply changes the coordinates of the infinitesimal character. (This generalizes the comments following Theorem 4.4.)

Lemma 8.12. *Suppose X is an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ, and let S be the ν-antitableau corresponding to* $\operatorname{Ann}(X)$. *Suppose $S=\coprod_{i=1}^{m+1} S_i$ is a good partition of S into $m+1$ difference-one skew columns S_i of length n_i. For $i=1,\ldots,m$ set*

$$\gamma_i = e_{t_i} - e_{t_i+1}, \qquad t_i = \sum_{j\leq i} n_i.$$

Consider integers $k_1,\ldots,k_m$, and define

$$\nu(k_1,\ldots,k_m) = \nu + \sum_{i\leq m} k_i(e_{t_i+1}+\cdots+e_n).$$

As usual, for any integer k, let $S_i(k)$ denote the skew column obtained from S_i by adding k to every entry, and consider the $\nu(k_1,\ldots,k_m)$-tableau

$$S(k_1,\ldots,k_m) = \coprod S_i(l_i), \qquad l_i = \sum_{j\leq i} k_j.$$

Suppose that this partition is nice (Definition 6.10). Then $\operatorname{Ann}(T_{\gamma_m}^{k_m}\circ\cdots\circ T_{\gamma_1}^{k_1}(X))$ *is nonzero if and only if*

$$\operatorname{overlap}(S_i,S_{i+1}) \geq \operatorname{sing}(S_i(l_i),S_{i+1}(l_{i+1})) \qquad \textit{for all } i;$$

in this case,

$$\operatorname{Ann}(T_{\gamma_m}^{k_m}\circ\cdots\circ T_{\gamma_1}^{k_1}(X)) = S(k_1,\ldots,k_m).$$

Pf. Again the lemma reduced to the case $\nu=\rho$. (In this case the nice hypothesis is equivalent to $k_i \leq \min(n_{i-1},n)$ for all i, and the condition for nonvanishing of the translation functor is that $\operatorname{overlap}(S_i,S_{i+1}) \leq k_i$, for all i.) The proof is an extremely complicated double induction on m and k_m using the ideas in the proof of Lemma 8.7. The idea is to use Lemma 8.5 to compute successive application of T_{γ_m}. By using using Lemma 8.11, we can reduce matters to locating the constituents of a single $w\cdot\Theta(Y)$ that have the correct τ-invariants. The 'nice' assumption of the lemma guarantees that we may proceed exactly as in the proof of Lemma 8.7 to locate a unique such constituent. The annihilator of this constituent can be explicitly computed by hook exchanges, the result of which is given in the statement of the lemma. We omit the horrendous details. □

Remark 8.13. By Theorem 6.4, the previous lemma applies with $X = A_{\mathfrak{q}}(\mathbb{C}_{triv})$, and (using Lemma 3.13) we recover another special case of Theorem 7.9. (When there are only two columns, this is subsumed by Lemma 8.7.) Moreover, we have deduced from Theorem 3.1(b)(iii) that any nice $A_{\mathfrak{q}}(\lambda)$ is nonzero and irreducible.

Now we have amassed all the tools to prove Theorem 7.9 for the weakly fair range. The proof is an induction on r; the $r=1$ case is trivial, and the $r=2$ case is Lemma 8.7 (see

Remark 8.8). So consider the $r = 3$ case; we are trying to compute $\mathrm{Ann}(A_{\mathfrak{q}}(\lambda))$ for λ in the mediocre range. Taking $X = A_{\mathfrak{q}}(\lambda')$ for an appropriate λ' (in the good range) of the form

$$\lambda = \lambda' + k_1\mu_{\gamma_1} + k_2\mu_{\gamma_2},$$

we are to compute $T_1^{k_1}T_2^{k_2}(X)$, where $T_i = T_{\gamma_i}$. (By changing λ', we can assume both k_1 and k_2 are positive.) Use Theorem 6.4 to compute $\mathrm{Ann}(X) = S_1 \coprod S_2 \coprod S_3$. Lemma 8.7 (that is, the $r = 2$ case) computes

$$\mathrm{Ann}(T_2^{k_2}(X)) = S_1 \coprod S_2' \coprod S_3',$$

where S_2' and S_3' are in nice position and obtained by applying Procedure 7.5 and the rest of the definition of the equivalence relation to $S_2 \coprod S_2(k_2)$. By Lemma 8.12, we can write $T_2^{k_2}(X)$ as $S_2^{m_2}(X')$ for an appropriate $S_2 = T_{i_2}$, where

$$\mathrm{Ann}(X') = S_1 \coprod S_2' \coprod S_3'(-m_2).$$

(It is important to note that X' may not be an $A_{\mathfrak{q}}(\lambda)$ module — this is the sense in which we must move outside the class of $A_{\mathfrak{q}}(\lambda)$ modules.) Taking m_2 large enough, we can assume S_1 and S_2' are isolated in the sense of Definition 8.9.

Now we are interested in computing the annihilator of $T_1^{k_1}T_2^{k_2}(X)$. By the above this is $T_1^{k_1}S_2^{m_2}(X')$, and by Corollary 8.4, this is $S_2^{m_2}T_1^{k_1}(X')$. Since the columns S_1 and S_2' of $\mathrm{Ann}(X')$ are isolated, we can use Proposition 8.10 to compute $T_1^{k_1}(X')$; the result is

$$\mathrm{Ann}(T_1^{k_1}T_2^{k_2}(X)) = \mathrm{Ann}(S_2^{m_2}(X'')),$$

where

$$\mathrm{Ann}(X'') = S_1' \coprod S_2'' \coprod S_3'(-m_2+k_1);$$

here $S_1' \coprod S_2''$ is obtained by applying Procedure 7.5 to $S_1 \coprod S_2'(k_1)$. Now the second two columns are in nice position, so we can isolate them using Lemma 8.12, interchange the order of translation, and use Lemma 8.7 on the first two columns, and so on.

It is clear that we are obtaining the see-saw algorithm described after Theorem 7.9. As remarked there, the algorithm must eventually either produce zero or a nice partition (which we know how to put together using Lemma 8.12). This finishes the $r = 3$ case. It is clear that the arguments just described suffice to handle the general case, and thus the proof of Theorem 7.9 is complete. We have also deduced Theorem 3.1b(iv) from Theorem 3.1b(iii).

9. Evidence for conjecture 1.1

In this section, we prove a small piece of Conjecture 1.1 (see Remark 3.7).

Theorem 9.1. *Let λ be in the mediocre range for $\mathfrak{q}$ (Definition 3.4). Then there exists λ' in the weakly fair range for some $\mathfrak{q}'$, so that*

$$A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{q}'}(\lambda').$$

As one might expect, we are going to reduce the theorem to the case of maximal $\mathfrak{q}$. This case turns out to follow from a simple application of Lemma 8.7, as the next example illustrates.

Example 9.2. Let $(p,q)=(5,2)$, let $\mathfrak{q}$ be attached to $\{(3,2),(2,0)\}$ and let $\lambda=(1,\dots,1,5,5)$, which is outside the weakly fair range, but inside the mediocre range for $\mathfrak{q}$. Theorem 6.4 attaches the following partition to $A_{\mathfrak{q}}(\lambda)$,

$$\begin{array}{|c|c|}\hline 4&3\\\hline 3&2\\\hline 2\\\cline{1-1} 1\\\cline{1-1} 0\\\cline{1-1}\end{array} \;=\; \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline\end{array} \;\coprod\; \begin{array}{|c|c|}\hline &3\\\hline &2\\\hline \\\cline{1-1} \\\cline{1-1} \\\cline{1-1}\end{array},$$

and Lemma 8.7 guarantees that this is the annihilator of $A_{\mathfrak{q}}(\lambda)$. On the other hand, we can compute the associated variety directly from Proposition 5.4 and Lemma 5.6, giving

$$\mathrm{AV}(A_{\mathfrak{q}}(\lambda)) \;=\; \begin{array}{|c|c|}\hline -&+\\\hline -&+\\\hline +\\\cline{1-1} +\\\cline{1-1} +\\\cline{1-1}\end{array}.$$

Thus we are looking for a weakly fair $A_{\mathfrak{q}'}(\lambda')$ with the above tableau parameters.

The idea is to move the first column of the above partition past the second one, in order to move from the mediocre range to the fair range. To make this precise, we notice that the previous partition is equivalent (in the sense of Definition 7.4) to the following one

$$(*)\qquad \begin{array}{|c|c|}\hline 3&4\\\hline 2&3\\\hline 2\\\cline{1-1} 1\\\cline{1-1} 0\\\cline{1-1}\end{array} \;=\; \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} \;\coprod\; \begin{array}{|c|c|}\hline &4\\\hline &3\\\hline 2\\\cline{1-1} 1\\\cline{1-1} 0\\\cline{1-1}\end{array}.$$

Now the partition $(*)$ is weakly fair, and if it is to correspond to some $A_{\mathfrak{q}'}(\lambda')$ with $\mathrm{AV}(A_{\mathfrak{q}}(\lambda)) = \mathrm{AV}(A_{\mathfrak{q}'}(\lambda'))$, Lemma 5.6 implies that $\mathfrak{q}'$ must be attached to $\{(0,2),(5,0)\}$. The data of $\mathfrak{q}'$ and the infinitesimal character imply that $\lambda'=(0,0,3,\dots,3)$, which is in the weakly fair range for $\mathfrak{q}$. In fact, one can check directly that this partition is the one that Theorem 6.4 attaches to the weakly fair $A_{\mathfrak{q}'}(\lambda')$. Hence we conclude that $A_{\mathfrak{q}}(\lambda)\cong A_{\mathfrak{q}'}(\lambda')$, as desired.

The argument given in the example easily leads to a general two column result.

Lemma 9.3. *Let $\mathfrak{q}$ be the maximal parabolic attached to $\{(p_1,q_1),(p_2,q_2)\}$, set $n_i=p_i+q_i$. Suppose*

$$\lambda=(\overbrace{\lambda_1,\dots,\lambda_1}^{n_1},\overbrace{\lambda_2,\dots,\lambda_2}^{n_2})$$

is inside the mediocre range (but outside the weakly fair range) for $\mathfrak{q}$ and that $A_{\mathfrak{q}}(\lambda)\neq 0$; explicitly (using Definition 3.4 and Lemma 8.7) these conditions become

$$-max(n_1,n_2)\le \lambda_1-\lambda_2 < -\frac{n}{2};$$

$$\text{if } p_1+q_1\ge p_2+q_2 \text{ then } p_1\ge q_2 \text{ and } q_1\ge p_2; \text{ and}$$

$$\text{if } p_1+q_1\le p_2+q_2 \text{ then } p_1\le q_2 \text{ and } q_1\le p_2.$$

Set

$$\lambda'=(\overbrace{\lambda_2,\dots,\lambda_2}^{n_2},\overbrace{\lambda_1,\dots,\lambda_1}^{n_1})+(\overbrace{-n_1,\dots,-n_1}^{n_2},\overbrace{n_2,\dots,n_2}^{n_1}),$$

and

(a) *if* $p_1 + q_1 \geq p_2 + q_2$*, let* $\mathfrak{q}'$ *be attached to*

$$\{(q_2, p_2), (p_1 + p_2 - q_2, q_1 + q_2 - p_2)\};$$

(b) *if* $p_1 + q_1 \geq p_2 + q_2$*, let* $\mathfrak{q}'$ *be attached to*

$$\{(p_1 + p_2 - q_1, q_1 + q_2 - p_1), (q_1, p_1)\};$$

Then λ' *is in the weakly fair range for* $\mathfrak{q}'$ *and* $A_{\mathfrak{q}}(\lambda) \cong A_{\mathfrak{q}'}(\lambda')$.

(Note that the hypothesis on the range of λ and the non-vanishing of $A_{\mathfrak{q}}(\lambda)$ imply that the sequence to which $\mathfrak{q}'$ is attached consists of pairs of nonnegative integers, as it must.)

An inductive argument using induction in stages now completes the proof of Theorem 9.1. The induction is not as trivial as it may first seem; in the multicolumn case, the application of Lemma 9.3 to two columns changes the relative position of other columns with respect to the original two. We leave the details to the reader.

Example 9.4. Let $\mathfrak{q}$ be attached to the sequence $\{(1,0), (p-1, q)\}$, assume $q > 1$, and let

$$\lambda = (\lambda_1, \overbrace{\lambda_2, \ldots, \lambda_2}^{p+q-1})$$

with

$$-(p+q-1) \leq \lambda_1 - \lambda_2 < -\frac{n}{2}.$$

The lemma shows that $A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}'}(\lambda')$ where $\mathfrak{q}'$ is attached to the sequence $\{(p, q-1), (0,1)\}$, and

$$\lambda' = (\overbrace{\lambda_2, \ldots, \lambda_2}^{p+q-1}, \lambda_1) + (\overbrace{-1, \ldots, -1}^{p+q-1}, p+q-1).$$

Now $\mathfrak{u} \cap \mathfrak{p}$ is an irreducible as a representation of $L \cap K$, and similarly for $\mathfrak{q}'$. Hence the Blattner formula implies that both modules $A_{\mathfrak{q}}(\lambda)$ and $A_{\mathfrak{q}'}(\lambda')$ are ladder representations whose (multiplicity-free) K type spectrums can be explicitly computed. The result of the computation shows the K types of both modules coincide, and hence one verifies directly that $A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}'}(\lambda')$.

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