Functors for unitary representations of classical real groups and affine Hecke algebras ☆

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Abstract

We define exact functors from categories of Harish–Chandra modules for certain real classical groups to finite-dimensional modules over an associated graded affine Hecke algebra with parameters. We then study some of the basic properties of these functors. In particular, we show that they map irreducible spherical representations to irreducible spherical representations and, moreover, that they preserve unitarity. In the case of split classical groups, we thus obtain a functorial inclusion of the real spherical unitary dual (with "real infinitesimal character") into the corresponding p-adic spherical unitary dual.
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Keywords: Unitary representations of reductive groups over local fields

1. Introduction

Many results in the representation theory of reductive groups over local fields may be formulated in such a way as to appear independent of the field under consideration. But the proofs of such results, while philosophically often the same, employ methods which differ substantially in the real and p-adic cases. This paper is motivated by an attempt to find functorial relationships between representations of real and p-adic groups, with the aim of carrying over results from one setting to the other.

More precisely, we define exact functors from categories of Harish–Chandra modules for certain real classical groups to modules over an associated graded affine Hecke algebra with parameters. Our main results, in increasing order of detail, are that the functors: (1) map spherical principal series of the real group to spherical principal series of the Hecke algebra; (2) map irreducible spherical representations to irreducible spherical representations; and (3) map irreducible Hermitian (resp. unitary) spherical representations to irreducible Hermitian (resp. unitary) spherical representations. In particular, (3) gives a functorial inclusion of the spherical unitary dual of the real group into the spherical unitary dual of an associated graded Hecke algebra. In the split cases, Lusztig's work [16] (together with the Borel–Casselman equivalence [8]) relates the latter category to Iwahori-spherical representations of a corresponding split p-adic group. Together with [6,7], the inclusion in (3) may thus be regarded as an inclusion of the real spherical unitary dual (with "real infinitesimal character") into the p-adic spherical unitary dual, one direction of an instance of Harish–Chandra's "Lefschetz Principle". Previously Barbasch [4,2] proved not only the inclusion but in fact equality for the cases under consideration. His methods relied on difficult and ingenious calculations and did not provide any hints toward the definition of functors implementing the equalities. In this paper, we find the functors and give a conceptually simple proof that they preserve unitarity.

Let $G_\mathbb{R}$ be a real form of a reductive algebraic group G and let $K_\mathbb{R}$ denote a maximal compact subgroup of $G_\mathbb{R}$. Using the restricted root space decomposition of $G_\mathbb{R}$, one may naturally define a graded affine Hecke algebra $H(G_\mathbb{R})$ associated to $G_\mathbb{R}$; see Definition 2.6.4. For instance, if $G_\mathbb{R}$ is split and connected, then $H(G_\mathbb{R})$ is simply the equal parameter algebra associated to the root system of G. (In general the rank of $H(G_\mathbb{R})$ coincides with the real rank k of $G_\mathbb{R}$.) Assume now that G is one of the classical groups $GL(V)$, $Sp(V)$, or $O(V)$. Given a Harish–Chandra module X for $G_\mathbb{R}$, the Schur–Weyl duality of Arakawa and Suzuki [1] suggests it is natural to look for an action of $H(G_\mathbb{R})$ on $X \otimes V^{\otimes k}$. In fact this space is too large to carry such an action in general. In the present setting, it makes sense cut down the space in question by considering

$$\mathrm{Hom}_{K_\mathbb{R}}\big(\mu, X \otimes V^{\otimes k}\big) \tag{1.1}$$

for a representation μ of $K_\mathbb{R}$. Such spaces were considered in [11,17,9] for $U(p,q)$ and $GL(n,\mathbb{R})$.

Again, in general, this space is too large for general μ. But for the groups $GL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, and $O(p,q)$, a very special choice of $\mu = \mu_0$ allows one to define an action of $H(G_\mathbb{R})$ on the space in (1.1). In fact, μ_0 may be characterized in an apparently completely different way as follows. Let $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ denote the Langlands decomposition of a minimal parabolic subgroup of $G_\mathbb{R}$, and write $W_\mathbb{R}$ for the Weyl group of the restricted roots of $\mathfrak{a}_\mathbb{R}$ in $\mathfrak{g}_\mathbb{R}$. Then the $M_\mathbb{R}$ fixed vectors in any representation of $K_\mathbb{R}$ is naturally a representation of $W_\mathbb{R}$. With this in mind, μ_0 is defined by the requirement that

$$\mathrm{Hom}_{M_\mathbb{R}}\big(\mu_0, V^{\otimes k}\big) = \big(\mu_0^* \otimes V^{\otimes k}\big)^{M_\mathbb{R}} \simeq \mathbb{C}[W_\mathbb{R}] \tag{1.2}$$

as representations of $W_\mathbb{R}$; see Proposition 2.4.3. In other words, μ_0 is a necessary twist needed to construct a model of the group algebra of $W_\mathbb{R}$ from the finite-dimensional representation theory of $K_\mathbb{R}$ (and $G_\mathbb{R}$). Such a μ_0 exists only for the classical real groups mentioned above; see Remark 2.4.4. It would be interesting to find a replacement for this condition in the remaining classical cases (and the exceptional ones too of course).

The model of $\mathbb{C}[W_\mathbb{R}]$ in (1.2) has a further subtle property: $(\mu_0^* \otimes V^{\otimes k})$ is single-petaled in the sense of Oda [18] (Definition 3.1.1). Single-petaled representations of $K_\mathbb{R}$ were independently considered by Barbasch and Vogan (e.g. [3]) for the purposes of matching certain intertwining operators in the real and p-adic case. Such matchings require extensive case-by-case analysis, and are one of the main tools for establishing inclusions of the spherical unitary dual of a real group in a p-adic one (e.g. [4,2]). But, roughly speaking, because of (1.2) and the single-petaled property, we can recast the intricate case-by-case matching in simple functorial statements.

Here is a more precise formulation of our main results. The statement about the action of $H(G_\mathbb{R})$ is given in Corollary 2.7.4 (a consequence of the more general Theorem 2.7.3). Part (1) is Theorem 3.0.4; parts (2) and (3) are contained in Theorem 4.2.2 and Corollary 4.2.3. (Here and elsewhere if A is an associative algebra with unit, A-mod denotes the category of finite-dimensional unital left A modules.)

Theorem 1.0.1. *Let $G_\mathbb{R}$ be one of the real groups $GL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, or $O(p, q)$, $p \geqslant q$. Let V be the defining ("standard") representation of $G_\mathbb{R}$, write $K_\mathbb{R}$ for the maximal compact subgroup of $G_\mathbb{R}$, and let k denote the real rank of $G_\mathbb{R}$. Let $\mathcal{HC}_1(G_\mathbb{R})$ denote the full subcategory of Harish–Chandra modules for $G_\mathbb{R}$ whose irreducible objects are irreducible subquotients of spherical principal series (Definition 2.4.1). Let μ_0 be the character of $K_\mathbb{R}$ given in Proposition 2.4.3. Then for any object X of $\mathcal{HC}_1(G_\mathbb{R})$, there is a natural action of the graded affine Hecke algebra $H(G_\mathbb{R})$ of Definition 2.6.4 on the space $\mathrm{Hom}_{K_\mathbb{R}}(\mu_0, X \otimes V^{\otimes k})$. This defines an exact covariant functor*

$$F_1 : \mathcal{HC}_1(G_\mathbb{R}) \longrightarrow H(G_\mathbb{R})\text{-mod}$$

with the following properties:

(1) *If $X_1^\mathbb{R}(\nu)$ is a spherical principal series for $G_\mathbb{R}$ with parameter ν (Definition 2.4.1), then $F_1(X_1^\mathbb{R}(\nu)) = X_1(\nu)$, the spherical principal series of $H(G_\mathbb{R})$ with the same parameter ν (Definition 3.0.3).*

(2) *If X is an irreducible spherical representation of $G_\mathbb{R}$, then $F_1(X)$ is an irreducible spherical representation of $H(G_\mathbb{R})$.*

(3) *If, in addition to the hypotheses in (2), X is Hermitian (resp. unitary), then so is $F_1(X)$.*

We remark that our proofs are essentially self-contained. (In Theorem 3.2.1 we do however rely on the main results of [18] to avoid some unpleasant case-by-case calculations.) We also remark that more general versions of parts (1) and (2) of the theorem were proved in [9], but only for $G_\mathbb{R} = GL(n, \mathbb{R})$ and by methods different from those used below.

The constructions in this paper potentially apply in greater generality. For example, when considering nonspherical principal series of a split group $G_\mathbb{R}$, it is natural to introduce a variant of $H(G_\mathbb{R})$ built from the "good roots" distinguished by the nonspherical inducing parameter (as in [5] for instance). In this setting, one expects the existence of functors from Harish–Chandra modules for $G_\mathbb{R}$ to modules for this other Hecke algebra; see Remark 2.4.2. We hope to return to this elsewhere.

2. Hecke algebra actions

The purpose of this section is to describe natural functors from Harish–Chandra modules for $G_\mathbb{R}$ to modules for a corresponding graded affine Hecke algebra (with parameters). Since it contains the core ideas essential for the rest of the paper, we give an overview. After some preliminaries in Section 2.1, we work in Section 2.2 in the general setting of arbitrary $U(\mathfrak{g})$ modules X and V, and recall the action of (a thickening of) the Lie algebra $\mathcal{A}_k$ of the affine braid group on $X \otimes V^{\otimes k}$ (Lemma 2.2.3). In Section 2.3, we recall the closely related graded affine Hecke algebra H_k of type A_{k-1}, a subquotient of $\mathcal{A}_k$. Using the results of Section 2.2 we are able to give a condition on V such that the action of $\mathcal{A}_k$ on $X \otimes V^{\otimes k}$ descends to one for H_k. The condition is automatic if $\mathfrak{g} = \mathfrak{gl}(V)$ (Lemma 2.3.3(a)), for instance, a fact used extensively in [1] and [9], but it is nontrivial in general. The question is to find a natural class of modules X and a natural subspace of $X \otimes V^{\otimes k}$ for which the condition holds.

At this point we impose the further hypothesis that X is a Harish–Chandra module for a real classical group $G_\mathbb{R}$ with maximal compact subgroup $K_\mathbb{R}$, that X arises as a subquotient of certain principal series with suitably "small" lowest $K_\mathbb{R}$ type, and that V is the defining representation of G. We then find a natural $K_\mathbb{R}$ isotypic component of $X \otimes V^{\otimes k}$ on which H_k acts (Proposition 2.4.5). In fact, as discussed in the introduction, the Lefschetz principle suggest that a larger Hecke algebra $\mathrm{H}(G_\mathbb{R})$ (containing H_k as a subalgebra) should act when k is taken to be the real rank of $G_\mathbb{R}$ (and certain parameters are introduced). The algebra $\mathrm{H}(G_\mathbb{R})$ is defined in Definition 2.6.4, and its action is obtained in Corollary 2.7.4, a consequence of the stronger Theorem 2.7.3. A final section contains details of an alternative presentation of $\mathrm{H}(G_\mathbb{R})$ needed in Sections 3 and 4.

2.1. The Casimir element and its image under comultiplication

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Let κ denote a fixed nondegenerate symmetric bilinear ad-invariant form on $\mathfrak{g}$. Let Δ denote the comultiplication for the natural Hopf algebra structure on $U(\mathfrak{g})$. More precisely, $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the composition of the diagonal embedding $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ with the natural isomorphism $U(\mathfrak{g} \oplus \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$. In particular:

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \text{for all } x \in \mathfrak{g},$$

$$\Delta(xy) = 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x, \quad \text{for all } x, y \in \mathfrak{g}.$$

If B is a basis of $\mathfrak{g}$, let $B^* = \{E^* \mid E \in B\}$ where E^* is defined by:

$$\kappa(E, F^*) = \begin{cases} 1, & \text{if } E = F, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C \in U(\mathfrak{g})$ denote the element

$$C = \sum_{E \in B} E E^*. \tag{2.1}$$

Then C is central in $U(\mathfrak{g})$ and does not depend on the choice of basis B. By Schur's lemma, it acts by a scalar in any irreducible representation of $\mathfrak{g}$. To make certain calculations below cleaner in the classical cases, we find it convenient to rescale κ as follows.

Convention 2.1.1. *Suppose $\mathfrak{g} = \mathfrak{g}(V)$ is a classical Lie algebra with defining representation V. Let κ_{tr} be the trace form of $\mathfrak{g}$. The form κ is a multiple of κ_{tr}, and we scale it as follows:*

$$\kappa = \begin{cases} \kappa_{\mathrm{tr}}, & \mathfrak{g} = \mathfrak{gl}(V), \\ \frac{\kappa_{\mathrm{tr}}}{2}, & \mathfrak{g} = \mathfrak{sp}(V), \\ \frac{\kappa_{\mathrm{tr}}}{2}, & \mathfrak{g} = \mathfrak{so}(V). \end{cases}$$

With this normalization, C acts on V by the scalar:

$$\chi_V(C) = \begin{cases} \dim V, & \mathfrak{g} = \mathfrak{gl}(V), \\ \dim V + 1, & \mathfrak{g} = \mathfrak{sp}(V), \\ \dim V - 1, & \mathfrak{g} = \mathfrak{so}(V). \end{cases}$$

Define the tensor $\Omega \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by:

$$\Omega = \frac{1}{2}\big(\Delta(C) - 1 \otimes C - C \otimes 1\big) = \frac{1}{2} \sum_{E \in B} \big(E \otimes E^* + E^* \otimes E\big). \tag{2.2}$$

Note that if $B = B^*$, in other words if the basis B is closed under taking κ-duals, then $\Omega = \sum_E E \otimes E^*$. It is always possible to find a such basis, for example by using a root decomposition of $\mathfrak{g}$. For simplicity of notation (but without loss of generality), we will assume this from now on, and write $\Omega = \sum_E E \otimes E^*$.

It is clear that the tensor Ω is symmetric, meaning that $R_{12} \circ \Omega = \Omega$, where $R_{12} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the flip $R_{12}(x \otimes y) = y \otimes x$. As a consequence of the fact that C is a central element, one can verify that Ω is $\mathfrak{g}$-invariant,

$$\big[\Delta(x), \Omega\big] = 0, \quad \text{for all } x \in \mathfrak{g}. \tag{2.3}$$

2.2. The algebra $\mathcal{A}_k$

Algebraic constructions related to the material in this subsection may be found in [14, Chapters XVII and XIX], for instance.

Let $k \geqslant 2$ be fixed, and fix a κ self-dual basis B of $\mathfrak{g}$ as above. For every $0 \leqslant i \neq j \leqslant k$, define a tensor in $U(\mathfrak{g})^{\otimes(k+1)}$:

$$\Omega_{i,j} = \sum_{E \in B} (E)_i \otimes \big(E^*\big)_j, \tag{2.4}$$

with notation as follows. Given $x \in U(\mathfrak{g})$ and $0 \leqslant l \leqslant k$, $(x)_l \in U(\mathfrak{g})^{\otimes k+1}$ denotes the simple tensor with x in the $(l+1)$st position and 1's in the remaining positions.

Lemma 2.2.1. *In $U(\mathfrak{g})^{\otimes(k+1)}$, we have the following identities:*

(1) $[\Omega_{i,j}, \Omega_{m,l}] = 0$, *for all distinct i, j, m, l.*
(2) $[\Omega_{i,j}, \Omega_{i,m} + \Omega_{j,m}] = 0$, *for all distinct i, j, m.*

Proof. (1) is clear. For (2), we have:

$$\left[\sum_E (E)_i \otimes (E^*)_j \otimes (1), \sum_F (F)_i \otimes (1)_j \otimes (F^*)_m + (1)_i \otimes (F)_j \otimes (F^*)_m\right]$$

$$= \sum_F\left[\sum_E (E)_i \otimes (E^*)_j, (F)_i \otimes (1)_j + (1)_i \otimes (F)_j\right] \otimes (F^*)_m$$

$$= \sum_F [\Omega, \Delta(F)]_{i,j} \otimes (F^*)_m = 0,$$

by the $\mathfrak{g}$-invariance (2.3) of Ω. $\quad\square$

It is convenient to package the relations of Lemma 2.2.1 together with various permutations into an abstract algebra. Let S_k be the symmetric group in k letters, and let $s_{i,j}$ the transposition (i, j), for $1 \leqslant i < j \leqslant k$.

Definition 2.2.2. Let $\mathcal{A}_k$ be the complex associative algebra with unit generated by S_k and elements $\omega_{i,j}$, $0 \leqslant i \neq j \leqslant k$, subject to the relations:

(1) $[\omega_{i,j}, \omega_{m,l}] = 0$, for all distinct i, j, m, l.
(2) $[\omega_{i,j}, \omega_{i,m} + \omega_{j,m}] = 0$, for all distinct i, j, m.
(3) $s_{i,j}\omega_{i,l} = \omega_{j,l}s_{i,j}$, for all distinct i, j, l and $s_{i,j}\omega_{l,m} = \omega_{l,m}s_{i,j}$, for all distinct i, j, l, m.

Next let X be a $U(\mathfrak{g})$-module, and assume that V is a representation of $\mathfrak{g}$. Denote the action of $U(\mathfrak{g})^{k+1}$ on $X \otimes V^{\otimes k}$ by π_k. By a slight abuse of notation, also let π_k denote the signed action of S_k permuting the factors in $V^{\otimes k}$,

$$\pi_k(\sigma) : x \otimes v_1 \otimes \cdots \otimes v_k \mapsto \mathrm{sgn}(\sigma)(x \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}), \quad \sigma \in S_k. \quad (2.5)$$

Lemma 2.2.3. *With the notation and definitions as above, there is a natural action of $\mathcal{A}_k$ on $X \otimes V^{\otimes k}$ defined on generators by*

$$\sigma \mapsto \pi_k(\sigma),$$

$$\omega_{i,j} \mapsto \pi_k(\Omega_{i,j}).$$

This action commutes with the action of $U(\mathfrak{g})$ on $X \otimes V^{\otimes k}$.

Proof. An easy verification shows that, as operators on $X \otimes V^{\otimes k}$, $\pi_k(\Omega_{i,j})$ and $\pi_k(\sigma)$ satisfy the commutation relations of Definition 2.2.2. This is the first part of the lemma.

It remains to check that the two actions commute. The action of $x \in \mathfrak{g}$ is of course given by $\sum_{l=0}^{k} \pi_k((x)_l)$, and it is clear that this commutes with the action of σ for $\sigma \in S_k$. To check that it commutes with the action of $\omega_{i,j}$ we verify

$$\left[\pi_k(\Omega_{i,j}), \sum_{l=0}^{k} \pi_k((x)_l)\right] = \left[\pi_k\left(\sum_E (E)_i \otimes (E^*)_j\right), \pi_k\left(\sum_{l=0}^{k+1}(x)_l\right)\right]$$

$$= \pi_k \left[\sum_E (E)_i \otimes (E^*)_j, (x)_i \otimes (1)_j + (1)_i \otimes (x)_j \right]$$

$$= \pi_k \left([\Omega, \Delta(x)]_{i,j} \right) = 0,$$

by (2.3). □

2.3. The type A affine graded Hecke algebra and its action on $X \otimes V^{\otimes k}$ for $\mathfrak{g} = \mathfrak{gl}(V)$

Definition 2.3.1. The algebra H_k for $\mathfrak{gl}(k)$ is the complex associative algebra with unit generated by S_k and ϵ_l, $1 \leqslant l \leqslant k$, subject to the relations:

- $[\epsilon_l, \epsilon_m] = 0$, for all $1 \leqslant l, m \leqslant k$.
- $s_{i,j}\epsilon_l = \epsilon_l s_{i,j}$, for all distinct i, j, l.
- $s_{i,i+1}\epsilon_i - \epsilon_{i+1}s_{i,i+1} = 1$, $1 \leqslant i \leqslant k-1$.

We next investigate when it is possible to define an action of H_k on $X \otimes V^{\otimes k}$ from the action of $\mathcal{A}_k$. Set

$$\varepsilon_l = \omega_{0,l} + \omega_{1,l} + \cdots + \omega_{l-1,l} \in \mathcal{A}_k. \tag{2.6}$$

Lemma 2.3.2. *In $\mathcal{A}_k$, we have:*

(1) $[\varepsilon_l, \varepsilon_m] = 0$, *for all l, m.*
(2) $s_{i,j}\varepsilon_l = \varepsilon_l s_{i,j}$, *for all distinct i, j, l.*
(3) $s_{i,i+1}\varepsilon_i - \varepsilon_{i+1}s_{i,i+1} = -\omega_{i,i+1}s_{i,i+1}$.

Proof. To prove (1), we have for $l < m$:

$$[\varepsilon_l, \varepsilon_m] = \sum_{0 \leqslant i < l} \left[\omega_{i,l}, \sum_{0 \leqslant j < m} \omega_{j,m} \right] = \sum_{0 \leqslant i < l} [\omega_{i,l}, \omega_{i,m} + \omega_{l,m}] = 0,$$

where we have used the first two defining relations of Definition 2.2.2. The assertion in (2) is obvious. To prove (3), we use the third defining relation repeatedly. □

Because we are ultimately interested in the defining an action of H_k, the right-hand side of the equality in Lemma 2.3.2(3) makes it natural to examine the action of $\Omega_{i,i+1}$ on $X \otimes V^{\otimes k}$. This comes down to computing the action of Ω on $V \otimes V$. The Casimir element C acts by a scalar $\chi_U(C)$ on each irreducible $U(\mathfrak{g})$-module U. Therefore, from (2.2), one sees that, on $V \otimes V$, we have

$$\pi_2(\Omega) = \frac{1}{2} \bigoplus_U \chi_U(C)\mathrm{pr}_U - \frac{1}{2} \bigoplus_{U'} \left(1 \otimes \chi_{U'}(C)\mathrm{pr}_{U'} + \chi_{U'}(C)\mathrm{pr}_{U'} \otimes 1\right), \tag{2.7}$$

where $V = \bigoplus U'$ and $V \otimes V = \bigoplus U$ are the decompositions into irreducible $U(\mathfrak{g})$-modules, and pr_U, $\mathrm{pr}_{U'}$ denote the corresponding projections. In the special case that $\mathfrak{g}$ is classical, (2.7) can be made very explicit.

Lemma 2.3.3. *Let $\mathfrak{g}$ be a classical Lie algebra and let V denote its defining representation. Scale the form κ as in Convention 2.1.1. Recall the flip operation $R_{12}(x \otimes y) = y \otimes x$ and the element $\Omega \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined in (2.2). Then:*

(1) *If $\mathfrak{g} = \mathfrak{gl}(V)$, $\pi_2(\Omega) = R_{12}$ as operators on $V \otimes V$.*
(2) *If $\mathfrak{g} = \mathfrak{sp}(V)$ or $\mathfrak{so}(V)$, $\pi_2(\Omega) = R_{12} - (\dim V)\mathrm{pr}_1$ as operators on $V \otimes V$; here pr_1 denotes the projection onto the trivial $U(\mathfrak{g})$ isotypic component of $V \otimes V$.*

Proof. The decomposition of $V \otimes V$ is well known, and using (2.7) we conclude, as an operator on $V \otimes V$,

$$\pi_2(\Omega) = \frac{1}{2}\chi_{S^2V}(C)\mathrm{pr}_{S^2V} + \frac{1}{2}\chi_{\wedge^2V}(C)\mathrm{pr}_{\wedge^2V} - \chi_V(C), \quad \text{for } \mathfrak{gl}(V),$$

$$\pi_2(\Omega) = \frac{1}{2}\chi_{S^2V/1}(C)\mathrm{pr}_{S^2V/1} + \frac{1}{2}\chi_{\wedge^2V}(C)\mathrm{pr}_{\wedge^2V} + \frac{1}{2}\chi_1(C)\mathrm{pr}_1 - \chi_V(C), \quad \text{for } \mathfrak{so}(V),$$

$$\pi_2(\Omega) = \frac{1}{2}\chi_{S^2V}(C)\mathrm{pr}_{S^2V} + \frac{1}{2}\chi_{\wedge^2V/1}(C)\mathrm{pr}_{\wedge^2V/1} + \frac{1}{2}\chi_1(C)\mathrm{pr}_1 - \chi_V(C), \quad \text{for } \mathfrak{sp}(V).$$

Note also that $R_{12} = \mathrm{pr}_{S^2V} - \mathrm{pr}_{\wedge^2V}$. The scalars by which C acts are easily computable (on highest weight spaces, for instance). The lemma follows. $\quad\square$

The following appears as Theorem 2.2.2 in [1].

Corollary 2.3.4. *Suppose $\mathfrak{g} = \mathfrak{gl}(V)$ and X is any $U(\mathfrak{g})$ module. Then there is a natural action of H_k on $X \otimes V^{\otimes k}$ defined on generators by*

$$\epsilon_l \mapsto \pi_k(\Omega_{0,l} + \Omega_{1,l} + \cdots + \Omega_{l-1,l}),$$

$$s_{i,i+1} \mapsto \pi_k(s_{i,i+1}).$$

Proof. This follows immediately from Lemmas 2.2.3, 2.3.2(2), and 2.3.3(1). $\quad\square$

Lemma 2.3.3(2) (and more generally (2.7)) show how the corollary can fail for general $\mathfrak{g}$. In the next section, for other classical algebras, we find a natural subspace of $X \otimes V^{\otimes k}$ and a natural class of modules X on which H_k indeed acts.

2.4. Action of H_k on subspaces of $X \otimes V^{\otimes k}$ for Harish–Chandra modules for classical $\mathfrak{g}$

Begin by assuming G is a complex reductive algebraic group and $G_{\mathbb{R}}$ is a real form with Cartan involution θ. Write $K_{\mathbb{R}}$ for the maximal compact subgroup consisting of the fixed points of θ on $G_{\mathbb{R}}$, and K for its complexification in G. Assume that X and V are objects in the category $\mathcal{HC}(G_{\mathbb{R}})$ of Harish–Chandra modules for $G_{\mathbb{R}}$ with V finite-dimensional. From the G-invariance of κ, and in particular $K_{\mathbb{R}}$-invariance, we see that the action of operators $\Omega_{i,j}$ defined in (2.4) on $X \otimes V^{\otimes k}$ commutes with the diagonal action of $K_{\mathbb{R}}$. This implies that for every representation (μ, V_μ) of $K_{\mathbb{R}}$, we obtain an exact functor

$$\widetilde{F}_{\mu,k,V} : \mathcal{H}(G_{\mathbb{R}}) \longrightarrow \mathcal{A}_k\text{-mod}, \quad \widetilde{F}_{\mu,k,V}(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big[V_\mu, X \otimes V^{\otimes k}\big]. \tag{2.8}$$

When $\mathfrak{g} = \mathfrak{gl}(n)$ (or $\mathfrak{sl}(n)$), Corollary 2.3.4 shows this functor descends to one which has the image in the category H_k-mod. By making a judicious choice of μ, we seek to make the same conclusion for a natural class of modules for other groups outside of type A.

We need some more notation. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition of the Lie algebra of $G_{\mathbb{R}}$, and fix a maximal abelian subspace $\mathfrak{a}_{\mathbb{R}}$ of $\mathfrak{p}_{\mathbb{R}}$. (To denote the corresponding complexified algebras, we drop the subscript $\mathbb{R}$.) Fix a maximal abelian subspace $\mathfrak{a}_{\mathbb{R}}$ of $\mathfrak{p}_{\mathbb{R}}$, let k denote the dimension of $\mathfrak{a}_{\mathbb{R}}$ (i.e. the real rank of $G_{\mathbb{R}}$), and let $M_{\mathbb{R}}$ denote the centralizer of $\mathfrak{a}_{\mathbb{R}}$ in $K_{\mathbb{R}}$. Let Φ denote the system of (potentially nonreduced) restricted roots of $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$. We thus have a decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \Phi} (\mathfrak{g}_{\mathbb{R}})_{\alpha}$, where $(\mathfrak{g}_{\mathbb{R}})_{\alpha}$ is the root space for α and $\mathfrak{m}_{\mathbb{R}}$ is the Lie algebra of $M_{\mathbb{R}}$. Write $W_{\mathbb{R}}$ for the Weyl group of Φ,

$$W_{\mathbb{R}} = N_{K_{\mathbb{R}}}(A_{\mathbb{R}})/M_{\mathbb{R}}. \tag{2.9}$$

We fix once and for all a choice of simple roots $\Pi_{\circ}$ in the reduced part $\Phi_{\circ}$ of Φ, and let $M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ denote the corresponding minimal parabolic subgroup of $G_{\mathbb{R}}$.

We next introduced a restricted class of Harish–Chandra modules on which we will eventually define our functors.

Definition 2.4.1. Suppose δ is a one-dimensional representation of $K_{\mathbb{R}}$ (and, by restriction, $M_{\mathbb{R}}$). Fix $\nu \in \mathfrak{a}^*$ and assume ν is dominant with respect to the roots of $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{n}_{\mathbb{R}}$. Let $X_{\delta}^{\mathbb{R}}(\nu)$ denote the minimal principal series $\text{Ind}_{M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}} (\delta \otimes e^{\nu} \otimes 1)$ where the induction is normalized as in [15, Chapter VII]. Let $\overline{X}_{\delta}^{\mathbb{R}}(\nu)$ denote the unique irreducible subquotient of $X_{\delta}^{\mathbb{R}}$ containing the $K_{\mathbb{R}}$ representation δ.

Define the full subcategory $\mathcal{HC}_{\delta}(G_{\mathbb{R}})$ of $\mathcal{HC}(G_{\mathbb{R}})$ to consist of objects which are subquotients of the various $X_{\delta}^{\mathbb{R}}(\nu)$ as ν ranges over all (suitably dominant) elements $\mathfrak{a}^*$.

Remark 2.4.2. Since our main results in Sections 3 and 4 are about spherical representations, the setting of Definition 2.4.1 is entirely appropriate. (Note also that in the case of $\delta = 1$, $\mathcal{HC}_1(G_{\mathbb{R}})$ is a real analogue of the category of Iwahori-spherical representation of a split p-adic group [8].) For applications to the nonspherical case (as mentioned in the introduction), the natural setting is to assume $G_{\mathbb{R}}$ is quasisplit and the $K_{\mathbb{R}}$-type δ is fine in the sense of Vogan.

Our next task is to find the appropriate μ and V so that $\widetilde{F}_{\mu,k,V}(X)$ (for X in some $\mathcal{HC}_{\delta}(G_{\mathbb{R}})$) has a chance of carrying an H_k action. Given any representation U of $K_{\mathbb{R}}$, the $M_{\mathbb{R}}$ fixed vectors $U^{M_{\mathbb{R}}}$ are naturally a representation of $W_{\mathbb{R}}$. In the proofs in Sections 3 and 4 below, we need the existence of a representation V of $G_{\mathbb{R}}$ and a character μ_0 of $K_{\mathbb{R}}$ such that

$$\left(\mu_0^* \otimes V^{\otimes k} \right)^{M_{\mathbb{R}}} \simeq \mathbb{C}[W_{\mathbb{R}}] \tag{2.10}$$

as $W_{\mathbb{R}}$ representations. This is admittedly a rather mysterious requirement at this point, but it emerges naturally (and as explained in the introduction is related to results of Barbasch and Oda). So we are left to investigate it.

Proposition 2.4.3. *Suppose $G_{\mathbb{R}} = GL(n, \mathbb{R}), U(p, q), Sp(2n, \mathbb{R})$ or $O(p, q)$. Let V denote the defining representation of $G_{\mathbb{R}}$. Then there exists a character μ_0 satisfying* (2.10). *Explicitly, we have:*

(i) $G_\mathbb{R} = GL(n, \mathbb{R})$. *Then μ_0 is the nontrivial ("sign of the determinant") character* sgn *of $K_\mathbb{R} = O(n)$.*

(ii) $G_\mathbb{R} = U(p, q)$, $p \geqslant q$. *Then μ_0 is the character* $1 \boxtimes \det$ *of $K_\mathbb{R} \simeq U(p) \times U(q)$.*

(iii) $G_\mathbb{R} = Sp(2n, \mathbb{R})$. *Then μ_0 is the determinant character* $\det$ *of $K_\mathbb{R} \simeq U(n)$.*

(iv) $G_\mathbb{R} = O(p, q)$, $p \geqslant q$. *Then μ_0 is the character* $1 \boxtimes$ sgn *of $K_\mathbb{R} \simeq O(p) \times O(q)$.*

Remark 2.4.4. No such character μ_0 as in Proposition 2.4.3 exists for the quaternionic series of classical groups $GL(n, \mathbb{H})$, $Sp(p, q)$, and $O^*(2n)$.

Proposition 2.4.5. *Let $G_\mathbb{R}$ be one of the groups $GL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, $O(p, q)$, $p \geqslant q$, and V be the defining representation of $G_\mathbb{R}$. Let k be the real rank of $G_\mathbb{R}$. Fix a one-dimensional representation μ of $K_\mathbb{R}$ and fix μ_0 as in Proposition 2.4.3. Recall the functor $\widetilde{F}_{\mu,k,V}$ from (2.8). Then for each object $X \in \mathcal{HC}_{\mu \otimes \mu_0^*}(G_\mathbb{R})$, there is a natural H_k action on $\widetilde{F}_{\mu,k,V}(X)$ defined on generators by*

$$\epsilon_l \mapsto \textit{composition with } \pi_k(\Omega_{0,l} + \Omega_{1,l} + \cdots + \Omega_{l-1,l}),$$

$$s_{i,i+1} \mapsto \textit{composition with } \pi_k(s_{i,i+1})$$

(cf. Corollary 2.3.4). Hence we obtain an exact functor

$$F_{\mu,k,V} : \mathcal{HC}_{\mu \otimes \mu_0^*}(G_\mathbb{R}) \longrightarrow \mathrm{H}_k\text{-mod},$$

$$X \longrightarrow \mathrm{Hom}_{K_\mathbb{R}}\left(\mu, X \otimes V^{\otimes k}\right).$$

Before turning the proofs, we remark outside of type A, the definition on generators given in Proposition 2.4.5 fails to extend to an action of H_k if k is not taken to be the real rank of $G_\mathbb{R}$.

2.5. Proofs of Propositions 2.4.3 and 2.4.5

We begin with a lemma whose proof constructs an explicit basis for the image of $\widetilde{F}_{\mu,m,V}$. The proof also shows the importance of choosing k to be the real rank of $G_\mathbb{R}$.

Lemma 2.5.1. *Retain the setting of Proposition 2.4.5.*

(1) *If $m < k$ then $\widetilde{F}_{\mu,m,V}(X_{\mu \otimes \mu_0^*}^\mathbb{R}(\nu)) = 0$.*

(2) *If $m = k$, then $\dim \widetilde{F}_{\mu,m,V}(X_{\mu \otimes \mu_0^*}^\mathbb{R}(\nu)) = |W_\mathbb{R}|$.*

Proof. For (1), we use Frobenius reciprocity:

$$\widetilde{F}_{\mu,k,V}\left(X_{\mu \otimes \mu_0^*}^\mathbb{R}(\nu)\right) = \mathrm{Hom}_{K_\mathbb{R}}\left[\mu, X_{\mu \otimes \mu_0^*}^\mathbb{R}(\nu) \otimes V^{\otimes m}\right] \tag{2.11}$$

$$= \mathrm{Hom}_{M_\mathbb{R}}\left[1, \mu_0^* \otimes V^{\otimes m}\right] = \left(\mu_0^* \otimes V^{\otimes m}\right)^{M_\mathbb{R}}. \tag{2.12}$$

We describe bases for this latter vector space corresponding to some explicit realizations of $G_\mathbb{R}$ and $K_\mathbb{R}$. Let $e_1, e_2, \ldots,$ denote the standard basis of V used to realize $G_\mathbb{R}$ as matrices.

(a) $GL(n, \mathbb{R})$. We may arrange the group $M_{\mathbb{R}} = O(1)^n$ to be diagonal. We have

$$V|_{M_{\mathbb{R}}} = \bigoplus_{i=1}^{n} 1 \boxtimes \cdots \boxtimes \underbrace{\text{sgn}}_{i\text{th}} \boxtimes \cdots \boxtimes 1,$$

where $1 \boxtimes \cdots \boxtimes \underbrace{\text{sgn}}_{i\text{th}} \boxtimes \cdots \boxtimes 1$ is generated by e_i. If $m < n$, then there can't be any copies of $\mu_0|_{M_{\mathbb{R}}} = \text{sgn} \boxtimes \cdots \boxtimes \text{sgn}$ in $V^{\otimes m}$. When $m = n$, the basis vectors are

$$\{e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} : \sigma \in S_n\}.$$

For the other groups, the analysis is similar, so we only give the realizations and the basis elements when $m = k$.

(b) $U(p, q)$, $p \geqslant q$. Let the form J defining $U(p, q)$ be given (in the basis $\{e_i\}$) by the diagonal matrix $(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$. Since the elements of $M_{\mathbb{R}}$ are block diagonal matrices of the form $(x; x_1, \ldots, x_q, x_q, \ldots, x_1)$, with $x \in U(p - q)$ and $x_1, \ldots, x_q \in U(1)$, the space $(\mu_0^* \otimes V^{\otimes q})^{M_{\mathbb{R}}}$ is spanned by

$$\left\{ f_{\sigma(1)}^{\eta_1} \otimes \cdots \otimes f_{\sigma(q)}^{\eta_q} : \eta_j \in \{\pm 1\}, \ \sigma \in S_q \right\},$$

where $f_j^{\eta} = e_{p-j+1} + \eta e_{p+j}$, $1 \leqslant j \leqslant q$.

(c) $O(p, q)$, $p \geqslant q$. Let the form J defining $O(p, q)$ once again be given by the diagonal matrix $(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$. Thus we naturally realize $O(p, q)$ as a subgroup of the realization $U(p, q)$ given in (b). So we can choose the same basis elements for $(\mu_0^* \otimes V^{\otimes q})^{M_{\mathbb{R}}}$.

(d) $Sp(2n, \mathbb{R})$. We choose the form $J = \begin{pmatrix} 0_n & J_n \\ -J_n & 0_n \end{pmatrix}$ where J_n is matrix with 1's along the antidiagonal and zeros everywhere else. This realization of $Sp(2n, \mathbb{R})$ is naturally a subgroup of the realization of $U(n, n)$ given in (b). So we may once again choose the same basis elements. $\quad \square$

Proof of Proposition 2.4.3. Using the explicit bases constructed in the proof of Lemma 2.5.1, the proposition becomes very easy to verify. We omit the details. $\quad \square$

Proof of Proposition 2.4.5. Because of Lemma 2.2.3, Lemma 2.3.2(3), and the relations defining H_k, the proposition reduces to showing that composition with $\pi_k(s_{i,i+1})$ coincides with composition with $\pi_k(\Omega_{i,i+1})$ on $\text{Hom}_{K_{\mathbb{R}}}(\mu, X \otimes V^{\otimes k})$. Because of Corollary 2.3.4, we may assume $\mathfrak{g}$ is not of type A. Let $\mathbf{J}$ denote the quadratic form on V corresponding to J defining $G_{\mathbb{R}}$ as in the proof of Lemma 2.5.1. In light of Lemma 2.3.3(b) and the proof of Lemma 2.5.1, we need only check for each i that the kernel of the projection pr_i from $V^{\otimes k}$ to $V^{\otimes(k-2)}$ defined by

$$\text{pr}_i(v_1 \otimes \cdots \otimes v_k) = \mathbf{J}(v_i, v_{i+1})(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+2} \otimes \cdots \otimes v_k)$$

contains $(\mu_0^* \otimes V^{\otimes k})^{M_\mathbb{R}}$. But this is a simple verification using the explicit bases given in the proof of Lemma 2.5.1. $\square$

2.6. The graded affine Hecke algebra attached to $G_\mathbb{R}$

We first recall Lusztig's definition of the general affine graded Hecke algebra with parameters, and then distinguish certain parameters using the real group $G_\mathbb{R}$. An alternative presentation will be given in Section 2.8.

Definition 2.6.1. (See [16].) Let R be a root system in a complex vector space V. (We do not assume R spans V.) Denote the action of the Weyl group W of R on V^* by $w \cdot f$ for $w \in W$ and $f \in V^*$. Set $\Psi = (R, V)$ and let $\mathbf{c} : R \to \mathbb{Z}$ be a W-invariant function. The affine graded Hecke algebra $H = H(\Psi, \mathbf{c})$ is the unique complex associative algebra on the vector space $S(V^*) \otimes \mathbb{C}[W]$ such that:

(1) The map $S(V^*) \to H$ sending f to $f \otimes 1$ is an algebra homomorphism.
(2) The map $\mathbb{C}[W] \to H$ sending w to $1 \otimes w$ is an algebra homomorphism.
(3) In H, we have $(f \otimes 1)(1 \otimes w) = f \otimes w$.
(4) For each simple root α in a fixed choice of simple roots $S \subset R$ and for each $f \in V^*$,

$$(1 \otimes s_\alpha)(f \otimes 1) - [s_\alpha \cdot f] \otimes s_\alpha = \mathbf{c}(\alpha) f(\alpha);$$

here $s_\alpha \in \mathbb{C}[W]$ is the reflection corresponding to α.

The choice of S does not affect the isomorphism class of $H(\Psi, \mathbf{c})$. As usual, we identify $S(V^*)$ and $\mathbb{C}[W]$ with their images under the maps (1) and (2), and write (4) as

$$s_\alpha f - (s_\alpha \cdot f) s_\alpha = \mathbf{c}(\alpha) f(\alpha). \tag{2.13}$$

Remark 2.6.2. For the set of roots $R_k \subset V_k := \mathfrak{h}_k^*$ of a Cartan subalgebra $\mathfrak{h}_k$ of $\mathfrak{gl}(k, \mathbb{C})$ and constant parameters $\mathbf{c} \equiv 1$, the algebra $H((R_k, V_k), \mathbf{c})$ of Definition 2.6.1 coincides with the algebra H_k of Definition 2.3.1.

Example 2.6.3. Suppose $\Psi = (R, V)$ is of type C_k. Explicitly take $R = \{\pm 2e_i \mid 1 \leqslant i \leqslant k\} \cup \{\pm(e_i \pm e_j) \mid 1 \leqslant i < j \leqslant k\}$, $V = \mathrm{span}(e_1, \ldots, e_k)$, and choose $S = \{e_1 - e_2, \ldots, e_{k-1} - e_k, 2e_k\}$. Let $\mathbf{c}$ be 1 on short roots and a fixed value $c \in \mathbb{C}$ on long roots. We let $\widetilde{H}_k(c)$ denote the algebra $H(\Psi, \mathbf{c})$ defined by Definition 2.6.1. If we write $\{\epsilon_1, \ldots, \epsilon_k\} \subset V^*$ for the basis dual to $\{e_1, \ldots, e_k\}$, $\widetilde{H}_k(c)$ is generated by the simple reflections $s_{i,i+1}$, $1 \leqslant i < k-1$, in the short simple roots, the reflection $\bar{s}_k$ in the long simple root, and $\{\epsilon_1, \ldots, \epsilon_k\}$ with the commutation relations:

$$s_{i,i+1} \epsilon_j - \epsilon_j s_{i,i+1} = 0, \quad j \neq i, i+1;$$

$$s_{i,i+1} \epsilon_i - \epsilon_{i+1} s_{i,i+1} = 1;$$

$$\bar{s}_k \epsilon_j - \epsilon_j \bar{s}_k = 0, \quad j \neq k;$$

$$\bar{s}_k \epsilon_k + \epsilon_k \bar{s}_k = 2c. \tag{2.14}$$

Clearly H_k of Definition 2.3.1 is naturally a subalgebra of $\widetilde{H}_k(c)$.

Table 1
Examples of H($G_\mathbb{R}$) for various classical groups.

G	$G_\mathbb{R}$	Φ	$\Phi_\circ$	$\mathbf{c}(\alpha)$	H($G_\mathbb{R}$)
$GL(n,\mathbb{C})$	$GL(n,\mathbb{R})$	A_{n-1}	A_{n-1}	$\mathbf{c} \equiv 1$	H_n
	$U(q,q)$	C_q	C_q	$\mathbf{c}(\alpha_{\mathrm{short}}) = 2$ $\mathbf{c}(\alpha_{\mathrm{long}}) = 1$	$\widetilde{\mathrm{H}}_q(1/2)$
	$U(p,q), p > q$	BC_q	B_q	$\mathbf{c}(\alpha_{\mathrm{short}}) = 1$ $\mathbf{c}(\alpha_{\mathrm{long}}) = p - q + 1$	$\widetilde{\mathrm{H}}_q((p-q+1)/2)$
$Sp(2q,\mathbb{C})$	$Sp(2q,\mathbb{R})$	C_q	C_q	$\mathbf{c} \equiv 1$	$\widetilde{\mathrm{H}}_q(1)$
$O(n,\mathbb{C})$	$O(p,q), p > q$	B_q	B_q	$\mathbf{c}(\alpha_{\mathrm{short}}) = 1$ $\mathbf{c}(\alpha_{\mathrm{long}}) = p - q$	$\widetilde{\mathrm{H}}_q((p-q)/2)$
	$O(q,q)$	D_q	D_q	$\mathbf{c} \equiv 1$	$\widetilde{\mathrm{H}}_q(0)$

Return to the setting of an arbitrary real reductive group $G_\mathbb{R}$ (as considered at the beginning of Section 2.4). Recall Φ denotes the roots of $\mathfrak{a}_\mathbb{R}$ in $\mathfrak{g}_\mathbb{R}$ and $\Phi_\circ$ denotes those α such that 2α is not such a root. For every $\alpha \in \Phi_\circ$, set

$$\mathbf{c}(\alpha) = \dim(\mathfrak{g}_\mathbb{R})_\alpha + 2\dim(\mathfrak{g}_\mathbb{R})_{2\alpha}, \tag{2.15}$$

the sum of the (real) dimensions of the generalized α and 2α eigenspaces of $\mathrm{ad}(\mathfrak{a}_\mathbb{R})$ in $\mathfrak{g}_\mathbb{R}$. Then $\mathbf{c}$ is constant on $W_\mathbb{R}$ orbits. As usual, we let $\mathfrak{a}$ denote the complexification of $\mathfrak{a}_\mathbb{R}$.

Definition 2.6.4. (Cf. [18, Section 4].) In the setting of the previous paragraph, suppose in addition that $G_\mathbb{R}$ is connected. Then define H($G_\mathbb{R}$) to be the algebra H($\Psi, \mathbf{c}$) attached by Definition 2.6.1 to $\Psi = (\Phi_\circ, \mathfrak{a}^*)$ and $\mathbf{c}$ of (2.15).

For disconnected groups the correct definition of H($G_\mathbb{R}$) incorporates the action of the component group of $G_\mathbb{R}$ on H($\Psi, \mathbf{c}$). (The reason that extra care is required is because we are aiming for statements like Theorem 1.0.1(3). Since the action of the component group of $G_\mathbb{R}$ affects the notion of being a Hermitian representation of $G_\mathbb{R}$, we need to balance this effect on the Hecke algebra side by also incorporating the action of the component group there.) We make this explicit in the next example for $GL(n,\mathbb{R})$ and $O(p,q)$, the only disconnected cases of interest to us here.

Example 2.6.5. For particular $G_\mathbb{R}$, Table 1 lists the restricted root system Φ, its reduced part $\Phi_\circ$, the values $\mathbf{c}(\alpha)$ of (2.15), and H($G_\mathbb{R}$). In each case, the latter algebra is isomorphic to one of the form H_k or $\widetilde{\mathrm{H}}_k(c)$ as considered in Example 2.6.3. As remarked in Definition 2.6.4, Table 1 *defines* H($G_\mathbb{R}$) for the disconnected groups under consideration. For $GL(n,\mathbb{R})$ and $O(p,q)$ with $p \neq q$, we simply take H($G_\mathbb{R}$) = H($\Psi, \mathbf{c}$) as in Definition 2.6.4. (The relevant action of the component group is trivial in these cases.) But if $p = q$, there is a natural identification

$$\widetilde{\mathrm{H}}_q(0) \simeq \mathrm{H}(D_q, 1) \rtimes \mathbb{Z}/2\mathbb{Z}$$

of the graded Hecke algebra of type B_q with parameters $\mathbf{c}(\alpha_{\mathrm{long}}) = 1, \mathbf{c}(\alpha_{\mathrm{short}}) = 0$, with the semidirect product of $\mathbb{Z}/2\mathbb{Z}$ with the graded Hecke algebra of type D_q with parameters $\mathbf{c} \equiv 1$. Here the finite group $\mathbb{Z}/2\mathbb{Z}$ acts in the usual way (by permuting the roots $\epsilon_{k-1} \pm \epsilon_k$) on the Dynkin

diagram of type D_k. Thus, in the setting of Definition 2.6.4 for $O(q,q)$, $H(\Psi, \mathbf{c})$ is naturally an index two subalgebra of $\widetilde{H}_q(0)$, and we define $H(O(q,q)) = \widetilde{H}_q(0)$.

2.7. The action of $\widetilde{H}(c)$

Given a character μ of K and $X \in \mathcal{HC}_{\mu \otimes \mu_0^*}(G_{\mathbb{R}})$, Proposition 2.4.5 defined an action of H_k on $\mathrm{Hom}_{K_{\mathbb{R}}}(\mu, X \otimes V^{\otimes k})$. When $\mu \otimes \mu_0^* = 1$, the Lefschetz principle discussed in the introduction suggests that the algebras listed in the last column of the table in Example 2.6.5 should act. We prove this in Corollary 2.7.4 below. Initially however we work in the setting of an arbitrary scalar $K_{\mathbb{R}}$ type μ and investigate when $\widetilde{H}_k(c)$ (for arbitrary c) acts. The main result is Theorem 2.7.3 (from which Corollary 2.7.4 follows trivially).

For $p + q = n$, consider the element of $GL(n, \mathbb{C})$:

$$\xi_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \tag{2.16}$$

where I_l denotes the identity $l \times l$-matrix. We assume that $G_{\mathbb{R}}$ is one of the classical groups $U(p,q)$, $Sp(2n, \mathbb{R})$, or $O(p,q)$, that we realize explicitly as in the proof of Lemma 2.5.1:

$$U(p,q) = \{g \in GL(n,\mathbb{C}): \bar{g}^{tr} \xi_{p,q} g = \xi_{p,q}\};$$

$$O(p,q) = \{g \in GL(n,\mathbb{R}): g^{tr} \xi_{p,q} g = \xi_{p,q}\};$$

$$Sp(2n, \mathbb{R}) = \{g \in GL(2n,\mathbb{R}): g^{tr} J g = J\}, \quad J = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \tag{2.17}$$

where g^{tr} denotes the transpose of g, and J_n is the matrix with 1's on the antidiagonal, and zeros everywhere else. Thus, if $G_{\mathbb{R}} = U(p,q)$, the complexification K of $K_{\mathbb{R}}$ is the fixed points in G of $\mathrm{Int}(\xi_{p,q})$. Moreover, with these realizations, $O(p,q)$ (resp. $Sp(2n, \mathbb{R})$) sits inside $U(p,q)$ (resp. $U(n,n)$), such that K commutes with $\xi_{p,q}$ (resp. $\xi_{n,n}$). In the case of $Sp(2n, \mathbb{R})$, we embed the maximal compact $K_{\mathbb{R}} = U(n)$, $A \in U(n) \mapsto \begin{pmatrix} (A^{tr})^{-1} & 0 \\ 0 & A \end{pmatrix}$, into the maximal compact $U(n) \times U(n)$ of $U(n,n)$.

In the following, we suppress the subscript of ξ from notation.

For every $1 \leqslant i \leqslant k$, define $\pi_k(\bar{s}_i) \in \mathrm{End}_{\mathbb{C}}(X \otimes V^{\otimes k})$ by

$$\pi_k(\bar{s}_i)(x \otimes v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) = -x \otimes v_1 \otimes \cdots \otimes \xi v_i \otimes \cdots \otimes v_k. \tag{2.18}$$

Since $\xi^2 = \mathrm{Id}$, we have $\pi_k(\bar{s}_i)^2 = \mathrm{Id}$, and it is easy to see that together with $\pi_k(s_{i,j})$ of (2.5), $\pi_k(\bar{s}_i)$ generate an action of $W(B_k)$, the Weyl group of type B_k on $X \otimes V^{\otimes k}$. Since $K_{\mathbb{R}}$ commutes with ξ we see that this action factors through to $\widetilde{F}_{\mu,k,V}(X)$.

We need to examine how $\pi_k(\bar{s}_i)$ interacts with the various $\pi_k(\Omega_{i,j})$. To make computations, we choose a κ-dual basis B of $\mathfrak{g}$ such that each individual $E \in B$ is either in $\mathfrak{k}$ or $\mathfrak{p}$. This is possible via the complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Moreover, we choose the basis elements in $\mathfrak{k}$, such that every $E \in \mathfrak{k} \cap B$ lies either in the center $\mathfrak{z}$ of $\mathfrak{k}$ or in the semisimple part of $\mathfrak{k}$. This will be important for the computations around (2.21). As before, we adhere to Convention 2.1.1.

We have the following calculation in $\text{End}(X \otimes V^{\otimes k})$:

$$-\pi_k(\bar{s}_j)\pi_k(\Omega_{i,j}) = \pi_k\left(\sum_{E \in B}(E)_i \otimes \left(\xi E^*\right)_j\right)$$

$$= \pi_k\left(\sum_{E \in B \cap \mathfrak{k}}(E)_i \otimes \left(E^*\xi\right)_j - \sum_{E \in B \cap \mathfrak{p}}(E)_i \otimes \left(E^*\xi\right)_j\right).$$

This implies

$$\pi_k(\bar{s}_j)\pi_k(\Omega_{i,j}) + \pi_k(\Omega_{i,j})\pi_k(\bar{s}_j) = -2\pi_k(\bar{s}_j)\pi_k\left(\Omega^{\mathfrak{k}}_{i,j}\right),$$

$$\text{where } \Omega^{\mathfrak{k}}_{i,j} = \sum_{E \in B \cap \mathfrak{k}}(E)_i \otimes \left(E^*\right)_j. \tag{2.19}$$

Lemma 2.7.1. *Let $G_{\mathbb{R}} = U(p,q)$, $Sp(2n, \mathbb{R})$ or $O(p,q)$. Fix a basis B for $\mathfrak{g}$ as described above, and let $C^{\mathfrak{k}}$ denote the Casimir-type element for $\mathfrak{k}$, $\sum_{E \in B \cap \mathfrak{k}} EE^* \in U(\mathfrak{k})$. Fix a one-dimensional representation μ of $K_{\mathbb{R}}$ and for $x \in \mathfrak{k}$ write $\mu(x)$ for the value of the complexified differential of μ, evaluated at x. Recall the definition of π_k given in (2.5) and (2.18); also write $\pi_k(\epsilon_i)$ for the action of ϵ_i given in Proposition 2.4.5. We have the following identity (as operators on $\widetilde{F}_{\mu,k,V}(X)$):*

$$\pi_k(\bar{s}_k)\pi_k(\epsilon_k) + \pi_k(\epsilon_k)\pi_k(\bar{s}_k)$$

$$= 2\pi_k(\bar{s}_k)\pi_k\left(\left(C^{\mathfrak{k}}\right)_k - \sum_{E \in B \cap \mathfrak{k}}\left(E^*\right)_k \mu(E)\right). \tag{2.20}$$

Proof. Recall $\pi_k(\epsilon_k) = \Omega_{0,k} + \Omega_{1,k} + \cdots + \Omega_{k-1,k}$. So we apply (2.19) repeatedly to find:

$$\pi_k(\bar{s}_k)\pi_k(\epsilon_k) + \pi_k(\epsilon_k)\pi_k(\bar{s}_k) = -2\pi_k(\bar{s}_k)\pi_k\left(\sum_{i=0}^{k-1}\Omega^{\mathfrak{k}}_{i,k}\right)$$

$$= -2\pi_k(\bar{s}_k)\pi_k\left(\sum_{E \in B \cap \mathfrak{k}}\left(E^*\right)_k E - \sum_{E \in B \cap \mathfrak{k}}\left(E^*E\right)_k\right)$$

$$= -2\pi_k(\bar{s}_k)\pi_k\left(\sum_{E \in B \cap \mathfrak{k}}\left(E^*\right)_k \mu(E) - \left(C^{\mathfrak{k}}\right)_k\right). \qquad \square$$

It remains to compute the operator in the right-hand side of (2.20). Since μ is one-dimensional, $\mu(E) = 0$ unless $E \in \mathfrak{z}$, the center of $\mathfrak{k}$. Eq. (2.20) becomes $\pi_k(\bar{s}_k)\pi_k(\epsilon_k) + \pi_k(\epsilon_k)\pi_k(\bar{s}_k) = 2\pi_k(\bar{s}_k)\pi_k((Q^{\mathfrak{k}}_\mu)_k)$, where

$$Q^{\mathfrak{k}}_\mu = C^{\mathfrak{k}} - \sum_{E \in B \cap \mathfrak{z}} \mu(E)E^*. \tag{2.21}$$

Table 2
List of characters μ and corresponding parameters.

$G_{\mathbb{R}}$	$K_{\mathbb{R}}$	μ	r_μ	c_μ
$U(p,q)$	$U(p) \times U(q)$	$\det_{U(p)}^{m_p} \boxtimes \det_{U(q)}^{m_q}$	$\frac{p+q-(m_p+m_q)}{2}$	$\frac{p-q+m_q-m_p}{2}$
$Sp(2n,\mathbb{R})$	$U(n)$	$\det^m$	n	m
$O(p,q)$	$O(p) \times O(q)$	$\operatorname{sgn}_{O(p)}^{m_p} \boxtimes \operatorname{sgn}_{O(q)}^{m_q}$	$\frac{p+q}{2} - 1$	$\frac{p-q}{2}$

Lemma 2.7.2. *Assume that $G_{\mathbb{R}}$ is one of the groups $U(p,q)$, $Sp(2n,\mathbb{R})$ or $O(p,q)$, $p \geqslant q$, and let V be the defining representation of $G_{\mathbb{R}}$. Recall the element ξ defined in (2.16). Let μ be a character of $K_{\mathbb{R}}$. Then there exist unique scalars r_μ and c_μ such that, as operators on V,*

$$Q_\mu^{\mathfrak{k}} - r_\mu = c_\mu \xi;$$

that is, the action of the left-hand side on V coincides with multiplication in V by $c_\mu \xi$. The explicit cases are listed in Table 2.

Sketch. The proof amounts to the explicit computation of $Q_\mu^{\mathfrak{k}}$ from (2.21), in each of the cases in Table 2. For instance, if $G_{\mathbb{R}} = U(p,q)$ (as in (2.17)), we may choose $B \cap \mathfrak{k} = \{E_{i,j} : 1 \leqslant i, j \leqslant p \text{ or } p+1 \leqslant i, j \leqslant n\}$, where $E_{i,j}$ denote the matrix with 1 on the (i,j)-position, and 0 everywhere else. One calculates $C^{\mathfrak{k}} = \begin{pmatrix} pI_p & 0 \\ 0 & qI_q \end{pmatrix}$, and $\sum_{E \in B \cap \mathfrak{z}} \mu(E) E^* = \sum_{i=1}^n \mu(E_{i,i}) E_{i,i} = \begin{pmatrix} m_p I_p & 0 \\ 0 & m_q I_q \end{pmatrix}$; thus, one finds $r_\mu = \frac{(p+q)-(m_p+m_q)}{2}$, $c_\mu = \frac{p-q+m_q-m_p}{2}$. The calculations are similar for $Sp(2n,\mathbb{R})$ and $O(p,q)$. Notice that the differential of μ in the case of $O(p,q)$ is zero; this explains why the entries for r_μ and c_μ do not depend on m_p, m_q. $\square$

Theorem 2.7.3. *Assume that we are in the setting of Lemma 2.7.2. Set k to be the real rank of $G_{\mathbb{R}}$ and recall the character μ_0 of Proposition 2.4.3. Then for each object $X \in \mathcal{HC}_{\mu \otimes \mu_0^*}(G_{\mathbb{R}})$, there is a natural $\widetilde{\mathrm{H}}_k(c_\mu)$ action on $\widetilde{F}_{\mu,k,V}(X)$ defined on generators by:*

$$\epsilon_l \mapsto \text{composition with } \pi_k(\Omega_{0,l} + \Omega_{1,l} + \cdots + \Omega_{l-1,l} + r_\mu),$$

$$s_{i,i+1} \mapsto \text{composition with } \pi_k(s_{i,i+1}),$$

$$\bar{s}_k \mapsto \text{composition with multiplication by } -\xi \text{ in the } k\text{th copy of } V$$

(cf. Proposition 2.4.5). Hence we obtain an exact functor

$$F_{\mu,k,V} : \mathcal{HC}_{\mu \otimes \mu_0^*}(G_{\mathbb{R}}) \longrightarrow \widetilde{\mathrm{H}}_k(c_\mu)\text{-mod},$$

$$X \longrightarrow \operatorname{Hom}_{K_{\mathbb{R}}}\big(\mu, X \otimes V^{\otimes k}\big).$$

Proof. The claim follows from Proposition 2.4.5 and Lemma 2.7.2 combined with Eq. (2.20). $\square$

In particular, by taking $\mu \otimes \mu_0^* = 1$, we obtain functors for constituents of spherical principal series. Combined with Corollary 2.3.4 for the type A cases, we have the following corollary.

Corollary 2.7.4. *Assume $G_{\mathbb{R}}$ is one of the groups $GL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, $O(p, q)$, $p \geqslant q$, and let V be the defining representation of $G_{\mathbb{R}}$. Let k be the real rank of $G_{\mathbb{R}}$, let μ_0 be the character of Proposition 2.4.3, recall the category $\mathcal{HC}_1(G_{\mathbb{R}})$ of subquotients of spherical minimal principal series of $G_{\mathbb{R}}$ (Definition 2.4.1), and finally recall the Hecke algebra $\mathrm{H}(G_{\mathbb{R}})$ from Definition 2.6.1. There exists an exact functor $F_1 : \mathcal{HC}_1(G_{\mathbb{R}}) \to \mathrm{H}(G_{\mathbb{R}})$-mod, $F_1(X) = \mathrm{Hom}_{K_{\mathbb{R}}}[\mu_0, X \otimes V^{\otimes k}]$, given by taking $\mu = \mu_0$ in Theorem 2.7.3.*

2.8. An alternative presentations of $\mathrm{H}(\Psi, c)$

The results in this subsection will be needed in Sections 3 and 4. In the setting of Lemma 2.7.2, let m_ξ denote multiplication by ξ in V. As usual, given an operator on V, let $(A)_i$ denote the operator on $V^{\otimes k}$ which is the identity in all positions except the ith where it is A.

Lemma 2.8.1. *Retain the setting of Theorem 2.7.3. As operators on $V \otimes V$, we have:*

(1) $\pi_2(\Omega^{\mathfrak{k}}) = \frac{1}{2}(R_{12} + (m_\xi)_2 R_{12}(m_\xi)_2)$, *if $G_{\mathbb{R}} = U(p, q)$;*
(2) $\pi_2(\Omega^{\mathfrak{k}}) = \frac{1}{2}(R_{12} + (m_\xi)_2 R_{12}(m_\xi)_2) - \frac{1}{2}(\dim V)\mathrm{pr}_1$, *if $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ or $O(p, q)$, where pr_1 denotes the projection onto the trivial representation isotypic component of $V \otimes V$.*

Proof. The proof is an analogous calculation with Casimir elements as in Lemma 2.3.3. We skip the details. $\square$

As in the proof of Proposition 2.4.5, we then find that, under the assumptions of Theorem 2.7.3, we have the following identity of operators on $F_{\mu,k,V}$:

$$\pi_k\big(\Omega^{\mathfrak{k}}_{i,j}\big) = -\frac{1}{2}\pi_k(s_{i,j} + \bar{s}_j s_{i,j} \bar{s}_j); \tag{2.22}$$

on the right-hand side, we have once again used π_k to denote the action of $\widetilde{\mathrm{H}}(c_\mu)$ given in Theorem 2.7.3. Define

$$\Omega^{\mathfrak{p}}_{i,j} = \sum_{E \in B \cap \mathfrak{p}} (E)_i \otimes \big(E^*\big)_j. \tag{2.23}$$

Clearly, we have $\Omega_{i,j} = \Omega^{\mathfrak{k}}_{i,j} + \Omega^{\mathfrak{p}}_{i,j}$. We compute the action of $\pi_k(\Omega^{\mathfrak{p}}_{i,j})$ on $F_{\mu,k,V}$ in terms of that of ϵ_j and s_β (for β a positive restricted root) as follows:

$$\begin{aligned}
\pi_k\big(\Omega^{\mathfrak{p}}_{i,j}\big) &= \pi_k\big(\Omega_{0,j} - \Omega^{\mathfrak{k}}_{0,j}\big) \\
&= \pi_k(\epsilon_j) - r_\mu + \sum_{i<j} \pi_k(s_{i,j}) - \sum_{E \in B \cap \mathfrak{k}} \pi_k\big((E)_0 \otimes (E^*)_j\big) \quad \text{by Theorem 2.7.3} \\
&= \pi_k(\epsilon_j) + \pi_k\big((Q^{\mathfrak{k}}_\mu)_j\big) - r_\mu \\
&\quad + \sum_{1 \leqslant i \neq j \leqslant k} \pi_k\big(\Omega^{\mathfrak{k}}_{i,j}\big) + \sum_{i<j} \pi_k(s_{i,j}) \quad \text{by the same calculation as for (2.20)}
\end{aligned}$$

$$= \pi_k(\epsilon_j) - c_\mu \pi_k(\bar{s}_j) + \frac{1}{2} \sum_{i<j} \pi_k(s_{i,j})$$

$$- \frac{1}{2} \sum_{i>j} \pi_k(s_{j,i}) - \frac{1}{2} \sum_{i \neq j} \bar{\pi}_k(s_j s_{i,j} \bar{s}_j) \quad \text{by Lemma 2.7.2 and (2.22).} \qquad (2.24)$$

We have proved:

Lemma 2.8.2. *In the setting of Theorem* 2.7.3, *the elements*

$$\epsilon_j - \frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) \epsilon_j(\beta) s_\beta \qquad (2.25)$$

of the Hecke algebra $\widetilde{\mathbb{H}}_k(c_\mu)$ *act by* $\pi_k(\Omega_{0,j}^{\mathfrak{p}})$ *on* $F_{\mu,k,V}(X)$.

The elements of the lemma will be relevant for us in Section 4 in the context of Hermitian forms and the natural $*$-operation on $\widetilde{\mathbb{H}}_k(c_\mu)$. They are also significant for a different reason which we now explain. In the setting of Definition 2.6.1, for any $f \in V^*$, define a new element $\widetilde{f} \in \mathrm{H} = \mathrm{H}(\Psi, \mathbf{c})$ by

$$\widetilde{f} = f - \frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) f(\beta) s_\beta \qquad (2.26)$$

where R^+ is the system of positive roots corresponding to the fixed simple system $S \subset R$. In H, these elements satisfy

$$s_\alpha \widetilde{f} = \widetilde{s_\alpha \cdot f},$$
$$[\widetilde{f}, \widetilde{f'}] = -\left[\frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) f(\beta) s_\beta, \frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) f'(\beta) s_\beta \right]. \qquad (2.27)$$

In fact the elements $\widetilde{f}$ for $f \in V^*$ together with $\mathbb{C}[W]$ generate H and the relations (2.27) provide an alternative presentation of H. This is sometimes called the Drinfeld presentation of $\mathrm{H}(\Psi, \mathbf{c})$ (after its introduction in a more general setting [10]).

3. Images of spherical principal series

The following is a companion to Definition 2.4.1.

Definition 3.0.3. Let $\mathrm{H} = \mathrm{H}(G_\mathbb{R})$ be the Hecke algebra attached to $G_\mathbb{R}$ by Definition 2.6.4. Let $\nu \in \mathfrak{a}^*$ be dominant with respect to the fixed simple roots $\Pi_\circ \subset \Phi_\circ$, and write $\mathbb{C}_\nu$ for the corresponding $S(\mathfrak{a})$ modules. Define

$$X_1(\nu) = \mathrm{H} \otimes_{S(\mathfrak{a})} \mathbb{C}_\nu.$$

Let $\overline{X}_1(\nu)$ denote the unique irreducible subquotient of $X_1(\nu)$ containing the trivial representation of $W_\mathbb{R}$. (Alternatively $\overline{X}_1(\nu)$ is characterized as the unique irreducible quotient of $X_1(\nu)$.)

The main result of this section is as follows, and the remainder of the section is devoted to proving it. (A more general result for $GL(n, \mathbb{R})$ is proved by a different method in [9, Theorem 3.5].)

Theorem 3.0.4. *Retain the setting of Corollary 2.7.4 and recall the standard modules of Definitions* 2.4.1 *and* 3.0.3. *Then*

$$F_1\big(X_1^{\mathbb{R}}(\nu)\big) = X_1(\nu).$$

3.1. Preliminary details on $(\mu_0^* \otimes V^{\otimes k})$ as a representation of $K_{\mathbb{R}}$

We retain the notation from Section 2.4. Fix root vectors $X_\alpha \in \mathfrak{g}_\alpha$, for every $\alpha \in \Pi_\circ$, normalized such that

$$\kappa\big(X_\alpha, \theta(X_\alpha)\big) = -2/\|\alpha\|^2, \tag{3.1}$$

where $\|\alpha\|$ is the length of α induced by κ. Set

$$Z_\alpha = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}_{\mathbb{R}}. \tag{3.2}$$

For later use define

$$k_\alpha = \exp(\pi Z_\alpha/2) \in K_{\mathbb{R}}. \tag{3.3}$$

Then $k_\alpha^2 \in M_{\mathbb{R}}$, and k_α induces in $W_{\mathbb{R}}$ the reflection corresponding to the root α.

Let (τ, V_τ) be any representation of $K_{\mathbb{R}}$. From the definitions (cf. (2.9)), $W_{\mathbb{R}}$ acts naturally on $V_\tau^{M_{\mathbb{R}}}$. For every Z_α, the operator $\tau(Z_\alpha)^2$ preserves $V_\tau^{M_{\mathbb{R}}}$ and it acts with negative square integer eigenvalues on it.

Definition 3.1.1. (See Oda [18].) A representation (τ, V_τ) of $K_{\mathbb{R}}$ is called quasi-spherical if $V_\tau^{M_{\mathbb{R}}} \neq 0$. A quasi-spherical representation (τ, V_τ) is called single-petaled if

$$V_\tau^{M_{\mathbb{R}}} \subset \mathrm{Ker}\big[\tau(Z_\alpha)\big(\tau(Z_\alpha)^2 + 4\big)\big], \tag{3.4}$$

for all $\alpha \in \Pi_\circ$. (A similar definition appears in [2] and [3], where the terminology "petite" is instead used. See also [12].)

Proposition 3.1.2. *Let $G_{\mathbb{R}}$ be one of the groups $GL(n, \mathbb{R})$, $U(p, q)$, $Sp(2n, \mathbb{R})$, $O(p, q)$, $p \geqslant q$, and recall the $K_{\mathbb{R}}$-character μ_0 from Proposition 2.4.3. Let V be the defining representation of $G_{\mathbb{R}}$, and let k be the real rank of $G_{\mathbb{R}}$. Then $(\mu_0^* \otimes V^{\otimes k})$ is a single-petaled representation of $K_{\mathbb{R}}$ (Definition* 3.1.1).

Sketch. One verifies the claims in every case, by considering explicit realizations for the elements Z_α. We use the same realizations and notation as in the proof of Lemma 2.5.1, where we computed the basis elements for $(\mu_0^* \otimes V^{\otimes k})^{M_{\mathbb{R}}}$. The verification is reduced to the case of real rank one groups corresponding to the restricted roots for these groups.

(a) $GL(2, \mathbb{R})$. Here $\mathfrak{p}_{\mathbb{R}}$ consists of symmetric matrices, and we can choose $\mathfrak{a}_{\mathbb{R}}$ to be the diagonal matrices. In the usual coordinates, the simple restricted root is $\alpha = \epsilon_1 - \epsilon_2$, with corresponding $X_\alpha = E_{12}$. (Here and below $E_{i,j}$ denotes a matrix with a single nonzero entry of 1 in the (i, j)th position.) We get $Z_\alpha = E_{12} - E_{21}$. We compute the action of $Z_\alpha(Z_\alpha^2 + 4)$ step-by-step as follows,

$$e_1 \otimes e_2 \xrightarrow{Z_\alpha} e_1 \otimes e_1 - e_2 \otimes e_2 \xrightarrow{Z_\alpha} -2e_1 \otimes e_2 - 2e_2 \otimes e_1 \xrightarrow{+4} 2(e_1 \otimes e_2 - e_2 \otimes e_1) \xrightarrow{Z_\alpha} 0,$$

and find $Z_\alpha(Z_\alpha^2 + 4)$ indeed acts by zero on the basis vector $e_1 \otimes e_2$ appearing in the proof of Lemma 2.5.1(a). The identical considerations give the same conclusion for $e_2 \otimes e_1$, as desired.

(b) $GL(2, \mathbb{C})$. The notation and calculation are identical to the case of $GL(2, \mathbb{R})$.

(c) $U(p, 1)$, $p \geqslant 1$. In this case, $K_{\mathbb{R}}$ consists of the block-diagonal $U(p) \times U(1)$, and $\mathfrak{p}_{\mathbb{R}}$ is formed of matrices $\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$ where B is a $p \times 1$ complex matrix and B^* denotes its conjugate transpose. We take $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$ to consist of matrices where B is of the form $(0, 0, \ldots, x)^{tr}$ for $x \in \mathbb{C}$. Let $\epsilon \in \mathfrak{a}_{\mathbb{R}}^*$ denote the functional which takes value x on such an element of $\mathfrak{a}_{\mathbb{R}}$. Then the roots of $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ are simply $\pm\epsilon$ and $\pm 2\epsilon$, and so we take $\Pi_\circ = \{\epsilon\}$. Then $Z_\epsilon = 2(E_{1,p} + E_{p,1})$. We compute the action of $Z_\epsilon(Z_\epsilon^2 + 4)$ on $f_1^\pm = e_p \pm e_{p+1}$ sequentially as

$$f_1^\pm \xrightarrow{Z_\epsilon} 2e_1 \xrightarrow{Z_\epsilon} -4e_p \xrightarrow{+4} -4e_p + 4f_1^\pm = \pm 4e_{p+1} \xrightarrow{Z_\epsilon} 0.$$

(d) $O(p, 1)$, $p > 1$. We have realized $O(p, 1)$ naturally as a subgroup of $U(p, 1)$, and so the calculation is essentially the same as in (c). □

In the setting of Proposition 3.1.2, we next note that $(\mu_0^* \otimes V^{\otimes k})^{M_{\mathbb{R}}}$ has two natural actions of $W_{\mathbb{R}}$ on it. One comes from the definitions and is computed in Proposition 2.4.3. The other comes from identifying the space $(\mu_0^* \otimes V^{\otimes k})^{M_{\mathbb{R}}}$ with $F_1(X_1^{\mathbb{R}}(\nu))$ using (2.11), and then restricting the action of $H(G_{\mathbb{R}})$ given in Corollary 2.7.4 to the subalgebra $\mathbb{C}[W_{\mathbb{R}}]$.

Proposition 3.1.3. *The two natural actions of $W_{\mathbb{R}}$ on $(\mu_0^* \otimes V^{\otimes k})$ described in the previous paragraph coincide.*

Sketch. On one hand, the action of $\mathbb{C}[W_{\mathbb{R}}] \subset H(G_{\mathbb{R}})$ is given in Theorem 2.7.3 and can be easily computed on the explicit bases constructed in the proof of Lemma 2.5.1.

On the other hand, the elements $k_\alpha \in K_{\mathbb{R}}$ defined in (3.3) give the other action of $W_{\mathbb{R}}$. To prove the proposition, we need to compute the k_α explicitly and check that their action on the bases in the proof of Lemma 2.5.1 coincides with that in the previous paragraph. This is an explicit case-by-case calculation (reducing again to real rank one) and we simply sketch the details in certain representative cases. With notation (especially for the elements Z_α) as in the proof of Proposition 3.1.2, we have:

(a) $GL(2, \mathbb{R})$. We have $k_\alpha = \exp(\frac{\pi}{2} Z_\alpha) = E_{12} - E_{21} \in O(2)$. It takes $e_1 \otimes e_2$ to $-e_2 \otimes e_1$, and so s_α acts by $-R_{12}$ just as in the restriction of the Hecke algebra action detailed in Corollary 2.3.4 (cf. (2.5)).

(c) $U(p, 1)$. In this case, k_ϵ is the diagonal matrix $(-1, 1, \ldots, 1, -1, 1)$ with negative entries in the first and pth entries. Thus k_{ϵ_1} takes $f_1^\pm = e_p \pm e_{p+1}$ to $-f_1^\mp$. This coincides with the action of multiplication by $-\xi$ as in the restriction of the Hecke algebra action described in the third displayed equation of Theorem 2.7.3. □

3.2. A natural vector space isomorphism $F_1(X_1^{\mathbb{R}}(\nu)) \to X_1(\nu)$

From Lemma 2.5.1(2), we know that $F_1(X_1^{\mathbb{R}}(\nu))$ and $X_1(\nu)$ have the same dimension (namely that of $|W_{\mathbb{R}}|$). In this section, we find a natural map giving the isomorphism of vector spaces. In the next section, we finish the proof of Theorem 3.0.4 by showing that this natural map is an H-module map.

We begin by working in the setting of general $G_{\mathbb{R}}$. Consider the "universal" spherical $(\mathfrak{g}, K)$-module

$$\mathcal{X}^{\mathbb{R}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} 1. \tag{3.5}$$

In particular, there is an obvious map $\mathcal{X}^{\mathbb{R}} \to X_1^{\mathbb{R}}(\nu)$ for any $X_1^{\mathbb{R}}(\nu)$ as in Definition 2.4.1. By the dominance condition on ν, the map is surjective. Write $\mathcal{J}^{\mathbb{R}}(\nu)$ for its kernel. So $X_1^{\mathbb{R}}(\nu) \simeq \mathcal{X}^{\mathbb{R}}/\mathcal{J}^{\mathbb{R}}(\nu)$.

Set $H = H(G_{\mathbb{R}})$ (Definition 2.6.4) and consider the H-module analog

$$\mathcal{X} = H \otimes_{C[W_{\mathbb{R}}]} 1. \tag{3.6}$$

We will now work exclusively with the Drinfeld presentation of H given around (2.27). In the present case $V^* = (\mathfrak{a}^*)^* = \mathfrak{a}$. Given $h = x_1 x_2 \cdots x_l \in S(\mathfrak{a})$ with each $x_i \in \mathfrak{a}$, we define

$$\widetilde{h} \otimes 1 = (\widetilde{x}_1 \widetilde{x}_2 \cdots \widetilde{x}_l) \otimes 1 \in \mathcal{X}. \tag{3.7}$$

Because the commutator of any $\widetilde{x}_i$ and $\widetilde{x}_j$ is contained in $\mathbb{C}[W_{\mathbb{R}}]$ by (2.27), the notation is well defined (independent of the ordering of the x_i). Thus, as sets, $\mathcal{X} = \{\widetilde{h} \otimes 1 \mid h \in S(\mathfrak{a})\}$. With this identification, for $f \in \mathfrak{a}$ and correspondingly $\widetilde{f} \in H$, the action of $\widetilde{f}$ on $\mathcal{X}$ is given

$$\widetilde{f} \cdot (\widetilde{h} \otimes 1) = (\widetilde{fh} \otimes 1). \tag{3.8}$$

Again there is an obvious surjection $\mathcal{X} \to X_1(\nu)$ for any $X_1(\nu)$ as in Definition 3.0.3. Write $\mathcal{J}^{\mathbb{R}}(\nu)$ for its kernel, so that $X_1(\nu) \simeq \mathcal{X}/\mathcal{J}(\nu)$.

We now introduce

$$\gamma : \mathcal{X}^{\mathbb{R}} \longrightarrow \mathcal{X}. \tag{3.9}$$

Let $\gamma_\circ : U(\mathfrak{g}) \to U(\mathfrak{a}) \simeq S(\mathfrak{a})$ denote the projection with respect to the decomposition $U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$. Define

$$\gamma(x \otimes 1) = \widetilde{\gamma_\circ(x)} \otimes 1; \tag{3.10}$$

here we are again using the identification $\mathcal{X} = \{\widetilde{h} \otimes 1 \mid h \in S(\mathfrak{a})\}$. (The elements $\widetilde{h}$ have a ρ-shift built into them, cf. (2.26), so γ is really a kind of Harish–Chandra homomorphism.)

Suppose now that (τ, V_τ) is any representation of $K_{\mathbb{R}}$. Consider the map

$$\Gamma : \mathrm{Hom}_{K_{\mathbb{R}}}(V_\tau, \mathcal{X}^{\mathbb{R}}) \longrightarrow \mathrm{Hom}_{W_{\mathbb{R}}}(V_\tau^{M_{\mathbb{R}}}, \mathcal{X}) \tag{3.11}$$

defined as the composition of restriction to $V_\tau^{M_\mathbb{R}}$ and composition with γ of (3.9) and (3.10). Here is the connection with the previous section.

Theorem 3.2.1. *(See [18].) Let (τ, V_τ) be a single-petaled representation of $K_\mathbb{R}$ (Definition 3.1.1). Then the map Γ of (3.11) is an isomorphism. Moreover it factors to an injection*

$$\Gamma_\nu : \operatorname{Hom}_{K_\mathbb{R}}\big(V_\tau, \mathcal{X}^\mathbb{R}/\mathcal{J}^\mathbb{R}(\nu)\big) \hookrightarrow \operatorname{Hom}_{W_\mathbb{R}}\big(V_\tau^{M_\mathbb{R}}, \mathcal{X}/\mathcal{J}(\nu)\big) \tag{3.12}$$

for each $\nu \in \mathfrak{a}^*$.

Proof. The first assertion is [18, Theorem 1.4]. The second follows from [18, Remark 1.5]. □

Return to the setting of Theorem 3.0.4. Take $V_\tau = (\mu_0^* \otimes V^{\otimes k})^*$. This is single-petaled by Proposition 3.1.2, and so Theorem 3.2.1 applies to give an inclusion

$$\operatorname{Hom}_{K_\mathbb{R}}\big((\mu_0^* \otimes V^{\otimes k})^*, \mathcal{X}^\mathbb{R}/\mathcal{J}^\mathbb{R}(\nu)\big) \hookrightarrow \operatorname{Hom}_{W_\mathbb{R}}\big([(\mu_0^* \otimes V^{\otimes k})^*]^{M_\mathbb{R}}, \mathcal{X}/\mathcal{J}(\nu)\big).$$

The left-hand side clearly identifies with $F_1(X_1^\mathbb{R}(\nu))$. Because of Proposition 2.4.3, the right-hand side identifies with $X_1(\nu)$. The dimension count of Lemma 2.5.1(2) implies that the injection is an isomorphism. We thus obtain a vector space isomorphism

$$\Gamma_\nu : \operatorname{Hom}_{K_\mathbb{R}}\big((\mu_0^* \otimes V^{\otimes k})^*, \mathcal{X}^\mathbb{R}/\mathcal{J}^\mathbb{R}(\nu)\big) \xrightarrow{\sim} \operatorname{Hom}_{W_\mathbb{R}}\big(\mathbb{C}[W_\mathbb{R}], \mathcal{X}/\mathcal{J}(\nu)\big). \tag{3.13}$$

3.3. Γ_ν is an H-module map

Corollary 2.7.4 gives a natural action of H on the left-hand side of (3.13). Because the right-hand side identifies with $X_1(\nu)$, it also has a natural H action. In this section we make those actions explicit, and then finish the proof of Theorem 3.0.4 by showing (3.13) respects the H-action.

To begin, we recall Proposition 2.4.3 and fix a vector $\mathbf{v} \in (\mu_0^* \otimes V^{\otimes k})^* \simeq \mathbb{C}[W_\mathbb{R}]$ which is cyclic for the action of $W_\mathbb{R}$. Though not essential, we save ourselves some notation by noting that in each of the classical cases under consideration, it is not difficult to verify using the bases introduced in the proof of Lemma 2.5.1(2) that $\mathbf{v}$ may be taken to be a simple tensor,

$$\mathbf{v} = \lambda \otimes \lambda_1 \otimes \cdots \otimes \lambda_k;$$

here $\lambda \in \mathbb{C}_{\mu_0^*}^*$ and $\lambda_i \in V^*$. For later use, we fix a basis $\{E_1, \ldots, E_k\}$ of $\mathfrak{a}$ such that $E_i \lambda_j = \delta_{ij}$, the Kronecker delta. Fix Υ in the left-hand side of (3.13). We need to explicitly understand the action of H on Υ. Again we work with the presentation of H given in (2.27). In particular, we have the elements $\widetilde{f_i} := \widetilde{E_i}$ whose span over $\mathbb{C}$ in H coincides with the span of all the elements of the form $\widetilde{f}$, $f \in \mathfrak{a}$. Then Lemma 2.8.2 (and the discussion at the end of Section 2.8) show that $\widetilde{f_i}$ acts in the H-module $F_1(X)$ by operator $\Omega_{0,i}^{\mathfrak{p}}$ defined in (2.23). Unwinding the natural vector space isomorphism between $F_1(X_1^\mathbb{R}(\nu))$ and the left-hand side of (3.13), the value at $\mathbf{v}$ of the function obtained by action by $\widetilde{f_i}$ on Υ is

$$[\widetilde{f_i} \cdot \Upsilon](\mathbf{v}) = \sum_{E \in B \cap \mathfrak{p}} E \Upsilon\big(\lambda \otimes \lambda_1 \otimes \cdots \otimes E^* \lambda_i \otimes \cdots \otimes \lambda_k\big), \tag{3.14}$$

where the actions of $E \in \mathfrak{p}$ on the right-hand side are the obvious ones. Recall, as in Section 2.7, that the elements $E \in B \cap \mathfrak{p}$ are a basis of $\mathfrak{p}$ which are (orthonormal) simultaneous eigenvectors for the action of $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Beyond that requirement and normalization considerations, we are free to choose them as we wish, and so we may assume that (possibly after rescaling) $E_1, \dots, E_k$ are among them.

Because of the appearance of the projection $\gamma_\circ$ in the definition of Γ_ν, many terms in (3.14) will not contribute to $\Gamma_\nu(\widetilde{f_i} \cdot \Upsilon)$. More precisely, note that for any Ψ in the left-hand side of (3.13), $\Gamma_\nu(\Psi)$ is determined by its value at $\mathbf{v}$; moreover, this value is $[\gamma_\circ(\Psi(\mathbf{v}))]^{\widetilde{}} \otimes 1$. Because the terms involving $E \notin \mathfrak{a}$ will not survive $\gamma_\circ$, we see

$$\big[\Gamma_\nu(\widetilde{f_i} \cdot \Upsilon)\big](\mathbf{v}) = \left[\gamma_\circ\bigg(\sum_{E \in B \cap \mathfrak{a}} E\Upsilon\big(\lambda \otimes \lambda_1 \otimes \cdots \otimes E^*\lambda_i \otimes \cdots \otimes \lambda_k\big)\bigg)\right]^{\widetilde{}} \otimes 1.$$

Thus we have

$$\big[\Gamma_\nu(\widetilde{f_i} \cdot \Upsilon)\big](\mathbf{v}) = \big[\gamma_\circ\big(E_i\Upsilon(\mathbf{v})\big)\big]^{\widetilde{}} \otimes 1 = \big[E_i\gamma_\circ\big(\Upsilon(\mathbf{v})\big)\big]^{\widetilde{}} \otimes 1$$
$$= \widetilde{f_i} \cdot \big[\Gamma_\nu(\Upsilon)\mathbf{v}\big];$$

the last equality is by (3.8). Thus

$$\Gamma_\nu(\widetilde{f_i} \cdot \Upsilon) = \widetilde{f_i} \cdot \Gamma_\nu(\Upsilon)$$

and Γ_ν respects the action of the elements $\widetilde{f_i}$, hence all $\widetilde{f}$, in H.

It remains to check that Γ_ν is equivariant for the action of $W_{\mathbb{R}}$. But this reduces to Proposition 3.1.3. We omit further details. This completes the proof of Theorem 3.0.4. $\quad\square$

4. Hermitian forms

This section can be interpreted as a generalization of results of [19] for category $\mathcal{O}$ for $\mathfrak{gl}(n)$. The main result is Theorem 4.2.2.

4.1. Generalities on Hermitian forms

Definition 4.1.1. (See [20].) In the setting of a general real reductive group $G_{\mathbb{R}}$, let X be a $(\mathfrak{g}, K)$-module. A Hermitian form $\langle\,,\,\rangle : X \times X \to \mathbb{C}$ is called invariant if it satisfies:

(a) $\langle A \cdot x, y \rangle = -\langle x, \overline{A} \cdot y \rangle$, for every $x, y \in X$, $A \in \mathfrak{g}$; here, $\overline{A}$ denotes the complex conjugate of A (with respect to $\mathfrak{g}_{\mathbb{R}}$).
(b) $\langle k \cdot x, y \rangle = \langle x, k^{-1} \cdot y \rangle$, for every $x, y \in X$, $k \in K_{\mathbb{R}}$.

A $(\mathfrak{g}, K)$ module X is called Hermitian if there exists an invariant form on X. (In this case, the form is unique up to scalar.) If the form is positive definite, then X is called unitary.

Definition 4.1.2. (See [7].) In the setting of Definition 2.6.1, set $H = H(\Psi, \mathbf{c})$ and let V be an H-module. A Hermitian form on V is H-invariant if

$$\langle x \cdot u, v \rangle = \langle u, x^* \cdot v \rangle, \quad \text{for all } u, v \in V, \ x \in H, \tag{4.1}$$

where $*$ is an involutive anti-automorphism on H defined on generators in the Lusztig presentation by:

$$w^* = w^{-1}, \quad \text{for all } w \in W,$$

$$f^* = -f + \sum_{\beta \in R^+} c_\beta \langle f, \beta \rangle s_\beta, \quad \text{for all } f \in \mathfrak{a},$$

where the sum ranges over the positive roots, and s_β denotes the reflection with respect to β. Because of (2.26), if we instead use the Drinfeld presentation (2.27) of H, $*$ is defined by

$$w^* = w^{-1}, \quad \text{for all } w \in W,$$

$$\widetilde{f}^* = -\widetilde{f}, \quad \text{for all } f \in \mathfrak{a}_\mathbb{R}.$$

Example 4.1.3. In the case of the type C_k Hecke algebra $\widetilde{H}_k(c)$ appearing in Example 2.6.3, we get:

$$s_{i,j}^* = s_{i,j}, \quad 1 \leqslant i < j \leqslant k;$$

$$\bar{s}_j^* = \bar{s}_j, \quad 1 \leqslant j \leqslant k;$$

$$\widetilde{\epsilon}_j^* = -\widetilde{\epsilon}_j, \quad 1 \leqslant j \leqslant k. \tag{4.2}$$

4.2. Preservation of unitarity

Return to the setting of Corollary 2.7.4. Given a $(\mathfrak{g}, K)$-invariant Hermitian form $\langle\ ,\ \rangle_X$ on an object X in $\mathcal{HC}_1(G_\mathbb{R})$, we wish to construct an $H(G_\mathbb{R})$-invariant form on $F_1(X) = \text{Hom}_{K_\mathbb{R}}[\mu_0, X \otimes V^{\otimes k}]$. To get started, we fix a positive definite Hermitian inner product $(\ ,\ )_V$ on V such that:

$$(k \cdot u, v)_V = \left(u, k^{-1} \cdot v\right)_V, \quad \text{for all } k \in K_\mathbb{R},$$

$$(E \cdot u, v)_V = (u, E \cdot v)_V, \quad \text{for all } E \in \mathfrak{p}_\mathbb{R}. \tag{4.3}$$

Define a Hermitian form on $\text{Hom}_\mathbb{C}(\mu_0, X \otimes V^{\otimes k})$ as follows. Given two elements φ, ψ with images $\varphi(1) = x \otimes u_1 \otimes \cdots \otimes u_n$ and $\psi(1) = y \otimes v_1 \otimes \cdots \otimes v_n$, set

$$\langle \varphi, \psi \rangle = \langle x, y \rangle_X (u_1, v_1)_V \cdots (u_n, v_n)_V. \tag{4.4}$$

Extend linearly to arbitrary φ and ψ. For every $a \in K_\mathbb{R}$ and $A \in \mathfrak{k}_\mathbb{R}$, we have:

$$\langle a \cdot \varphi, \psi \rangle = \langle \varphi, a^{-1} \cdot \psi \rangle, \tag{4.5}$$

$$\langle A \cdot \varphi, \psi \rangle = -\langle \varphi, A \cdot \psi \rangle. \tag{4.6}$$

Thus the form (4.4) induces a Hermitian form, denoted $\langle\ ,\ \rangle_1$, on the trivial $K_\mathbb{R}$ isotypic component $F_1(X)$ of $\text{Hom}_\mathbb{C}(\mu_0, X \otimes V^{\otimes k})$. It remains to check that this is $H(G_\mathbb{R})$-invariant.

Lemma 4.2.1. *In the setting of Corollary 2.7.4, assume that $F_1(X) \neq 0$. Assume further that there exists an invariant form $\langle \, , \, \rangle_X$ on X (Definition 4.1.2). Then the Hermitian form $\langle \, , \, \rangle_1$ defined on $F_1(X)$ induced by (4.4) is $\mathrm{H}(G_{\mathbb{R}})$-invariant. Moreover, if $\langle \, , \, \rangle_X$ is nondegenerate, then $\langle \, , \, \rangle_1$ is nondegenerate.*

Proof. We verify that $\langle \, , \, \rangle_1$ preserves the $*$-operation on the generators of $\mathrm{H}(G_{\mathbb{R}})$ in the Drinfeld presentation as in Example 4.1.3. Since $s_{i,j}$ acts by permuting the factor of $V^{\otimes k}$-factors, we clearly have

$$\langle s_{i,j} \cdot \varphi, \psi \rangle = \langle \varphi, s_{i,j} \cdot \psi \rangle.$$

Also, since $\xi \in K_{\mathbb{R}}$ and $\xi^2 = 1$, we have

$$\langle \bar{s}_i \cdot \varphi, \psi \rangle = \langle \varphi, \bar{s}_i \cdot \psi \rangle.$$

It remains to check the claim for each $\widetilde{\epsilon}_i$. Note that for every $1 \leqslant i \leqslant k$,

$$
\begin{aligned}
\big\langle \pi_k\big(\Omega^{\mathfrak{p}}_{0,i}\big)&(x \otimes \cdots \otimes u_i \cdots), (y \otimes \cdots \otimes v_i \cdots)\big\rangle \\
&= \sum_{E \in \mathfrak{p}_{\mathbb{R}}} \big\langle Ex \otimes \cdots \otimes E^* u_i \cdots, y \otimes \cdots \otimes v_i \cdots \big\rangle \\
&= \sum_{E \in \mathfrak{p}_{\mathbb{R}}} \langle Ex, y \rangle \cdots \big(E^* u_i, v_i\big)_V \cdots \\
&= -\sum_{E \in \mathfrak{p}_{\mathbb{R}}} \langle x, Ey \rangle \cdots \big(u_i, E^* v_i\big)_V \cdots \\
&= -\big\langle (x \otimes \cdots \otimes u_i \cdots), \pi_k\big(\Omega^{\mathfrak{p}}_{0,i}\big)(y \otimes \cdots \otimes v_i \cdots)\big\rangle.
\end{aligned}
$$

Since $\widetilde{\epsilon}_i$ acts by $\Omega^{\mathfrak{p}}_{0,i}$ (Lemma 2.8.2 and (2.26)), this implies that

$$\langle \widetilde{\epsilon}_i \cdot \varphi, \psi \rangle = -\langle \varphi, \widetilde{\epsilon}_i \cdot \psi \rangle.$$

Hence the form is invariant.

Now suppose that $\langle \, , \, \rangle_X$ is nondegenerate, and suppose the induced form $\langle \, , \, \rangle_1$ on $F_1(X)$ were degenerate. Notice that (4.5) shows that the trivial isotypic component $F_1(X)$ is orthogonal with (respect to $\langle \, , \, \rangle_X$) to every nontrivial $K_{\mathbb{R}}$ isotypic component of $\mathrm{Hom}_{\mathbb{C}}(\mu_0, X \otimes V^{\otimes k})$. Thus if $F_1(X)$ were degenerate, then $\langle \, , \, \rangle_X$ would also be degenerate, a contradiction. $\square$

Theorem 4.2.2. *In the setting of Corollary 2.7.4 and Theorem 3.0.4, let X be an irreducible Hermitian spherical $(\mathfrak{g}, K)$-module. Then:*

(1) *$F_1(X)$ is an irreducible Hermitian spherical $\mathrm{H}(G_{\mathbb{R}})$-module, and*
(2) *if, in addition, X is unitary, then $F_1(X)$ is unitary.*

Proof. In light of Lemma 4.2.1, the only claim that needs explanation is the preservation of irreducibility. So let X be an irreducible Hermitian spherical $(\mathfrak{g}, K)$-module in $\mathcal{H}_1(G_{\mathbb{R}})$. For the groups under consideration, X is of the form $\overline{X}_1^{\mathbb{R}}(\nu)$ with notation as in Definition 2.4.1; see [13], for instance. More precisely, there exists an invariant Hermitian form, say $\langle\,,\,\rangle_X$, on $X_1^{\mathbb{R}}(\nu)$ such that $\overline{X}_1^{\mathbb{R}}(\nu)$ is the quotient of $X_1^{\mathbb{R}}(\nu)$ by the radical of $\langle\,,\,\rangle_X$; that is, there is an exact sequence

$$0 \longrightarrow \mathrm{rad}\langle\,,\,\rangle_X \longrightarrow X_1^{\mathbb{R}}(\nu) \longrightarrow \overline{X}_1^{\mathbb{R}}(\nu) \longrightarrow 0,$$

where $\mathrm{rad}\langle\,,\,\rangle_X$ denotes the radical of $\langle\,,\,\rangle_X$. In particular, the form $\langle\,,\,\rangle_X$ is nondegenerate on $\overline{X}_1^{\mathbb{R}}(\nu)$. Applying the exact functor F_1 we have

$$0 \longrightarrow F_1\big(\mathrm{rad}\langle\,,\,\rangle_X\big) \longrightarrow X_1(\nu) \longrightarrow F_1\big(\overline{X}_1^{\mathbb{R}}(\nu)\big) \longrightarrow 0$$

where we have used Theorem 3.0.4 on the middle term. Lemma 4.2.1 gives an invariant form $\langle\,,\,\rangle_1$ on $F_1(X_1^{\mathbb{R}}(\nu)) = X_1(\nu)$ and a corresponding nondegenerate form on $F_1(\overline{X}_1^{\mathbb{R}}(\nu))$. Thus the second exact sequence is really

$$0 \longrightarrow \mathrm{rad}\langle\,,\,\rangle_1 \longrightarrow X_1(\nu) \longrightarrow F_1\big(\overline{X}_1^{\mathbb{R}}(\nu)\big) \longrightarrow 0. \tag{4.7}$$

It is easy to check that a nonzero spherical vector in $X_1^{\mathbb{R}}(\nu)$ naturally gives rise to a nonzero spherical vector in $F_1(X_1^{\mathbb{R}}(\nu))$. Thus $F_1(\overline{X}_1^{\mathbb{R}}(\nu))$ is nonzero. Together with (4.7), this implies $F_1(\overline{X}_1^{\mathbb{R}}(\nu))$ is irreducible. $\quad\square$

As remarked in Definition 3.0.3, the standard module $X_1(\nu)$ (for ν dominant) has a unique irreducible quotient. So (4.7) gives the following more precise result.

Corollary 4.2.3. *Retain the setting of Corollary* 2.7.4 *and recall the standard modules of Definitions* 2.4.1 *and* 3.0.3. *Assume* $\overline{X}_1^{\mathbb{R}}(\nu)$ *is Hermitian (Definition* 4.1.1*). Then*

$$F_1\big(\overline{X}_1^{\mathbb{R}}(\nu)\big) = \overline{X}_1(\nu).$$

Remark 4.2.4. In fact, Corollary 4.2.3 remains true without the assumption that $\overline{X}_1^{\mathbb{R}}(\nu)$ is Hermitian. (One instead needs to consider the pairing of $\overline{X}_1^{\mathbb{R}}(\nu)$ with its Hermitian dual and step through the proof of Theorem 4.2.2.) Since the argument goes through without much change, we omit the details.

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