CHARACTERS OF SPRINGER REPRESENTATIONS ON ELLIPTIC CONJUGACY CLASSES

DAN CIUBOTARU AND PETER E. TRAPA

ABSTRACT. For a Weyl group W, we investigate simple closed formulas (valid on elliptic conjugacy classes) for the character of the representation of W in the homology of a Springer fiber. We also give a formula (valid again on elliptic conjugacy classes) of the W-character of an irreducible discrete series representation with real central character of a graded affine Hecke algebra with arbitrary parameters. In both cases, the Pin double cover of W and the Dirac operator for graded affine Hecke algebras play key roles.

1. INTRODUCTION

Let W denote a Weyl group acting by the reflection representation in a real vector space V_0, and recall that an element of W is called elliptic if it has no nonzero fixed points on V_0. This paper investigates simple closed formulas, valid on the set of elliptic elements in W, for the character of the W representation on the homology of a Springer fiber. Our formulas depend on two ingredients — the computation of the Springer correspondence (in the top degree of homology) and the character table of a certain double cover of W — both of which are known. In particular, our approach is independent of the Lusztig-Shoji algorithm [L4, Sh].

In more detail, let $\Phi = (R, X, R^\vee, Y)$ be a crystallographic root system as in Section 2.1, and let W denote the corresponding Weyl group. Let e denote a nilpotent element in the complex semisimple Lie algebra $\mathfrak{g}$ attached to the root system Φ, and let $A(e)$ denote the component group of centralizer in $\mathrm{Ad}(\mathfrak{g})$ of e. Then Springer has defined an action of $W \times A(e)$ on the homology $H_\bullet(\mathcal{B}_e, \mathbb{Z})$ where $\mathcal{B}_e$ denotes the Springer fiber over e. For a fixed irreducible representation ϕ of $A(e)$, write $\chi_{e,\phi}$ for the character of the W representation on the ϕ-isotypic component of $H_\bullet(\mathcal{B}_e, \mathbb{Z})$, and assume this space is nonzero. Write $\sigma(e, \phi)$ for the irreducible representation of W in the top degree of homology.

Suppose e is distinguished in the sense of Bala-Carter (e.g [CM, Section 8.2]). Then Theorem 1.1 gives a formula for the value of $\chi_{e,\phi}$ on any elliptic element of W. The proof relies on a conceptual connection between the representation theory of the Pin double cover of W and harmonic analysis on the affine Hecke algebra attached to Φ (in the the guise of Theorem 3.5).

To state our formula, we first fix a choice of positive roots R^+ and a W-invariant inner product $\langle \ , \ \rangle$ on $V_0 = X \otimes_\mathbb{Z} \mathbb{R}$. The group $\mathsf{Pin}(V_0)$ is a subgroup of units in the Clifford algebra $C(V_0)$, and maps surjectively onto the orthogonal group $\mathsf{O}(V_0) = \mathsf{O}(V_0, \langle \ , \ \rangle)$ with kernel of order two. Write p for this surjection. In order

to state our results uniformly, let $W' = W$ if the dimension of V_0 is odd; and let W' denote the kernel of the sign character of W if the dimension of V_0 is even. Since W' acts by orthogonal transformation on V_0, we can consider its preimage $\widetilde{W}'$ in $\mathsf{Pin}(V_0)$.

If $\dim(V_0)$ is odd, $C(V_0)$ has exactly two inequivalent complex simple modules, denoted S^+ and S^-, which remain irreducible and inequivalent when restricted to $\widetilde{W}'$. Both restrictions are genuine in the sense that they do not factor to W'. If $\dim(V_0)$ is even, there is a single complex simple module S for $C(V_0)$, whose restriction to the even part of $C(V_0)$ is a sum of two nonisomorphic simple modules S^+ and S^-. Since elements in $\widetilde{W}'$ are in the even part of $C(V_0)$ in this case, it makes sense to restrict S^+ and S^- to $\widetilde{W}'$. These restrictions are again irreducible, inequivalent, and genuine. In both cases, we write χ_{S^+} and χ_{S^-} for the characters of the corresponding $\widetilde{W}'$ modules.

Propositions 2.1 and 2.2 below show that (up to sign)

$$\chi_{S^+}(\tilde{w}) - \chi_{S^-}(\tilde{w}) = \det(1 - p(\tilde{w}))^{1/2}, \tag{1.1}$$

for $w \in W'$. On the other hand, the elliptic elements in W are always contained in W'; see Section 3.1 below. Hence (1.1) is nonzero if and only if $p(\tilde{w})$ is elliptic. The final ingredient we need is the Casimir element

$$\Omega_{\widetilde{W}} = -\frac{1}{4} \sum_{\substack{\alpha,\beta>0 \\ s_\alpha(\beta)<0}} \frac{|\alpha^\vee|}{|\alpha|} \frac{|\beta^\vee|}{|\beta|} \alpha\beta \in C(V_0).$$

In fact $\Omega_{\widetilde{W}}$ is a central element in $\mathbb{C}[\widetilde{W}']$ which acts by a real scalar in any irreducible representation of $\widetilde{W}'$. Our main result, proved at the end of Section 3.3 below, is as follows.

Theorem 1.1. *Suppose e is a distinguished nilpotent element and ϕ is an irreducible representation of $A(e)$ such that $\chi_{e,\phi}$ is nonzero. Then there exist two distinct irreducible genuine representation $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ of $\widetilde{W}'$ (depending only on e and ϕ) such that for any elliptic element $w \in W$,*

$$\chi_{e,\phi}(w) = \frac{\chi_{\tilde{\sigma}^+}(\tilde{w}) - \chi_{\tilde{\sigma}^-}(\tilde{w})}{\chi_{S^+}(\tilde{w}) - \chi_{S^-}(\tilde{w})}; \tag{1.2}$$

here $\chi_{\tilde{\sigma}^\pm}$ denotes the character of $\tilde{\sigma}^\pm$, and $\tilde{w}$ is either element of $\widetilde{W}$ so that $p(\tilde{w}) = w$. The representations $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ are related by the involution Sg of (2.5),

$$\tilde{\sigma}^+ = \mathsf{Sg}(\tilde{\sigma}^-).$$

Moreover, $\tilde{\sigma}^\pm$ may be characterized as the unique (multiplicity one) constituents of

$$\sigma(e,\phi) \otimes (S^+ - S^-)$$

for which the scalar by which $\Omega_{\widetilde{W}}$ acts is as small as possible.

Note that the denominator in (1.2) is nonzero by (1.1) (and again any elliptic element is automatically in W'). Note also that the quotient in (1.2) is independent of the choice of $\tilde{w}$ since both $S^\pm$ and $\tilde{\sigma}^\pm$ are genuine representations of $\widetilde{W}$. Finally, the characterization of $\tilde{\sigma}^\pm$ in terms of eigenvalues of $\Omega_{\widetilde{W}}$ is computed explicitly in [C2].

The simplest example of Theorem 1.1 is when e is a regular nilpotent orbit and ϕ is trivial. In this case, $\sigma^+ = S^+$ and $\sigma^- = S^-$, and $\chi_{e,\phi}$ is identically one on the elliptic set.

According to the classification of Kazhdan-Lusztig [KaL] and Lusztig [L1, L3], the characters $\chi_{e,\phi}$ are the restrictions to W of the irreducible discrete series modules with real central character for a (graded) affine Hecke algebra $\mathbb{H}$ with equal parameters. So Theorem 1.1 may be interpreted as a result about discrete series $\mathbb{H}$ modules whose W characters do not vanish on the elliptic set. With this in mind, the idea of the proof of Theorem 1.1 is as follows. Fix such a discrete series $\mathbb{H}$ module X. The key object of study for us is

$$I(X) := X \otimes (S^+ - S^-)$$

introduced in Section 2.10. On one hand, in Lemma 3.4, we related $I(X)$ to the elliptic representation theory developed by Schneider-Stuhler [ScSt], Reeder [R], and Opdam-Solleveld [OS]. From this we deduce that $I(X)$ has the simple form $\tilde{\sigma}^+ - \tilde{\sigma}^-$, and dividing by $\chi_{S^+} - \chi_{S^+}$ leads to (1.2). On the other hand, in Proposition 2.5, we relate $I(X)$ to the index of the Dirac operator defined in [BCT]. A formula for the square of the Dirac operator then imposes strict limitations on the possibilities for $\tilde{\sigma}^{\pm}$. Together with [C2], they lead to the explicit description of $\tilde{\sigma}^{\pm}$ given in the theorem.

In the setting of arbitrary parameters, our argument leads to the following result. It bears a strong formal resemblance to Harish-Chandra's character formulas for the discrete series of a semisimple Lie group.

Theorem 1.2. *Let $\mathbb{H}$ be a graded affine Hecke algebra attached to the root system Φ and arbitrary real positive parameters as defined in Section 2.7. Let X be an irreducible discrete series module for $\mathbb{H}$, and write χ_X for the character of the W representation afforded by X. Fix $w \in W$ elliptic and choose $\tilde{w}$ such that $p(\tilde{w}) = w$. Then there exist two inequivalent genuine irreducible representations $\tilde{\sigma}^+ = \mathsf{Sg}(\tilde{\sigma}^-)$ $\widetilde{W}'$ such that*

$$\chi_X(w) = \frac{\chi_{\tilde{\sigma}^+}(\tilde{w}) - \chi_{\tilde{\sigma}^-}(\tilde{w})}{\chi_{S^+}(\tilde{w}) - \chi_{S^-}(\tilde{w})} \tag{1.3}$$

When the parameters are of geometric origin (as computed in [L3]), then the results of [C2] again apply to give an explicit description of $\tilde{\sigma}^{\pm}$ by a procedure similar to the one given in Theorem 1.1.

It is natural to ask to what extent Theorem 1.2 can be extended to modules outside the discrete series or (according to the Kazhdan-Lusztig classification) the extent to which Theorem 1.1 can be extended to nilpotent elements which are not distinguished. Reeder [R] has shown that $\chi_{e,\phi}$ is not identically zero on the set of elliptic elements if and only if e is quasidistinguished in the sense of Definition 3.8. Theorem 1.1 nearly holds assuming only e is quasidistinguished, but there are some fascinating exceptions. We summarize the situation in Section 3.4.

2. PRELIMINARIES

2.1. **Root data.** Fix a (reduced) root datum $\Phi = (R \subset X, R^{\vee} \subset Y)$. Thus X and Y are finite-rank lattices, there exists a perfect $\mathbb{Z}$-linear pairing denoted $(\ ,\ )$ from $X \times Y$ to $\mathbb{Z}$, and a bijection $\alpha \mapsto \alpha^{\vee}$ from R to $R^{\vee}$ such that

$$(\alpha, \alpha^{\vee}) = 2$$

and so that

$$s_\alpha(v) := v - (v, \alpha^\vee)\alpha$$
$$s_{\alpha^\vee}(v') := v' - (\alpha, v')\alpha^\vee$$

preserve R and $R^\vee$ respectively. We further assume R spans $V_0 := X \otimes_{\mathbb{Z}} \mathbb{R}$.

Let W denote the subgroup of $GL(X)$ generated by $\{s_\alpha \mid \alpha \in R\}$. Let sgn denote the character of W obtained by composing the inclusion $W \subset GL(X)$ with the determinant. Set $W_{\mathsf{ev}} = \ker(\mathsf{sgn})$, the even subgroup of W. Because of a well-known dichotomy which appears below for simple modules of the Clifford algebra, we find it convenient to define

$$W' = \begin{cases} W & \text{if } \dim(V_0) \text{ is odd;} \\ W_{\mathsf{ev}} & \text{if } \dim(V_0) \text{ is even.} \end{cases} \tag{2.1}$$

Choose a system of positive roots $R^+ \subset R$ and let Π denote the corresponding simple roots in R^+. As usual, write $\alpha > 0$ or $\alpha < 0$ in place of $\alpha \in R^+$ or $\alpha \in (-R^+)$, respectively.

Finally we fix a W-invariant inner product $\langle \cdot, \cdot \rangle$ on V_0 and use the same notation for its extension to a bilinear form on $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$.

2.2. The Clifford algebra and the Pin double cover $\widetilde{W}$.

Let $C(V_0)$ denote the Clifford algebra defined by V_0 and $\langle \, , \, \rangle$. Thus $C(V_0)$ is the quotient of the tensor algebra of V_0 by the ideal generated by

$$\omega \otimes \omega' + \omega' \otimes \omega + 2\langle \omega, \omega' \rangle, \quad \omega, \omega' \in V_0.$$

Let $\mathsf{O}(V_0)$ denote the group of orthogonal transformation of V_0 with respect to $\langle \, , \, \rangle$. The action of $-1 \in \mathsf{O}(V_0)$ induces a grading

$$C(V_0) = C(V_0)_{\mathsf{ev}} + C(V_0)_{\mathsf{odd}}.$$

Let ϵ be the automorphism of $C(V_0)$ which is $+1$ on $C(V_0)_{\mathsf{ev}}$ and -1 on $C(V_0)_{\mathsf{odd}}$. Let ^{t} be the transpose antiautomorphism of $C(V_0)$ characterized by

$$\omega^t = -\omega, \ \omega \in V_0, \quad (ab)^t = b^t a^t, \ a, b \in C(V_0).$$

The Pin group is defined as

$$\mathsf{Pin}(V_0) = \{a \in C(V_0) \mid \epsilon(a)V_0 a^{-1} \subset V_0, \ a^t = a^{-1}\}.$$

and we have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathsf{Pin}(V_0) \overset{p}{\longrightarrow} \mathsf{O}(V_0) \longrightarrow 1,$$

where the projection p is given by $p(a)(\omega) = \epsilon(a)\omega a^{-1}$. The image under p of $\alpha/|\alpha|$ is s_α, and the elements

$$\tilde{s}_\alpha := \alpha/|\alpha|, \tag{2.2}$$

for α simple, generate $\widetilde{W}$.

Since W acts by orthogonal transformations of V_0, we may define

$$\widetilde{W} = p^{-1}(W) \tag{2.3}$$

and

$$\widetilde{W'} = p^{-1}(W') \tag{2.4}$$

where W' is defined in (2.1).

2.3. **Notation for representation rings.** Given a ring R, we let $\mathcal{R}(R)$ denote the complex Grothendieck group of finite-length left R-modules. In the special case that $R = \mathbb{C}[G]$, the group algebra of a finite group G, we write $\mathcal{R}(G)$ instead of $\mathcal{R}(\mathbb{C}[G])$. As usual, define a bilinear pairing on $\mathcal{R}(G)$ via

$$(\sigma_1, \sigma_2)_G = \dim \operatorname{Hom}_G(\sigma_1, \sigma_2)$$

for simple $\mathbb{C}[G]$ modules σ_i (and extended linearly to $\mathcal{R}(G)$). In terms of the characters χ_i of σ_i,

$$(\sigma_1, \sigma_2)_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g).$$

2.4. **An involution on $\mathcal{R}(\widetilde{W}')$.** We define

$$\mathsf{Sg} \; : \; \mathcal{R}(\widetilde{W}') \to \mathcal{R}(\widetilde{W}') \tag{2.5}$$

as follows. If $\dim(V_0)$ is odd, then $W' = W$ by definition, and we define Sg by linearly extending

$$\mathsf{Sg}(M) = M \otimes \mathsf{sgn}.$$

for simple modules M.

Suppose instead that $\dim(V_0)$ is even so that $\widetilde{W}'$ is an index two subgroup of $\widetilde{W}$. Given a simple module M, Frobenius reciprocity implies that there exists a unique simple module $\mathsf{Sg}(M)$ such that

$$\operatorname{Res}_{\widetilde{W}'} \left(\operatorname{Ind}_{\widetilde{W}'}^{\widetilde{W}}(M) \right) \;=\; M + \mathsf{Sg}(M)$$

in $\mathcal{R}(\widetilde{W}')$. This defines $\mathsf{Sg}(M)$ on simple modules, and the general definition is obtained by extending linearly.

2.5. **Simple $C(V_0)$ modules and their restrictions to $\widetilde{W}'$.** The point of this subsection is to define two inequivalent irreducible representations $(\gamma^{\pm}, S^{\pm})$ of the group $\widetilde{W}'$ defined in (2.4).

Suppose first that $\dim(V_0) = 2r$ is even (so in particular $\widetilde{W}'$ is a subgroup of $C(V_0)_{\mathsf{ev}}$). Then, up to equivalence, $C(V_0)$ has a unique simple (complex) module S which remains irreducible when restricted to $\widetilde{W}$. Its restriction to $C(V_0)_{\mathsf{ev}}$, however, splits into two inequivalent modules, each of dimension 2^{r-1}, and each of which restrict irreducibly to the group $\widetilde{W}'$. We denote these two $\widetilde{W}'$ representations by (γ^+, S^+) and (γ^-, S^-). By construction the action of $\widetilde{W}/\widetilde{W}'$ interchanges S^+ and S^-. The notational superscripts $\pm$ are arbitrary, and a choice must be made when fixing them.

Next suppose $\dim(V_0^{\vee}) = 2r + 1$ is odd. Then, up to equivalence, there is unique complex simple module S for $C(V_0)_{\mathsf{ev}}$ (of dimension 2^r) which may be extended naturally in two distinct ways to obtained inequivalent simple module structures S^+ and S^- on the same space S. (The choice of superscript labels is again arbitrary.) Each of these modules remains irreducible when restricted to $\widetilde{W} = \widetilde{W}'$, and we continue to denote these two $\widetilde{W}'$ representations by (γ^+, S^+) and (γ^-, S^-). By construction, the action of $\widetilde{w} \in \widetilde{W}$ in S^+ and S^- differ by tensoring with sgn,

$$\gamma^+(\widetilde{w})s = \mathsf{sgn}(p(\widetilde{w}))\gamma^-(\widetilde{w})s, \tag{2.6}$$

and, in particular, the representations are inequivalent.

In the notation of Section 2.4, note that $\mathsf{Sg}(S^{\pm}) = S^{\mp}$.

Proposition 2.1. *Recall the construction of $S^{\pm}$ given above, and the notation of (2.4) and Section 2.3. Set*

$$\wedge^{\pm} V = \sum_i (-1)^i \wedge^i V, \tag{2.7}$$

regarded as an element of $\mathcal{R}(\widetilde{W'})$. Then

$$\left(S^+ - S^-\right) \otimes \left(S^+ - S^-\right)^* = \frac{2}{[W:W']} \wedge^{\pm} V \tag{2.8}$$

as elements of $\mathcal{R}(\widetilde{W'})$. (Here $[W:W']$ denotes the index of W' in W, which is one or two according to whether $\dim(V)$ is odd or even, and $(S^+ - S^-)^*$ is the contragredient of $(S^+ - S^-)$.)

Proof. This is a direct consequence of the construction of the simple modules for $C(V_0)$. See, for example, the discussion around [BoW, Lemma II.6.5]. $\square$

Proposition 2.2. *Given $w \in W$ let $\det(1-w)$ denote the determinant of $Id_V - w$ acting on V. Let $\chi_{\wedge^{\pm}}$ denote the character of $\wedge^{\pm} V$ (defined in (2.7)) regarded now as a virtual representation of W. Then*

$$\chi_{\wedge^{\pm}}(w) = \det(1-w)$$

for all w in W.

Proof. This is [R, Lemma 2.1.1]. $\square$

2.6. **The affine Hecke algebra.** In the setting of Section 2.1, let $W_{\text{ex}} = X \rtimes W$ denote the extended Weyl group. We denote the elements of W_{ex} by $w\mathsf{a}^x$, where $w \in W$, $x \in X$ and a is a symbol. The length function ℓ on W_{ex} is defined as in [L1, 1.4(a)]:

$$\ell(w\mathsf{a}^x) = \sum_{\alpha \in R^+, w(\alpha) \in R^-} |(x, \check{\alpha}) + 1| + \sum_{\alpha \in R^+, w(\alpha) \in R^+} |(x, \check{\alpha})|.$$

Let $\leq$ be the order on $R^{\vee}$: $\check{\alpha} \leq \check{\beta}$ if and only if $\check{\beta} - \check{\alpha} \in \mathbb{Z}_{\geq 0}\langle \check{\alpha} : \alpha \in \Pi \rangle$, and set $\Pi_m = \{\beta \in R : \check{\beta} \text{ minimal for } \leq\}$.

Let $Q = \mathbb{Z}R$ denote the integral span of R. Then $W_{\text{aff}} := Q \rtimes W$ is a Coxeter group generated by $S = \{s_{\alpha} : \alpha \in \Pi\} \cup \{s_{\alpha}\mathsf{a}^{\alpha} : \alpha \in \Pi_m\}$. Fix an indeterminate q and a function $L : S \to \mathbb{N}$ such that $L(s) = L(s')$ whenever s and s' are conjugate by W_{ex}. If $\alpha \in \Pi$, let $S(\alpha)$ be the connected component of α in the Coxeter graph (W_{aff}, S). If $\check{\alpha} \in 2Y$, then $S(\alpha)$ is an affine diagram of type C, and therefore it has a unique nontrivial automorphism. Let $\widetilde{s}_{\alpha}$ denote the image of s_{α} under this automorphism.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\Phi, \mathsf{q}, L)$ is the complex associative algebra with unit over $\mathbb{C}[\mathsf{q}, \mathsf{q}^{-1}]$ with basis $\{N_x \mid x \in W_{\text{ex}}\}$ subject to the relations:

$$N_x N_y = N_{xy} \qquad \text{if } \ell(xy) = \ell(x) + \ell(y)$$

and

$$(N_s - \mathsf{q}^{L(s)})(N_s + \mathsf{q}^{L(s)}) = 0 \qquad \text{for } s \in S_{\text{aff}}.$$

Set $T = \text{Hom}_{\text{alg}}(X, \mathbb{C}^{\times})$. The center of $\mathcal{H}$ is isomorphic to $\mathbb{C}[T]^W$, the W invariants in the coordinate ring of T. A version of Schur's Lemma implies that the center acts by scalars in any irreducible $\mathcal{H}$ module $\mathcal{X}$, and hence we can attached a W orbit in T to $\mathcal{X}$ called the central character of $\mathcal{X}$. If the orbit consists of real positive valued functions on X, then we say that $\mathcal{X}$ has real central character.

In the case that L is identically 1, the Borel-Casselman equivalence gives natural notions of tempered (and discrete series) $\mathcal{H}$ modules. In the general case, a version of Casselman's weight criterion can be formulated to define tempered and discrete series $\mathcal{H}$ modules. Opdam has shown that these definitions have the expected analytic interpretations. See [O, Section 2] for a summary.

2.7. **The graded affine Hecke algebra.** In the setting of Section 2.1, fix a W-invariant map $c : R \to \mathbb{N}$, and set $c_\alpha = c(\alpha)$. Let r denote an indeterminate. Lusztig has defined a graded affine Hecke algebra [L1] $\mathbb{H} = \mathbb{H}(\Phi, \mathsf{r}, c)$ attached to the root datum Φ and with parameter function c is the complex associative algebra over $\mathbb{C}[\mathsf{r}]$ with unit generated by the symbols $\{t_w \mid w \in W\}$ and $\{t_f \mid f \in S(V)\}$, subject to the relations:

(1) The linear map from the group algebra $\mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w$ to $\mathbb{H}$ taking w to t_w is an injective map of algebras.
(2) The linear map from the symmetric algebra $S(V)$ to $\mathbb{H}$ taking an element f to t_f is an injective map of algebras.

We write f instead of t_f in $\mathbb{H}$. The final relation is

(3)
$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = c_\alpha \mathsf{r}(\omega, \alpha^\vee), \quad \alpha \in \Pi, \ \omega \in V;$$

here $s_\alpha(\omega)$ is the element of V obtained by s_α acting on ω.

The center $Z(\mathbb{H})$ of $\mathbb{H}$ is $S(V)^W$. Again a version of Schur's Lemma implies that the center acts by scalars in any irreducible $\mathbb{H}$ module X, and hence determines a W orbit in $V^\vee$ called the central character of X. If the W orbit actually lies in $V_0^\vee$, then we say X has real central character.

Finally, any $\mathbb{H}$ module X can be restricted to $\mathbb{C}[W]$ to obtain a representation of W. This descends to a map

$$\mathsf{res}_W \ : \ \mathcal{R}(\mathbb{H}) \longrightarrow \mathcal{R}(W). \tag{2.9}$$

2.8. **Relation between $\mathcal{H}$ modules and $\mathbb{H}$ modules.** In the setting of Sections 2.6 and 2.7, specialize from now on $\mathsf{q} \in \mathbb{R}_{>1}$, $\mathsf{r} = \log q > 0$, and let c denote the parameter function for $\mathbb{H}$ defined as follows:

$$c(\alpha) = \begin{cases} 2L(s_\alpha), & \alpha \in \Pi, \check{\alpha} \notin 2Y, \\ L(s_\alpha) + L(\widetilde{s}_\alpha), & \alpha \in \Pi, \check{\alpha} \in 2Y; \end{cases}$$

here $s_\alpha, \widetilde{s}_\alpha, L$ are as in Section 2.6. The main results of [L1, Section 10.9] establish an equivalence between the category of $\mathcal{H}$ modules $\mathcal{R}^0(\mathcal{H})$ with real central character and the category of $\mathbb{H}$ modules with real central character. We simply collect the properties of this equivalence we shall need below:

(a) There is a bijection between tempered $\mathcal{H}$ modules with real central character and tempered $\mathbb{H}$ modules with real central character.
(b) If $\mathcal{X}$ and X correspond under the equivalence of categories, then define (with notation as in (2.9))

$$\mathsf{res}_W(\mathcal{X}) := \mathsf{res}_W(X). \tag{2.10}$$

This extends to a linear map

$$\mathsf{res}_W : \mathcal{R}^0(\mathcal{H}) \to \mathcal{R}(W). \tag{2.11}$$

2.9. **The Dirac operator.** Let $\omega_1, \ldots, \omega_r$ denote an orthonormal basis of V_0 (with respect to $\langle\,,\,\rangle$). Set

$$\widetilde{\omega}_i = \omega_i - \frac{1}{2}\sum_{\beta>0} c_\beta(\omega_i, \beta^\vee)s_\beta \in \mathbb{H}.$$

Following [BCT], define

$$\mathcal{D} = \sum_i \widetilde{\omega}_i \otimes \omega_i \in \mathbb{H} \otimes C(V_0). \tag{2.12}$$

Then it is easy to verify that $\mathcal{D}$ is well-defined independent of original choice of orthonormal basis. Given an $\mathbb{H}$-module X and a complex simple module S for $C(V_0)$, denote the image of $\mathcal{D}$ in endomorphisms of $X \otimes S$ by D,

$$D \in \mathrm{End}_{\mathbb{C}}(X \otimes S).$$

Then D is called the Dirac operator for X (and S). Define the Dirac cohomology of X to be

$$H_D(X) = \ker(D)/(\ker(D) \cap \mathrm{im}(D)).$$

Proposition 2.3. *Let ρ denote the inclusion*

$$\mathbb{C}[\widetilde{W}] \longrightarrow \mathbb{H} \otimes C(V_0)$$

obtained by linearly extending the map

$$\widetilde{w} \mapsto t_{p(\widetilde{w})} \otimes \widetilde{w}.$$

Then

$$\rho(\widetilde{w})\mathcal{D} = \mathsf{sgn}(p(\widetilde{w}))\mathcal{D}\rho(\widetilde{w})$$

as elements of $\mathbb{H} \otimes C(V_0)$. Thus left multiplication by $\rho(\widetilde{w})$ defines a representation of $\widetilde{W}$ on $H_D(X)$.

Proof. This is [BCT, Lemma 3.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Consider the Casimir element

$$\Omega_{\widetilde{W}} = -\frac{1}{4}\sum_{\substack{\alpha,\beta>0 \\ s_\alpha(\beta)<0}} c_\alpha c_\beta |\alpha^\vee||\beta^\vee|\,\tilde{s}_\alpha\tilde{s}_\beta \in C(V_0),$$

with $\tilde{s}_\alpha$ and $\tilde{s}_\beta$ as in (2.2). In fact $\Omega_{\widetilde{W}}$ is an element of $\mathbb{C}[\widetilde{W}']$, and it is central in $\mathbb{C}[\widetilde{W}]$; e.g. [BCT, Section 3.4]. Given an irreducible representation $\tilde{\sigma}$ of $\widetilde{W}$, let

$$a(\tilde{\sigma}) = \text{the scalar by which } \Omega_{\widetilde{W}} \text{ acts in } \tilde{\sigma}. \tag{2.13}$$

Proposition 2.4. *Suppose X is an irreducible $\mathbb{H}$ module with central character represented by $\nu \in V^\vee$. If $\tilde{\sigma}$ is an irreducible representation of $\widetilde{W}$ such that*

$$\mathrm{Hom}_{\widetilde{W}}(\tilde{\sigma}, H_D(X)) \neq 0,$$

then

$$\langle \nu, \nu \rangle = a(\tilde{\sigma}),$$

with notation as in (2.13).

Proof. Theorem 3.5 of [BCT] shows that

$$\mathcal{D}^2 = -\Omega_{\mathbb{H}} \otimes 1 + \rho\left(\Omega_{\widetilde{W}}\right) \tag{2.14}$$

where $\Omega_{\mathbb{H}}$ is a central element of $\mathbb{H}$ which acts by the squared length of the central character in any irreducible $\mathbb{H}$ module. Hence if $v \neq 0$ is in the $\tilde{\sigma}$ isotypic component of the kernel of $\mathcal{D}$ acting on $X \otimes S$, then applying both sides of (2.14) to v gives

$$0 = -\langle \nu, \nu \rangle + a(\tilde{\sigma}).$$

This proves the proposition. $\qquad\square$

2.10. **The Dirac index.** Retain the setting of Section 2.7, and recall the irreducible $\widetilde{W}'$ modules $S^{\pm}$ introduced in Section 2.5. Define the Dirac index

$$I \ : \ \mathcal{R}(\mathbb{H}) \longrightarrow \mathcal{R}(\widetilde{W}')$$

as

$$I(X) = \mathsf{res}_{W'}(X) \otimes (S^+ - S^-); \tag{2.15}$$

here $\mathsf{res}_{W'}(X)$ denote the restriction of X to W' pulled back to an element of $\mathcal{R}(\widetilde{W}')$. The remainder of this section will be devoted to explaining the relationship between $I(X)$ and $H_D(X)$.

Suppose first that $\dim(V_0)$ is even, and let (γ, S) denote the unique complex simple $C(V_0)$ module up to equivalence. For a fixed $\mathbb{H}$ module X, let D denote the Dirac operator defined in the previous section. As remarked in Section 2.5, the restriction of S to $C(V_0)_{\mathsf{ev}}$ splits into simple modules as $S = S^+ \oplus S^-$. Thus for $s \in S^{\pm}$ and $v \in V_0 \subset C(V_0)_{\mathsf{odd}}$, $\gamma(v)s \in S^{\mp}$. Hence D maps $X \otimes S^+$ to $X \otimes S^-$, and $X \otimes S^-$ to $X \otimes S^+$. We let D^+ and D^- denote the respective restrictions,

$$D^{\pm} \ : \ X \otimes S^{\pm} \longrightarrow X \otimes S^{\mp},$$

and set

$$H_D^{\pm}(X) = \ker(D^{\pm})/(\ker(D^{\pm}) \cap \mathrm{im}(D^{\mp})). \tag{2.16}$$

According to Proposition 2.3,

$$H_D^{\pm}(X) \in \mathcal{R}(\widetilde{W}'). \tag{2.17}$$

Next suppose $\dim(V_0)$ is odd, and recall the two simple $C(V_0)$ module structures S^+ and S^- (on the same complex vector space). Fix an $\mathbb{H}$ module X and define

$$D \ : \ X \otimes S^+ \longrightarrow X \otimes S^+.$$

We can compose this with the vector space identity map $S^+ \to S^-$ to obtain

$$D^+ \ : \ X \otimes S^+ \longrightarrow X \otimes S^-.$$

Reversing the roles of S^+ and $S-$ we obtain

$$D^- \ : \ X \otimes S^- \longrightarrow X \otimes S^+.$$

We then define $H_D^{\pm}(X)$ via (2.16). According to (2.6) and Proposition 2.3, we once again have

$$H_D^{\pm}(X) \in \mathcal{R}(\widetilde{W}'). \tag{2.18}$$

Proposition 2.5. *Fix an $\mathbb{H}$ module X, and recall the notation of (2.15), (2.17), and (2.18). Then*

$$I(X) = H_D^+(X) - H_D^-(X) \tag{2.19}$$

as elements of $\mathcal{R}(\widetilde{W}')$.

Proof. Equation (2.14) implies that $D^{\pm} \circ D^{\mp}$ are diagonalizable linear operators. The proposition then reduces to simple linear algebra. $\qquad\square$

Proposition 2.5 explains why $I(X)$ is called the Dirac index of X. Results like Proposition 2.4 place rather strict limitations on the possible structure of $I(X)$, and hence give nontrivial information about the structure of X as a W' module. In Section 3.3, we use this idea (together with the main results of [C2]) to explicitly identify the numerator in Theorem 1.1.

3. MAIN RESULTS

3.1. **Elliptic representations of W.** An element $w \in W$ is called elliptic if the action of w on V has no nontrivial fixed points; equivalently (in the notation of Proposition 2.2) if

$$\det(1 - w) \neq 0.$$

The set of elliptic elements of W will be denoted W_{ell}. We can make the same definition for the subgroup W' defined in (2.1). Then $W'_{\text{ell}} = W_{\text{ell}}$. (To see this, let $M(w)$ denote the image of w in $\mathsf{O}(V_0)$. Since w is elliptic, the only possible real eigenvalue of $M(w)$ is -1. Since the imaginary eigenvalues come in pairs, $\det(M(w)) = (-1)^{\dim(V_0)}$. See [Ca] for more refined results.)

Following [R, (2.2.1)], we define a bilinear pairing on $\mathcal{R}(W)$ defined on irreducible representations σ_1 and σ_2 via

$$e_W(\sigma_1, \sigma_2) = \sum_i (-1)^i \dim \operatorname{Hom}_W \left(\sigma_1 \otimes \wedge^i V, \sigma_2 \right).$$

Then [R, (2.2.2)] shows that taking characters induces an isomorphism

$$\overline{\mathcal{R}}(W) := \mathcal{R}(W)/\ker(e_W) \longrightarrow \mathbb{C}[W_{\text{ell}}]^W \tag{3.1}$$

onto the class functions on W vanishing off W_{ell}. (More intrinsically, the kernel of e_W consists of the span of representations induced from proper parabolic subgroups of W.) Continue to write e_W for the induced nondegenerate form on $\overline{\mathcal{R}}(W)$.

Remark 3.1. By taking characters of both sides of (2.15) and using Propositions 2.1-2.2, we see that $I(X) = 0$ if and only if the character of $\operatorname{res}_{W'}(X)$ vanishes on W'_{ell}. Since the representation obtained by restriction to W of a parabolically induced $\mathbb{H}$-module is induced from a corresponding parabolic subgroup of W, the parenthetic remark after (3.1) implies that $I(X)$ vanishes if and only if X is in the span of $\mathbb{H}$ modules which are induced from proper parabolic subalgebras of $\mathbb{H}$.

In the setting of Section 2.2 and 2.3, let $\mathcal{R}_g(\widetilde{W'})$ denote the complex linear combinations of genuine irreducible representations of $\widetilde{W'}$. (A representation of $\widetilde{W'}$ is called genuine is it does not factor to W'.) Recall the (genuine) $\widetilde{W'}$ modules $S^{\pm}$ defined in Section 2.5. Consider the map

$$i \; : \mathcal{R}(W) \longrightarrow \mathcal{R}_g(\widetilde{W'}) \tag{3.2}$$

defined by

$$\sigma \mapsto \operatorname{res}_{W'}(\sigma) \otimes (S^+ - S^-); \tag{3.3}$$

where $\operatorname{res}_{W'}(\sigma)$ denotes the restriction of σ to W' pulled back to $\widetilde{W'}$; c.f. (2.15). Proposition 2.1 shows that i vanishes on any element of $\mathcal{R}(W)$ whose character has

support in the complement of $W_{\mathsf{ell}} = W'_{\mathsf{ell}}$. Thus (3.1) implies that i descends to an injection

$$i \; : \overline{\mathcal{R}}(W) \longrightarrow \mathcal{R}_g(\widetilde{W'}). \tag{3.4}$$

Proposition 3.1. *The map i defined in (3.4) satisfies*

$$(i(\sigma_1), i(\sigma_2))_{\widetilde{W'}} = 2e_W(\sigma_1, \sigma_2)$$

for all $\sigma_i \in \overline{\mathcal{R}}(W')$.

Proof. For irreducible representations σ_1 and σ_2 of W', we compute

$$
\begin{aligned}
(i(\sigma_1), i(\sigma_2))_{\widetilde{W'}} &= \big(\sigma_1 \otimes (S^+ - S^-), \sigma_2 \otimes (S^+ - S^-)\big)_{\widetilde{W'}} \\
&= \big(\sigma_1 \otimes (S^+ - S^-) \otimes (S^+ - S^-)^*, \sigma_2\big)_{\widetilde{W'}} \\
&= \frac{2}{[W : W']} \big(\sigma_1 \otimes \wedge^{\pm} V, \sigma_2\big)_{\widetilde{W'}} \\
&= \frac{2}{[W : W']} \big(\sigma_1 \otimes \wedge^{\pm} V, \sigma_2\big)_{W'}
\end{aligned}
$$

for the third equality, we have used Proposition 2.1; for the fourth, we have used that the representations being paired all factor to W'. On the other hand, since the character $\chi_{\wedge^{\pm}(V)}$ is supported on W_{ell} (by Proposition 2.2) and since $W'_{\mathsf{ell}} = W_{\mathsf{ell}}$,

$$
\begin{aligned}
\frac{2}{[W : W']} \big(\sigma_1 \otimes \wedge^{\pm} V, \sigma_2\big)_{W'} &= \frac{2}{[W : W']} \frac{1}{|W'|} \sum_{x \in W'} \overline{\chi_{\sigma_1 \otimes \wedge^{\pm}(V)}(x)} \chi_{\sigma_2}(x) \\
&= \frac{2}{|W|} \sum_{x \in W} \overline{\chi_{\sigma_1 \otimes \wedge^{\pm}(V)}(x)} \chi_{\sigma_2}(x) \\
&= 2 \sum_i (-1)^i \dim \operatorname{Hom}_W \big(\sigma_1 \otimes \wedge^i V, \sigma_2\big) \\
&= 2e_W(\sigma_1, \sigma_2).
\end{aligned}
$$

This completes the proof. $\qquad\square$

3.2. **Relation with the Euler-Poincaré pairing; proof of Theorem 1.2.** In the setting of Section 2.6, let $\mathcal{X}$ and $\mathcal{Y}$ be two irreducible $\mathcal{H}$ modules. Following [ScSt] define

$$\mathrm{EP}(\mathcal{X}, \mathcal{Y}) = \sum_{i \geq 0} (-1)^i \dim \operatorname{Ext}^i_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}).$$

Theorem 3.2 (Schneider-Stuhler, Opdam-Solleveld). *Suppose $\mathcal{X}$ is an irreducible discrete series $\mathcal{H}$ module and $\mathcal{Y}$ is an irreducible tempered $\mathcal{H}$ module. Then*

    (a) $\mathrm{EP}(\mathcal{X}, \mathcal{X}) = 1$; *and*
    (b) $\mathrm{EP}(\mathcal{X}, \mathcal{Y}) = 0$ *if $\mathcal{X}$ is not equivalent to $\mathcal{Y}$.*

Proof. This is [OS, Theorem 3.8]. $\qquad\square$

The connection with the previous section is as follows. Let $\mathcal{R}^0_{\mathsf{temp}}(\mathcal{H})$ denote the subspace of $\mathcal{R}(\mathcal{H})$ generated by irreducible tempered modules with real central character. Set

$$\overline{\mathcal{R}}^0_{\mathsf{temp}}(\mathcal{H}) = \mathcal{R}^0_{\mathsf{temp}}(\mathcal{H})/\ker(\mathrm{EP}) \tag{3.5}$$

and continue to write EP for the induced nondegenerate form.

Theorem 3.3 (Reeder, Opdam-Solleveld). *Recall the map res_W of (2.11) and the notation of (3.1) and (3.5). Then res_W restricts to a linear map*

$$\mathsf{res}_W \; : \; \overline{\mathcal{R}}^0_{\mathsf{temp}}(\mathcal{H}) \longrightarrow \overline{\mathcal{R}}(W)$$

which satisfies

$$e_W\left(\mathsf{res}_W(\mathcal{X}), \mathsf{res}_W(\mathcal{Y})\right) = EP(\mathcal{X}, \mathcal{Y}) \tag{3.6}$$

for all $\mathcal{X}, \mathcal{Y} \in \overline{\mathcal{R}}(\mathcal{H})$.

Proof. For equal parameters, this is [R, Theorem 5.10.1]. The general case follows from Proposition 3.9(1) and Theorem 3.2(c) in [OS]. $\qquad\square$

Lemma 3.4. *In the setting of Section 2.7, let X and Y be two irreducible tempered $\mathbb{H}$ modules with real central character, and write $\mathcal{X}$ and $\mathcal{Y}$ for the corresponding tempered $\mathcal{H}$ modules (Section 2.8). Recall the Dirac index $I(X) = \mathsf{res}_{W'}(X) \otimes (S^+ - S^-)$ of (2.15). Then*

$$(I(X), I(Y))_{\widetilde{W}} = 2EP(\mathcal{X}, \mathcal{Y}).$$

If we further assume that X is a discrete series module, then

$$(I(X), I(X))_{\widetilde{W}} = 2. \tag{3.7}$$

Proof. From (2.15) and (3.2), it follows that $I(X) = i\left(\mathsf{res}_W(X)\right)$. Proposition 3.1 and Theorem 3.3 imply that

$$(I(X), I(Y))_{\widetilde{W'}} = 2EP(\mathcal{X}, \mathcal{Y}).$$

This is the first assertion of the lemma, and (3.7) then follows from Theorem 3.2. $\quad\square$

Theorem 3.5. *In the setting of Section 2.7, suppose X is an irreducible discrete series module for $\mathbb{H}$. Then there exist inequivalent genuine irreducible $\widetilde{W'}$ representations $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ such that*

$$I(X) = \tilde{\sigma}^+ - \tilde{\sigma}^-.$$

In the notation of Section 2.4,

$$\tilde{\sigma}^+ = \mathsf{Sg}(\tilde{\sigma}^-).$$

Proof. By (3.7), there exist inequivalent irreducible $\widetilde{W'}$ representations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and coefficients $a_i \in \{\pm 1\}$ such that

$$I(X) = a_1\sigma_1 + a_2\sigma_2.$$

As remarked before Proposition 2.1, $\mathsf{Sg}(S^\pm) = S^\mp$. Hence

$$\mathsf{Sg}(I(X)) = -I(X).$$

Thus Sg interchanges σ_1 and σ_2, and $a_1 = -a_2$. $\qquad\square$

Proof of Theorem 1.2. By definition, $I(X) = \mathsf{res}_{W'}(X) \otimes (S^+ - S^-)$. So applying Theorem 3.5, taking characters, and dividing by $\chi_{S^+} - \chi_{S^-}$ gives the conclusion of Theorem 1.2. $\qquad\square$

Example. Let Φ be the root system of type B_n, and let $\mathbb{H}_{n,m}$, $m > 0$, be the graded affine Hecke algebra for Φ with parameters $c(\alpha_l) = 1$, for a long root α_l, and $c(\alpha_s) = m$, for a short root α_s. Fix a partition σ of n. We attach to σ and m a real central character $c_m(\sigma)$ (see the procedure described at the beginning of Section 3 in [CKK], for example). Opdam [O, Lemma 3.31] showed that when

$m \notin \frac{1}{2}\mathbb{Z}$, there exists a unique discrete series $\mathbb{H}_{n,m}$-module $\mathsf{ds}_m(\sigma)$ with central character $c_m(\sigma)$.

Theorem 1.2 gives a simple formula for the $W(B_n)$-character of $\mathsf{ds}_m(\sigma)$ on the set of elliptic elements. Recall that the conjugacy classes of elliptic elements in $W(B_n)$ are in one-to-one correspondence with partitions λ of n. More precisely, to every partition $\lambda = (n_1 \geq \cdots \geq n_\ell > 0)$, one attaches the conjugacy class of w_λ, a Coxeter element in $W(B_{n_1}) \times \cdots \times W(B_{n_\ell}) \subset W(B_n)$. Since $W(B_n) = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, every $W(B_n)$-type is induced from a character χ of $(\mathbb{Z}/2\mathbb{Z})^n$ tensored with an irreducible representation of the stabilizer in S_n of χ. Let $(\sigma \times \emptyset)$ denote the $W(B_n)$-type obtained from the trivial character of $(\mathbb{Z}/2\mathbb{Z})^n$ and the S_n-type parameterized by σ. By [C2, Sections 3.7 and 3.10], the representation $\widetilde{\sigma}^+$ occuring in (1.3) when $X = \mathsf{ds}_m(\sigma)$ equals $(\sigma \times \emptyset) \otimes S^+$ (up to tensoring with sgn). Then, up to a sign depending only on $\mathsf{ds}_m(\sigma)$, Theorem 1.2 reduces to

$$\chi_{\mathsf{ds}_m(\sigma)}(w_\lambda) = \chi_{\sigma \times \emptyset}(w_\lambda) = \chi_\sigma^{S_n}((\lambda));$$

here $\chi_\sigma^{S_n}$ denotes the character of the S_n-representation labelled by the partition σ, and (λ) denotes the S_n-conjugacy class with cycle structure λ. In particular, this shows that, up to a sign, the character of $\mathsf{ds}_m(\sigma)$ on the elliptic set is independent of m. The same result also follows from [CKK, Algorithm 3.30].

It is natural to ask if the same "independence of parameter" holds for the characters (on the elliptic set) of families of discrete series (in the sense of [O, Section 3]) for Hecke algebras with unequal parameters attached to other multiply-laced root systems. Using (1.3) and the explicit characters $\widetilde{\sigma}^\pm$ given by [C2, Tables 1,2,6,7], one can easily verify that this is the case for the graded affine Hecke algebras of types G_2 and F_4 with geometric parameters.

3.3. **Proof of Theorem 1.1.** Let $\mathfrak{g}$ denote the complex semisimple Lie algebra constructed from the the root system Φ. In particular this construction fixes a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ isomorphic to $V^\vee$, and hence $\mathfrak{h}$ (and $\mathfrak{h}^*$) is equipped with the symmetric, nondegenerate, W-invariant bilinear form $\langle \, , \, \rangle$ which we may normalize so that long roots have length 2. Let e be a nilpotent element of $\mathfrak{g}$, and consider an $\mathfrak{sl}_2$-triple $\{e, f, s\}$ with $s \in \mathfrak{h}$. Then

$$h(e) := \langle s, s \rangle \tag{3.8}$$

is well-defined independent of the choice of triple. Let $\widehat{A}_0(e)$ denote the irreducible representations of the component group of the centralizer of e in $\mathrm{Ad}(\mathfrak{g})$ which appear in the Springer correspondence.

According to the classification of [KaL] and [L1], the pair (e, ϕ) parametrizes an irreducible tempered $\mathbb{H}$ module $X(e, \phi)$ with (real) central character represented by $\mathsf{rcs}/2$, where s is the semisimple element mentioned above, such that the character of $\mathsf{res}_W(X(e, \phi))$ is $\chi_{e,\phi}$. The module $X(e, \phi)$ is a discrete series module if and only if e is distinguished in the sense of Bala-Carter. Thus (1.2) now follows from Theorem 1.2. To complete the proof of Theorem 1.1, it remains to identify the numerator in (1.2) explicitly.

Recall from Section 1 the notation $\sigma(e, \phi)$ for the irreducible Springer representation in the top degree homology.

Lemma 3.6. *Suppose $\tilde{\sigma}$ is an irreducible representation of $\widetilde{W}'$ such that*

$$\left(\tilde{\sigma}, X(e, \phi) \otimes (S^+ - S^-)\right)_{\widetilde{W}'} \neq 0.$$

Then

$$\left(\tilde{\sigma}, \sigma(e,\phi) \otimes (S^+ - S^-)\right)_{\widetilde{W}'} \neq 0.$$

Proof. For nilpotent elements e and e' in $\mathfrak{g}$, write $e' > e$ if the closure of the $\mathrm{Ad}(\mathfrak{g})$ orbit through e' contains e. By the parametrization introduced above, this induces an order on the set of irreducible tempered representations with real central character and the set of irreducible representations of W. With these orders in place, consider the matrix whose rows are indexed by irreducible tempered modules with real infinitesimal character, whose columns are indexed by irreducible representations of W, and whose content measures the restriction of a tempered module to W. Corollaire 2 of [BM] shows that this matrix is upper triangular with 1's on the diagonal; see the exposition of [C1, Section 3], for example. Thus in $\mathcal{R}(W)$ we may write

$$\mathsf{res}_W\left(X(e,\phi)\right) = \sigma(e,\phi) + \sum_{e' > e} b_{e,e'}\mathsf{res}_W\left(X(e',\phi')\right),$$

for integers $b_{e,e'}$. Thus

$$I(X(e,\phi)) = \sigma(e,\phi) \otimes (S^+ - S^-) + \sum_{e' > e} b_{e,e'} I(X(e',\phi')). \tag{3.9}$$

By Propositions 2.4 and 2.19 applied to $I(X(e,\phi))$, the only irreducible $\widetilde{W}'$ representations appearing on the right-hand side of (3.9) arise as restrictions of irreducible $\widetilde{W}$ representations $\tilde{\sigma}$ for which $a(\tilde{\sigma}) = h(e)$. By Propositions 2.4 and 2.19 applied to $I(X(e',\phi'))$, the final sum on the right-hand side of (3.9) can only contribute $\widetilde{W}'$ modules arising as the irreducible $\widetilde{W}$ representations $\tilde{\sigma}$ for which $a(\tilde{\sigma}) = h(e')$ for $e' > e$. But $h(e') > h(e)$ whenever $e' > e$ (see for example [GS, Corollary 3.3]). The proof is complete. $\qquad\square$

Lemma 3.7. *Assume that e is a distinguished nilpotent element of $\mathfrak{g}$, and recall the notation of (2.1), (2.5), (2.13), and (3.8). Then there are exactly two inequivalent irreducible representations $\tilde{\sigma}^+ = \mathsf{Sg}(\sigma^-)$ of $\widetilde{W}'$ such that*

$$\left(\tilde{\sigma}^\pm, \sigma(e,\phi) \otimes (S^+ - S^-)\right)_{\widetilde{W}'} \neq 0 \text{ and } h(e) = a(\tilde{\sigma}^\pm). \tag{3.10}$$

Moreover $\tilde{\sigma}^\pm$ is the unique irreducible representation of $\widetilde{W}'$ appearing in $\sigma(e,\phi) \otimes (S^+ - S^-)$ for which the scalar by which $\Omega_{\widetilde{W}}$ acts is as small as possible.

Proof. This follows from Theorem 1.0.1(c) in [C2]. (The statement given there is in terms of $\widetilde{W}$, but it is an easy matter to deduce the corresponding result for $\widetilde{W}'$.) $\qquad\square$

Lemmas 3.6 and 3.7 thus specify the numerator as stated in Theorem 1.1. This completes the proof the theorem. $\qquad\square$

3.4. **The quasidistinguished case.** In this section, we explain how Theorem 1.1 can be extended to other nilpotent elements.

Definition 3.8 ([R, (3.2.2)]). *A nilpotent element $e \in \mathfrak{g}$ is called quasidistinguished if there exists a semisimple element $t \in \mathrm{Ad}(\mathfrak{g})$ centralizing e such that $t\exp(e)$ is not contained in any proper Levi subgroup of $\mathrm{Ad}(\mathfrak{g})$. (Notice that every distinguished nilpotent element e is quasidistinguished by taking $t = 1$.)*

Theorem 3.9 (Reeder [R]). *Suppose $\chi_{e,\phi}$ is nonzero. Then $\chi_{e,\phi}$ is nonzero on the set of elliptic elements in W if and only if e is quasidistinguished in the sense of Definition 3.8.*

We are therefore interested in extending Theorem 1.1 to the case of e quasidistinguished. Fix a maximal torus S in the reductive part of the centralizer of e in $\mathrm{Ad}(\mathfrak{g})$, and write $\mathfrak{s}_0$ for the real span of the corresponding coroots. Then $A(e)$ naturally acts on $\mathfrak{s}_0$ (e.g. [R, Section 3.2]) and we can therefore define a pairing (as in [R, (2.2.1)]) on $\mathcal{R}(A(e))$ defined on irreducible representations σ_1 and σ_2 via

$$f_{A(e)}(\sigma_1, \sigma_2) = \sum_i (-1)^i \dim \mathrm{Hom}_{A(e)} \left(\sigma_1 \otimes \wedge^i \mathfrak{s}_0, \sigma_2\right). \qquad (3.11)$$

We let $\overline{\mathcal{R}}(A(e))$ denote the quotient of $\mathcal{R}(A(e))$ by the kernel of this pairing, and $\overline{\mathcal{R}}_0(A(e))$ the image in $\overline{\mathcal{R}}(A(e))$ of the span of $\widehat{A}_0(e)$.

Here is the generalization of Theorem 1.1.

Theorem 3.10. *Suppose e is quasidistinguished and that ϕ is an irreducible representation of $A(e)$ so that $\chi_{e.\phi}$ is not identically zero. If e is not distinguished, suppose further that (in the notation of (3.11)),*

$$f_{A(e)}(\phi, \phi) = 1. \qquad (3.12)$$

Then the conclusion of Theorem 1.1 holds.

In this setting, the characterization of the numerator in (1.2) follows from Lemmas 3.6 and 3.7 as before. (Lemma 3.7 and its proof are valid with no change for e quasidistinguished.) Thus Theorem 3.10 is equivalent (via the Kazhdan-Lusztig classification) to the following result.

Theorem 3.11. *In the setting of Theorem 1.2, suppose the parameter function c in the definition of $\mathbb{H}$ is identically one. Suppose X is an irreducible tempered $\mathbb{H}$-module parametrized by a pair (e, ϕ) with e quasidistinguished. Suppose further that if e is not distinguished, then ϕ satisfies (3.12). Then the conclusion of Theorem 1.2 holds.*

Proof. We need to establish the conclusion of Theorem 3.5 in our setting; then the argument proceeds just as in the case of discrete series. From the proof of Theorem 3.5, everything thus comes down to establishing

$$(I(X), I(X))_{\widetilde{W}'} = 2, \qquad (3.13)$$

or, equivalently, by Proposition 3.1, that

$$e_W\left(\mathsf{res}_W(X), \mathsf{res}_W(X)\right) = 1. \qquad (3.14)$$

Because of (3.7) we may assume e is not distinguished. Let $\mathcal{R}_e(W)$ denote the subspace of $\mathcal{R}(W)$ spanned by $\{\mathsf{res}_W(X(e,\phi)) : \phi \in \widehat{A}_0(e)\}$, and let $\overline{\mathcal{R}}_e(W)$ denote the quotient of $\mathcal{R}_e(W)$ by kernel of e_W. By [R, (3.4.1),(3.4.3)], the map $\mathcal{R}_0(A(e)) \to \mathcal{R}_e(W)$, $\phi \mapsto \mathsf{res}_W(X(e,\phi))$, induces a linear isomorphism $\overline{\mathcal{R}}_0(A(e)) \to \overline{\mathcal{R}}_e(W)$ which is an isometry with respect to the elliptic pairings on the two spaces. Thus the hypothesis of (3.12) implies that (3.14) holds, and the theorem is proved. $\square$

Remark 3.2. A case-by-case analysis (which we do not reproduce here) shows that there are only a small number of cases where (3.12) fails for e quasidistinguished but not distinguished. There is one case in E_7 (when e is parametrized in the

Bala-Carter notation by $A_4 + A_1$); and there are a series of cases in D_{2n} (when e corresponds to a partition of the form $(a_1, a_1, a_2, a_2, \ldots, a_l, a_l)$ for distinct parts a_i). (See [COT, Section 5].) In each of these cases it transpires (again based on case-by-case considerations) that $f_{A(e)}(\phi, \phi) = 2$, and the arguments given above allow one to deduce that the conclusion Equation 1.2 and Theorem 1.2 are valid, except now $\tilde{\sigma}^+$ is the sum of *two* inequivalent irreducible representations. It would be interesting to give a conceptual explanation of this phenomenon.

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(D. Ciubotaru) Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: `ciubo@math.utah.edu`

(P. Trapa) Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: `ptrapa@math.utah.edu`