Linear Algebra

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Contents

Chapter 1

Vector Spaces

Definition 1.1. The statement that $\{\mathbf{X}, \Phi, +, \cdot\}$ is a vector (linear) space means that $\mathbf{X}$ is a collection of objects, usually called vectors, Φ is a scalar field (the real or complex numbers), called scalars as contrasted to vectors, $+$ and $\cdot$ are binary operations and the following hold between the objects in $\mathbf{X}$ and the scalars in Φ. We assume that for each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{X}$, $\mathbf{x}+\mathbf{y}$ is uniquely defined and is again in $\mathbf{X}$ and that if $\mathbf{x} \in \mathbf{X}$ and $k \in \Phi$, then $k \cdot \mathbf{x}$ is uniquely defined and is a vector in $\mathbf{X}$. Moreover,

(1). $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for each pair $\mathbf{x}$ and $\mathbf{y}$ from $\mathbf{X}$.

(2). $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ in $\mathbf{X}$.

(3). There exists Θ in $\mathbf{X}$ such that $\mathbf{x}+\Theta=\mathbf{x}$ for each $\mathbf{x}$ in $\mathbf{X}$.

(4). For each $x \in \mathbf{X}$, there exists $(-\mathbf{x})$ in $\mathbf{X}$ such that $\mathbf{x}+(-\mathbf{x})=\Theta$.

(5). $a \cdot (b \cdot \mathbf{x}) = (ab) \cdot \mathbf{x}$, $\forall \mathbf{x} \in \mathbf{X}$, $a \in \Phi$, and $b \in \Phi$.

(6). $a \cdot (\mathbf{x}+\mathbf{y}) = a \cdot \mathbf{x} + a \cdot \mathbf{y}$, $\forall \mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{X}$ and $a \in \Phi$.

(7). $(a+b) \cdot \mathbf{x} = a \cdot \mathbf{x} + b \cdot \mathbf{x}$, $\forall \mathbf{x} \in \mathbf{X}, a \in \Phi$ and $b \in \Phi$.

(8). $1 \cdot \mathbf{x} = \mathbf{x}$, $\forall \mathbf{x} \in \mathbf{X}$.

Remark. Absent something like (8), meaningful computations are not possible. Some refer to this definition as 'the axioms for a vector space.' We will sometimes follow that convention.

Exercise 1.1.

(1). Show that if $\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{z}$, then $\mathbf{y}=\mathbf{z}$.

(2). As a corollary to Exercise 1.1.1, show that Θ is unique, i.e., if $\mathbf{x}+\Theta=\mathbf{x}=x+\Theta'$, then $\Theta=\Theta'$.

(3). As another corollary to Exercise 1.1.1, show that for each $\mathbf{x} \in \mathbf{X}$, $(-\mathbf{x})$ is uniquely determined.

Example 1.1.

(1). $\{\mathbf{X}, \Phi, +, \cdot\} = \{\mathbb{R}, \mathbb{R}, +, \cdot\}$, the ordinary real numbers.

(2). $\{\mathbb{C}, \mathbb{R}, +, \cdot\}$, the complex numbers over the real scalar field.

(3). $\{\mathbb{C}, \mathbb{C}, +, \cdot\}$, the complex numbers over the complex scalar field.

Remark. As we shall see later, Examples 1.1.(2) and 1.1.(3) are quite different vector spaces even though the set $\mathbf{X}$ is the same in both cases.

(4). $\mathbf{F}[a,b]$ is the set of all real valued functions defined on the interval $[a,b]$ with $+$ defined by $(f+g)(x) = f(x) + g(x)$, $\Phi \equiv \mathbb{R}$ and scalar muliplication defined by $(k \cdot f)(x) = kf(x)$.

Exercise 1.2.
Show that $\{\mathbf{F}[a,b], \mathbb{R}, +, \cdot\}$ is a vector space.

(5). $C^{(0)}([a,b])$ is the set of all continuous real valued functions on $[a,b]$ with $\Phi = \mathbb{R}$ and $+$ and $\cdot$ defined as in example 1.1.(4).

Remark. $C^{(0)}([a,b])$ is a subset of $\mathbf{F}([a,b])$.

(6). $\mathbf{D}([a,b])(\equiv C^{(1)}([a,b]))$ is the set of all continuously differentiable real valued functions on $[a,b]$ with Φ, $+$ and $\cdot$ as in Example 1.1.(4).

(7). $C^{(n)}[a,b]$ is the set of all n-times continuously differentiable functions on $[a,b]$ with Φ, $+$, and $\cdot$ as in Example 1.1.(4).

Remark. $\mathbf{F}([a,b]) \supset C^{(0)}([a,b]) \supset \mathbf{D}([a,b]) \supset \ldots \supset C^{(n-1)}([a,b]) \supset C^{(n)}([a,b])$.

Exercise 1.3.
Show that $\{C^{(n)}([a,b]), \mathbb{R}, +, \cdot\}$ is a vector space, but don't do it yet; it will be easier later.

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Before proceeding further, there are some manipulative matters to which we must attend, namely, those arithmetic things having to do with $0, \Theta$ and negative signs.

Theorem 1.1. *Suppose* $\mathbf{X}$ *is a vector space over the scalar field* Φ, $\mathbf{x} \in \mathbf{X}$ *and* $a \in \Phi$, *then*

(a). $0 \cdot \mathbf{x} = \Theta$

(b). $a \cdot \Theta = \Theta$

(c). $(-1) \cdot \mathbf{x} = (-\mathbf{x})$

(d). if $a \cdot \mathbf{x} = \Theta$, *then either* $a = 0$ *or* $\mathbf{x} = \Theta$.

Proof of Theorem 1.1.

(a). By axiom (7) in Definition 1.1

$$0 \cdot \mathbf{x} + 0 \cdot \mathbf{x} = (0+0) \cdot \mathbf{x} = 0 \cdot \mathbf{x}$$

so that by axiom (3) we have

$$0 \cdot \mathbf{x} + 0 \cdot \mathbf{x} = 0 \cdot \mathbf{x} = 0 \cdot \mathbf{x} + \Theta$$

and by Exercise 1.1.(1), $0 \cdot \mathbf{x} = \Theta$.

(b).

$$a \cdot \Theta + a \cdot \Theta = a \cdot (\Theta + \Theta) = a \cdot \Theta = a \cdot \Theta + \Theta$$

and by Exercise 1.1, $a \cdot \Theta = \Theta$.

(c).

$$\mathbf{x} + (-1) \cdot \mathbf{x} = 1 \cdot \mathbf{x} + (-1) \cdot \mathbf{x} = [1 + (-1)] \cdot \mathbf{x} = 0 \cdot \mathbf{x} = \Theta = \mathbf{x} + (-\mathbf{x})$$

and by Exercise 1.1, $(-1) \cdot \mathbf{x} = (-\mathbf{x})$.

(d). Suppose $a \neq 0$, *then* $\frac{1}{a}$ *exists and*

$$\frac{1}{a} \cdot (a \cdot \mathbf{x}) = \frac{1}{a} \cdot \Theta = \Theta \qquad \text{by (b) above, then by axiom (5),}$$

$$\left(\frac{1}{a} \cdot a\right) \cdot \mathbf{x} = \Theta$$

$$1 \cdot \mathbf{x} = \Theta \qquad \text{so that by axiom (8),}$$

$$\mathbf{x} = \Theta, \qquad \text{i.e., if } a \neq 0, \text{ then } x = \Theta.$$

∎

Our next project is to make Exercise 1.3 and all similar exercises easy.

Definition 1.2. As subset $\mathbf{V}$ of $\mathbf{X}$, where $\{\mathbf{X}, \Phi, +, \cdot\}$ is a vector space, is called a (vector) subspace of $\mathbf{X}$ if $\{\mathbf{V}, \Phi, +, \cdot\}$ is also a vector space with the same element Θ.

In general, to verify that a subset of a vector space is again a vector space, one would have to verify that $+$ and $\cdot$ are still uniquely defined and that all eight axioms hold. Actually, as the next theorem shows, this is much more than one really needs to do.

Theorem 1.2. *If* $\mathbf{V}$ *is a (non-empty) subset of a vector space* $\{\mathbf{X}, \Phi, +, \cdot\}$, *then* $\{\mathbf{V}, \Phi, +, \cdot\}$ *is a vector subspace if and only if*

(a). $\mathbf{x} + \mathbf{y} \in \mathbf{V}, \forall \mathbf{x}, \mathbf{y} \in \mathbf{V}$ *and*

(b). $a \cdot \mathbf{x} \in \mathbf{V}, \forall a \in \Phi$ *and* $\forall \mathbf{x} \in \mathbf{V}$, *i.e,* $\mathbf{V}$ *is closed under vector addition and scalar multiplication.*

Proof of Theorem 1.2. *Suppose* $\{\mathbf{V}, \Phi, +, \cdot\}$ *is a subspace, then clearly (a) and (b) hold by the definition.*

Now suppose (a) and (b) hold and $\mathbf{V} \subset \mathbf{X}$, *then since axioms (1) and (2) hold in* $\mathbf{X}$, *they also hold in* $\mathbf{V}$ *and the same is true of (7) and (8).*

In order to obtain (3), we must show that $\Theta \in \mathbf{V}$. *By (b),* $0 \cdot \mathbf{x} \in \mathbf{V}, \forall \mathbf{x}$ *but by Theorem 1.1,* $0 \cdot \mathbf{x} = \Theta$, *thus* $\Theta \in \mathbf{V}$ *and (3) holds. Note that (4) holds by (b) also since* $(-1) \cdot \mathbf{x} = -\mathbf{x}$, *again by Theorem 1.1. Axiom (5) follows at once from (b). Axiom (6) holds in* $\mathbf{V}$ *by (a) and (b) and the fact that axiom (6) holds in* $\mathbf{X}$. ∎

Remark. Exercise 1.3 is now an immediate consequence of Exercise 1.2, Theorem 1.2 and elementary facts from a first course in calculus, namely that if f and g are differentiable, then so is $f + g$ and $k \cdot f$ where k is any scalar.

Example 1.2.

(1). Suppose n is a positive integer and denote by $\mathbf{P}_n$ the polynomials of degree less than or equal n, i.e., all functions $f(x) = a_0 + a_1x + \ldots + a_nx^n$.

Note that $\mathbf{P}_n$ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) and is a vector subspace of each $C^{(j)}(\mathbb{R}); j = 0, 1, 2, \ldots$ and we have hierarchies of subspaces:

$$\mathbf{F}([a,b]) \supset C^{(0)}([a,b]) \supset \cdots \supset C^{(n)}([a,b]) \supset C^{(n+1)}([a,b]) \supset \cdots \supseteq C^{(\infty)}([a,b]) \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} C^{(n)}([a,b]) \supset$$
$$\cdots \supset \mathbf{P}_{n+1}([a,b]) \supset \mathbf{P}_n([a,b]) \supset \mathbf{P}_1([a,b]) \supset \mathbf{P}_0([a,b]) \cong \mathbb{R}.$$

(2). The vector space $\mathbb{R}^n$ consists of the n-tupples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of real numbers with $\mathbf{x} + \mathbf{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $a \cdot \mathbf{x} = a \cdot (x_1, x_2, \ldots, x_n) \stackrel{\text{def}}{=} (ax_1, ax_2, \ldots, ax_n)$ and $\Theta = (0, 0, \ldots, 0)$.

Since the real numbers are commutative, associative, distributive and so on, it is easy to see that $\mathbb{R}^n$ is a vector space.

(3). $\mathbb{C}^n$ is like $\mathbb{R}^n$ except we use $\mathbb{C}$ rather than $\mathbb{R}$. Note that one could have either $\{\mathbb{C}^n, \mathbb{C}, +, \cdot\}$ or, $\{\mathbb{C}^n, \mathbb{R}, +, \cdot\}$ and as mentioned earlier, these have different characteristics.

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Exercise 1.4.

(1). Determine which of the following sets are vector spaces with the given operations. For those that are not, list the axioms that fail to hold.

(a) $\{(x,y,z) : x,y,z \text{ real }\}; (x,y,z)+(x',y',z') \stackrel{\text{def}}{=} (x+x', y+y', z+z')$ and $k(x,y,z) \stackrel{\text{def}}{=} (kx, ky, kz)$.

(b) $\{(x,x,\ldots,x) \in \mathbb{R}^n : x \text{ is real }\}$ with the usual $\mathbb{R}^n$ operations.

(c) $\{x > 0\}$ with $x + x' \stackrel{\text{def}}{=} x \cdot x'$ and $k \cdot x \stackrel{\text{def}}{=} x^k$.

(d) All 2×2 matrices of the form $\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right)$ with the usual matrix addition and scalar multiplication.

(e) All 2×2 matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with the usual matrix addition and scalar multiplication.

(f) $\{f : \mathbb{R} \to \mathbb{R} : f(2) = 0\}$ with the usual $\mathbf{F}([a,b])$ operations.

(2). Which of the following are subspaces of $\mathbb{R}^3$:

(a) All vectors of the form $(\xi, 0, 0)$?

(b) All vectors of the form $(\xi, 1, 0)$?

(c) All vectors of the form (ξ_1, ξ_2, ξ_3) with $\xi_2 = \xi_1 + \xi_3$?

(3). Which of the following are subspaces of $\mathbf{P}_3$:

(a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$?

(b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 2$?

(c) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_2 = 0$, and $a_0 + a_1 = a_3$?

(d) All polynomials of the form $a_0 + a_1x$?

One of the primary games which mathematicians play is the study of properties which are preserved under certain types of transformations or mappings or changes of variables. (Other scientists play similar games by studying conservation laws in physics and so on.) The basic reason we are interested in such things is that we can sometimes make an ugly problem pretty by a change of variables, then solve the pretty problem and change the solutions back to the ugly case. The problem is that changing the variables just might destroy the very property which we want to study. Therefore we need to know which transformations preserve the properties we need most and we then study those particular transformations to learn as much as we can about them.

The study of linear algebra is basically the study of such a collection of transformations, namely, the study of those transformations of one vector space into another which are well enough behaved that the algebraic structure is preserved by the transformation; more precisely:

Definition 1.3. Suppose $\{\mathbf{X}, \Phi, +, \cdot\}$ and $\{\mathbf{Y}, \Phi, \tilde{+}, \tilde{\cdot}\}$ are vector spaces over the same scalar field Φ and T is a (function) transformation from $\mathbf{X}$ to $\mathbf{Y}$ such that

(a). $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1)\tilde{+}T(\mathbf{x}_2)$ and

(b). $T(a \cdot \mathbf{x}) = a \,\tilde{\cdot}\, T(\mathbf{x})$

hold for $a \in \Phi, \mathbf{x}, \mathbf{x}_1$, and $\mathbf{x}_2 \in \mathbf{X}$, then T is called a linear transformation.

Note. If T is a linear transformation, then $T(\theta_{\mathbf{X}}) = T(0 \cdot \mathbf{x}) = 0 \tilde{\cdot} T(\mathbf{x}) = \Theta_{\mathbf{Y}}$ always holds; i.e., linear transformations always map Θ into Θ.

Exercise 1.5.

(1). Show that the mapping $T_d : f \to \dfrac{df}{dx}$ is a linear transformation of $C^{(n+1)}([a,b])$ into $C^{(n)}([a,b])$ for $n = 0, 1, 2, \ldots$.

(2). Show that the mapping $T_i : f \to \displaystyle\int_a^x f(t)dt$ is a linear transformation of $C^{(n)}([a,b])$ into $C^{(n+1)}([a,b])$.

Remark. We thus see that much of what is learned in the calculus is properly a part of linear algebra. Moreover, what we learned in the way of manipulative skills we learned because we knew how to compute $\dfrac{df}{dx}$, and $\displaystyle\int_a^x f(t)dt$, knowing f, that is to say, we knew how to manipulate these *specific* transformations. This raises the question whether we can do a similar thing in other cases. The answer is a qualified yes, ..., sometimes.

Representations of Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Case One: Suppose T is a linear map from $\mathbb{R}^1$ to $\mathbb{R}^1$ and set $m = T(1)$. Then if $x \in \mathbb{R}$,

$y = T(\mathbf{x}) = T(1 \cdot \mathbf{x}) = T(\mathbf{x} \cdot 1) = \mathbf{x}T(1) = \mathbf{x} \cdot m = m\mathbf{x}$, and the graph of T is a straight line through the origin which has slope m.

This is why such transformations are called *linear* transformations in Definition 1.3.

Case Two: Suppose T is a linear map from $\mathbb{R}^n \to \mathbb{R}^1$ and that we set $\mathbf{e}_1 = (1,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,0,\ldots,0)$, $\ldots, \mathbf{e}_n = (0,0,\ldots,0,1)$ and then denote $m_1 = T(\mathbf{e}_1), m_2 = T(\mathbf{e}_2), \ldots, m_n = T(\mathbf{e}_n)$. We may now write, for $\mathbf{x} = (x_1,\ldots,x_n)$

$$\begin{aligned}
\mathbf{y} = T(\mathbf{x}) &= T(x_1,x_2,\ldots,x_n) = T((x_1,0,\ldots,0) + (0,x_2,0,\ldots,0) + \ldots + (0,0,\ldots,0,x_n)) \\
&= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) = T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) + \ldots + T(x_n\mathbf{e}_n) \\
&= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \ldots + x_nT(\mathbf{e}_n) = x_1m_1 + x_2m_2 + \ldots + x_nm_n \\
&= m_1x_1 + x_2m_2 + \ldots + x_nm_n \overset{\text{def}}{=} \mathbf{m}\bullet\mathbf{x} = (m_1,m_2,\ldots,m_n)\bullet(x_1,x_2,\ldots,x_n)
\end{aligned}$$

where we *define* the "dot" product of two vectors in $\mathbb{R}^n$ by

$$\mathbf{m}\bullet\mathbf{x} = (m_1,m_2,\ldots,m_n)\bullet(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} m_ix_i.$$

We now again have a "representation" for T given by

$$\mathbf{y} = m\mathbf{x}$$

very much like the Case One representation.

Case Three: Suppose T is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$ and that we set

$$\begin{aligned}
T(\mathbf{e}_1) &= (a_{11},a_{21},\ldots,a_{m1}) \\
T(\mathbf{e}_2) &= (a_{12},a_{22},\ldots,a_{m2}) \\
\vdots \quad & \vdots \qquad\qquad \vdots \\
T(\mathbf{e}_n) &= (a_{1n},a_{2n},\ldots,a_{mn}).
\end{aligned}$$

Then if

$$\begin{aligned}
\mathbf{y} &= (y_1,y_2,\ldots,y_m) = \\
T(\mathbf{x}) &= T(x_1,x_2,\ldots,x_n) = T\left(\sum_{i=1}^{n} x_i\mathbf{e}_i\right) = \sum_{i=1}^{n} x_iT(\mathbf{e}_i) \\
&= \sum_{i=1}^{n} x_i(a_{1i},a_{2i},\ldots,a_{mi}) = \left(\sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{mi}x_i\right) \\
&= (y_1,y_2,\ldots,y_m).
\end{aligned}$$

Let's now write this out in greater detail.

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \quad & \vdots \qquad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n.
\end{aligned}$$

Note that each of these y components is a dot product. We write it in shorter form or notation (fewer "=" and "+" signs)

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{matrix} \| & \| & \cdots & \| \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{matrix}$$

Or, in even shorter form as

$$T(\mathbf{x}) = \mathbf{y} = M\mathbf{x}$$

where the rows of the array M are used to form the dot product with $\mathbf{x}$, the results of which are the components of the $\mathbf{y}$ vector. The columns of the array M are the images of the vectors $\mathbf{e}_i$ after T maps them into $\mathbf{Y}$. We write $M = (a_{ij})$.

Definition 1.4. The array M is called a *matrix*. Note that it has m rows and n columns. We shall refer to its shape by calling it an $m \times n$ matrix.

It is our intention to study such matrices. It is very important that we keep in mind that these matrices are just ways of writing formulas or representations for the linear transformations. It is really the transformations which attract our interest and the matrices are simply pictures or devices for carrying out manipulations or calculations on these functions. Such a collection of devices for calculations is usually called a "calculus". Let's begin now to develop such a calculus.

Theorem 1.3. *Suppose that each of T_1 and T_2 is a linear transformation from $\mathbf{X}$ to $\mathbf{Y}$ and define $(T_1 + T_2)(\mathbf{x}) \stackrel{def}{=} T_1(\mathbf{x}) + T_2(\mathbf{x})$, then define $(\alpha T_1)(\mathbf{x}) \stackrel{def}{=} \alpha \cdot (T_1(\mathbf{x}))$ for each $\alpha \in \Phi$, then*

(a). $T_1 + T_2$ is a linear transformation from $\mathbf{X}$ to $\mathbf{Y}$ and

(b). αT_1 is a linear transformation from $\mathbf{X}$ to $\mathbf{Y}$.

Proof of Theorem 1.3.

(a).

$$\begin{aligned} (T_1 + T_2)(\mathbf{x}_1 + \mathbf{x}_2) &= T_1(\mathbf{x}_1 + \mathbf{x}_2) + T_2(\mathbf{x}_1 + \mathbf{x}_2) \\ &= \{T_1(\mathbf{x}_1) + T_1(\mathbf{x}_2)\} + \{T_2(\mathbf{x}_1) + T_2(\mathbf{x}_2)\} \\ &= \{T_1(\mathbf{x}_1) + T_2(\mathbf{x}_1)\} + \{T_1(\mathbf{x}_2) + T_2(\mathbf{x}_2)\} \\ &= (T_1 + T_2)(\mathbf{x}_1) + (T_1 + T_2)(\mathbf{x}_2). \\ (T_1 + T_2)(k\mathbf{x}) &= T_1(k\mathbf{x}) + T_2(k\mathbf{x}) = kT_1(\mathbf{x}) + kT_2(\mathbf{x}) \\ &= k\{T_1(\mathbf{x}) + T_2(\mathbf{x})\} = k(T_1 + T_2)(\mathbf{x}). \end{aligned}$$

(b). The proof is quite similar and is left as an exercise.

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Exercise 1.6.

(a). Show that the arguments in Case One, Case Two, and Case Three above still work when applied to spaces $\{\mathbb{C}^{(m)}, \mathbb{C}, +, \cdot\}$ and that the same representations are obtained.

(b). What happens when one attempts those same arguments with $\{\mathbb{C}^{(m)}, \mathbb{R}, +, \cdot\}$?

(c). Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ in such a way that $T(\mathbf{e}_1) = (1,0,0)$ and $T(\mathbf{e}_2) = (0,1,0)$. Write its matrix representation.

(d). Suppose $S : \mathbb{R}^3 \to \mathbb{R}^2$ in such a way that $S(\mathbf{e}_1) = (1,0), S(\mathbf{e}_2) = (0,1)$, and $S(\mathbf{e}_3) = (0,0)$. Write a matrix representation for S.

(e). Suppose S maps $\mathbb{C}^3 \to \mathbb{C}^3$ in such a way that $S(1,0,0) = (i,0,0)$; $S(0,1,0) = (1,i,0)$ and $S(0,0,1) = (1,1,i)$. Write a matrix representation for S.

Remark (on Theorem 1.3). In case $\mathbf{X} = \mathbb{R}^n$ (or $\{\mathbb{C}^n, \mathbb{C}, +, \cdot\}$) and $\mathbf{Y} = \mathbb{R}^m$ (or $\{\mathbb{C}^m, \mathbb{C}, +, \cdot\}$), then T_1 and T_2 each have a matrix representation as an $m \times n$ matrix. Since every linear transformation has such a representation, it follows from Theorem 1.3 that $T_1 + T_2$ and αT_1 must also have such matrix representations.

Suppose that T_1 has as its matrix

$$\begin{aligned} A &= (T_1(\mathbf{e}_1), T_1(\mathbf{e}_2), \ldots, T_1(\mathbf{e}_n)) = (a_{ij}), \text{ i.e.,} \\ T_1(\mathbf{x}) &= A\mathbf{x} \\ &\quad \text{and that } T_2 \text{ has as its matrix} \\ B &= (T_2(\mathbf{e}_1), T_2(\mathbf{e}_2), \ldots, T_2(\mathbf{e}_n)) = (b_{ij}). \end{aligned}$$

Question: Can we obtain the matrices $(T_1 + T_2)$ and (αT_1) from these two matrices A and B?

Answer: Let's try.

The matrix for $(T_1 + T_2)$ is given by

$$((T_1 + T_2)(\mathbf{e}_1), (T_1 + T_2)(\mathbf{e}_2), \ldots (T_1 + T_2)(\mathbf{e}_n)$$

which (because $(T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x})$) is the same as

$$(T_1(\mathbf{e}_1) + T_2(\mathbf{e}_2), T_1(\mathbf{e}_2) + T_2(\mathbf{e}_2), \ldots, T_1(\mathbf{e}_n) + T_2(\mathbf{e}_n))$$

and we know all of these from the matrix A and the matrix B.

$$\text{Suppose}\quad T_1(\mathbf{x}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{matrix} \| & \| & \cdots & \| \\ T_1(\mathbf{e}_1) & T_1(\mathbf{e}_2) & \cdots & T_1(\mathbf{e}_n) \end{matrix}$$

$$\text{and}\quad T_2(\mathbf{x}) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{matrix} \| & \| & \cdots & \| \\ T_2(\mathbf{e}_1) & T_2(\mathbf{e}_2) & \cdots & T_2(\mathbf{e}_n) \end{matrix}$$

and thus if we add the corresponding columns, as above, we get that

$$(T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}) = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{matrix} \| & \| & \cdots & \| \\ T_1(\mathbf{e}_1)+T_2(\mathbf{e}_2) & T_1(\mathbf{e}_2)+T_2(\mathbf{e}_2) & \cdots & T_1(\mathbf{e}_n)+T_2(\mathbf{e}_n) \end{matrix}$$

$$= (a_{ij} + b_{ij})\mathbf{x}.$$

Definition 1.5. The *sum* of two matrices A and B is the matrix which represents the linear tranformation which is the sum of those represented by A and B respectively.

We have just shown that this matrix is obtained by adding the corresponding entries in A and B, that is if

$$A = (a_{ij}) \text{ and } B = (b_{ij}), \text{ then } A + B = (a_{ij} + b_{ij})$$

Please note that this last formula is not a definition, rather it is a *fact*. We could have called it a Theorem.

Since the elements of these matrices are scalars, that is numbers, and numbers satisfy the commutative and associative properties of addition, the matrices also satisfy these same properties. We now have addition of matrices uniquely defined (they must be of the same shape, i.e, $m \times n$) and also have that axioms (1) and (2) for vector spaces hold. Very interesting. These hold for the linear transformations in general and for their representing matrices *when they exist.* What about axiom (3)? Does there exist a Θ or zero transformation from $\mathbf{X}$ into $\mathbf{Y}$ which changes nothing when added to another transformation from $\mathbf{X}$ into $\mathbf{Y}$? If so, what must its matrix be?

We require a linear transformation ϑ (script theta) such that

$$T + \vartheta = T$$

for every linear transformation T from $\mathbf{X} \to \mathbf{Y}$. What does this mean? It means that

$$(T + \vartheta)(\mathbf{x}) \equiv T(\mathbf{x}) + \vartheta(\mathbf{x}) = T(\mathbf{x}) = T(\mathbf{x}) + \Theta$$

holds for each $\mathbf{x}$ in $\mathbf{X}$. Exercise 1.1.(1) tells us that

$$\vartheta(\mathbf{x}) = \Theta$$

must hold for each $\mathbf{x}$ and therefore $\vartheta(\mathbf{e}_i) = \Theta$ holds for each $\mathbf{e}_i$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) when we apply the result to that specific case. Thus the matrix which represents ϑ is an array whose columns contain only zeroes. We shall call the resulting matrix Θ, hoping no confusion will arise.

Note that the transformation ϑ maps the entire set $\mathbf{X}$ into the Θ element or origin in $\mathbf{Y}$, and that the resulting matrix formulation is

$$A + \Theta = A$$

so that axiom (3) is satisfied, both for the linear transformations and for their representing matrices *when such exist.*

Part (b) of Theorem 1.3 establishes scalar multiplication uniquely and shows that the set of all linear transformations from $\mathbf{X}$ to $\mathbf{Y}$ is closed under this scalar multiplication. Can it be that with this scalar product in place, the remaining properties of a vector space hold, i.e., axioms (4) through (8)?

If so do these also hold for the representing matrices when such exist?

First let's see what the matrix representation for (αT_1) might look like.

Since $(\alpha T_1)\mathbf{x} = \alpha(T_1(\mathbf{x}))$ for each $\mathbf{x}$, we see that $(\alpha T_1)(\mathbf{e}_i) = \alpha T_1(\mathbf{e}_i)$ and thus the matrix for αT_1 has columns $(\alpha T_1(\mathbf{e}_1), \alpha T_1(\mathbf{e}_2), \ldots, \alpha T_1(\mathbf{e}_n))$ which is exactly the result of multiplying each element of the matrix A which represents T_1 by the scalar α.

Definition 1.6. The scalar product αA is the matrix which represents the linear transformation which is α times the linear transformation which is represented by A.

We now have that if

$$A = (a_{ij}), \text{ then } \alpha A = (\alpha a_{ij}).$$

This seems to suggest that axioms (4) through (8) may work out. The only question is what to do about negatives of matrices. If we do it right, i.e., all axioms hold, then Theorem 1.1 will also follow and (c) of Theorem 1.1 states that $(-\mathbf{x}) = (-1) \cdot \mathbf{x}$, if we apply this to matrices, it tells us that we have no choice but have that

$$(-A) = (-1) \cdot A$$

and the right side of this expression being a scalar product, is already uniquely defined. In the case of the general linear transformation we must use

$$(-T) = (-1) \cdot T$$

as well and thus

$$(-T)(\mathbf{x}) = (-1) \cdot T(\mathbf{x}).$$

Theorem 1.4. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are vector spaces over the same scalar field* Φ*, then the set* $L[\mathbf{X}, \mathbf{Y}]$ *of all linear transformations from* $\mathbf{X}$ *into* $\mathbf{Y}$ *is also a vector space over* Φ.

Exercise 1.7.
Prove Theorem 1.4

Remark. General representation results for linear transformations in $L[\mathbf{X}, \mathbf{Y}]$ are not known. Site specific cases are understood and such knowledge has made major contributions to the success of our modeling of many aspects of the world around us.

Theorem 1.5. *The set* M_{mn} *of all matrices representing linear transformations in* $L[\mathbb{R}^n, \mathbb{R}^m]$ *(or* $L[\mathbb{C}^n, \mathbb{C}^m]$*) is a vector space over* $\mathbb{R}$ *(or* $\mathbb{C}$*).*

Exercise 1.8.

(1). Prove Theorem 1.5

(2). Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ in such a way that $L(\mathbf{e}_1) = (1, 2, 3)$, $L(\mathbf{e}_2) = (2, 3, 1)$, and $L(\mathbf{e}_3) = (3, 1, 2)$. Write a matrix which represents L and use it to compute $L(1, 1, 1)$.

(3). Show that the L in part (2) maps elements of the form (x, x, x) back into a scalar multiple of itself, i.e., $L(x, x, x) = \alpha(x, x, x)$. Compute that scalar multiple.

(4). Show that vectors of the form (x, x, x) form a vector subspace of $\mathbb{R}^3$.

Remark. Exercise 1.8.(3) shows that L, restricted to this subspace, amounts to a simple scalar multiplication. **Question**: Does this always happen in the case of matrices? We will revisit this question throughout the course.

(5). Suppose $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}$.

(a). Compute $4A - B$ and $6A + 2B$.

(b). Where does $4A - B$ map $\mathbf{e}_1$?

(c). Where does $6A + 2B$ map $\mathbf{e}_2$?

(d). Where does B map $\mathbf{e}_3$?

(e). Where does Θ map $\mathbf{e}_1$? Why? How do you know that?

(f). Where does A map $\mathbf{e}_1$? Where does it map $\mathbf{e}_1 + \mathbf{e}_2$?

The Composition of Linear Transformations

Suppose we have two linear transformations S and T where $S : \mathbf{X} \to \mathbf{Y}$ and $T : \mathbf{Y} \to \mathbf{Z}$ where $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ are vector spaces over the same scalar field Φ.

As with other functions, we define $(T \circ S)(\mathbf{x}) \stackrel{\text{def}}{=} T(S(\mathbf{x}))$, the composition of the two functions.

Proposition 1.1. *The map* TS *is a linear map from* $\mathbf{X}$ *into* $\mathbf{Z}$.

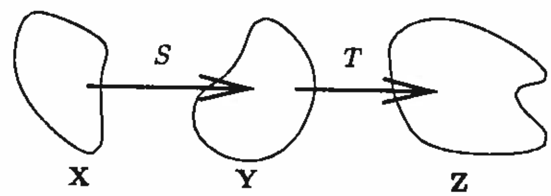

Figure 1.1:

Exercise 1.9.
Prove Proposition 1.1.

Now suppose that $\mathbf{X} = \mathbb{R}^n$, $\mathbf{Y} = \mathbb{R}^m$ and $\mathbf{Z} = \mathbb{R}^q$, then S, T and $T \circ S$ have representations as matrices which are $m \times n$, $q \times m$ and $q \times n$ respectively. Denote the matrix for S by A and that for T by B. The matrix for $T \circ S$ has as its columns,

$$(T \circ S)(\mathbf{e}_j) = T(S(\mathbf{e}_j)) = B[A(\mathbf{e}_j)]$$

and $A(\mathbf{e}_j)$ is the j^{th} column of A, i.e., the j^{th} column of the matrix for $T \circ S$ is given by

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qm} \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = \begin{bmatrix} \sum_{i=1}^{n} b_{ij} a_{ij} \\ \sum_{i=1}^{n} b_{2i} a_{ij} \\ \vdots \\ \sum_{i=1}^{n} b_{qi} a_{ij} \end{bmatrix},$$

thus the (i,j)-entry of the matrix for $T \circ S$ is $\sum_{k=1}^{n} b_{ik} a_{kj}$. The resulting matrix

$$\left(\sum_{k=1}^{n} b_{ik} a_{kj} \right)$$

is usually denoted BA and its (i,j)-entry is the dot product of the i^{th}-row of B and the j^{th}-column of A. This matrix, which represents the composition of T and S, is usually called ***the product*** BA. Referring back to Figure 1.1, one can see that the ***reverse*** composition may not have meaning and indeed will not unless $\mathbf{X} = \mathbf{Z}$. In that case, the diagram becomes as below in Figure 1.2 and the reverse composition is possible, but $T \circ S$ maps $\mathbf{X}$ into $\mathbf{X}$while $S \circ T$ maps $\mathbf{Y}$ into $\mathbf{Y}$. In the case $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^q \equiv \mathbb{R}^n$ the resulting matrices have shapes $\begin{cases} BA \text{ is } n \times n \\ AB \text{ is } m \times m \end{cases}$. Thus both are square but of different sizes unless $\mathbf{X}$ is $\mathbf{Y}$.

If $\mathbf{X} \equiv \mathbf{Y} \equiv \mathbb{R}^n$, then the question as to whether $BA = AB$ is a meaningful question, but the answer in general is not affirmative. Since it is seldom the case that two functions commute under composition, i.e., $T \circ S \neq S \circ T$, the product of their corresponding matrices suffers the same outcome, indeed, the product commutes only if the functions commute, which is hardly ever.

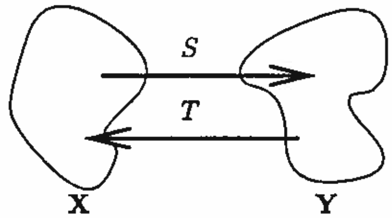

Figure 1.2:

Example 1.3.
If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, then $AB = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Ω

It is an interesting sport to find (or try to find) a class of matrices, say 2×2, all of which commute.

Now that we have a notion of product, at least in special cases, the question of division or inverses arises. We must of course have an identity (or more than one) in order to consider the notion of inverse.

In Figure 1.2 above, if $ST = I_{\mathbf{Y}}$, the identity on the space $\mathbf{Y}$ we say that S is a left inverse of T and that T is a right inverse of S. If $TS = I_{\mathbf{X}}$, the order is, of course, reversed.

Definition 1.7. If $S \in L[\mathbf{X}, \mathbf{Y}]$ and $T \in L[\mathbf{Y}, \mathbf{X}]$, we say that T is an inverse of S if and only if T is both a left and a right inverse of S. In that case we write $T = S^{-1}$.

Recall that by identity, we mean a transformation I which doesn't change things, i.e., $I(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$ in the space. If $\mathbf{X} \equiv \mathbb{R}^n$, then $I(\mathbf{e}_j) = \mathbf{e}_j$ must hold and the matrix for I has $\mathbf{e}_j$ for its j^{th} column, e.g., the identity matrix on $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The next theorem contains some subtleties not appearant on first reading nor from the proof.

Theorem 1.6. *If $S \in L[\mathbf{X}, \mathbf{Y}]$, $T \in L[\mathbf{Y}, \mathbf{X}]$, and $U \in L[\mathbf{Y}, \mathbf{X}]$ and $US = I_{\mathbf{X}}$ and $ST = I_{\mathbf{Y}}$, then $U \equiv T$.*

Proof of Theorem 1.6. $U = U \cdot I_{\mathbf{Y}} = U(ST) = (US)(T) = I_{\mathbf{X}}T = T$. ∎

Remark. The subtlety occurs at the last step, namely $(US)T = I_{\mathbf{X}}T = T$. In order for this step to hold, it is only necessary for US to behave like $I_{\mathbf{X}}$ on the range of T, not on all of $\mathbf{X}$. This actually does occur.

Example 1.4.
In Exercises 1.6.(3) and 1.6.(4), (take $\mathbf{X} \equiv \mathbb{R}^3$ and $\mathbf{Y} \equiv \mathbb{R}^2$) $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$TS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_{\mathbf{X}}\big|_{\mathbf{R}^2}$, i.e., this is *not* $I_{\mathbf{R}^3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $ST = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathbf{R}^2}$. Ω

One can check that as *functions*, S is not one-to-one and thus not invertible, but T *is* one-to-one and does have a *function* as its inverse but this function does not have all of $\mathbf{X}$ in its domain, only the range of T.

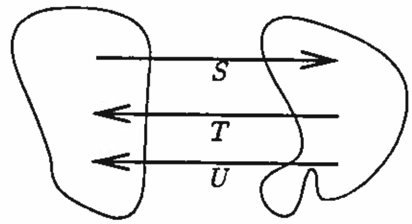

Figure 1.3:

Corollary 1.7. *If, in Theorem 1.6, $\mathbf{X} \equiv \mathbf{Y}$ and S has both a left and right inverse, then S is invertible.*

Proof of Corollary 1.7. *The left and right inverse are the same, hence are an inverse.* ∎

We list some properties which hold (when they make sense) for all linear maps and also for their matrix representations when there are such matrices.

1.	$A+B=B+A$	2.	$A+(B+C)=(A+B)+C$
3.	$A(BC)=(AB)C$	4.	$A(B+C)=AB+AC$
5.	$(B+C)A=BA+CA$	6.	$(-A)=(-1)A$
7.	$\alpha(B+C)=\alpha B+\alpha C$	8.	$(\alpha+\beta)C=\alpha C+\beta C$
9.	$(\alpha\beta)\cdot C=\alpha(\beta C)$	10.	$\alpha(BC)=(\alpha B)C$
11.	$A+\Theta=\Theta+A=A$	12.	$A+(-A)=\Theta$
13.	$A\Theta=\Theta A=\Theta$		

Now that a product is possible in some cases, the result in Theorem 1.1(d) must be asked again. If we know that the product is "zero," can we conclude that a factor must be "zero"? The answer is sometimes, but not always.

Theorem 1.8. *If A, B and C are linear transformations, A^{-1} exists and $AB = AC$, then $B = C$.*

Proof of Theorem 1.8. *Multilply $AB = AC$ by A^{-1}.* ∎

Remark. We can not say that if $A \neq \Theta$, then $B = C$; indeed it may be a false statement.

Example 1.5.

If $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 1 & 1 \\ 3 & 4 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} 2 & 5 \\ 3 & 4 \end{smallmatrix}\right)$, then $AB = AC = \left(\begin{smallmatrix} 3 & 4 \\ 6 & 8 \end{smallmatrix}\right)$, but $B \neq C$. Now set $D = \left(\begin{smallmatrix} 3 & 7 \\ 0 & 0 \end{smallmatrix}\right)$ and note that $AD = \Theta$, but neither $A = \Theta$ nor $D = \Theta$. Ω

Definition 1.8. If $AD = \Theta$, then A is called a *left zero divisor* of D and D is called a *right zero divisor* of A.

The burden of these remarks is that Θ is not the only non-invertible linear map and the only time that we know we may "cancel" a factor is when we have an invertible map as the factor to be canceled.

Theorem 1.9.

(1). If both A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

(2). A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(3). For any positive integer n, A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.

(4). For any non-zero scalar k, $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Proof of Theorem 1.9. *Just try them, they work.* ∎

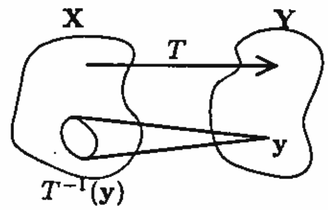

Figure 1.4:

Now suppose $T : \mathbf{X} \to \mathbf{Y}$ is a linear transformation and we are given $\mathbf{y}$ in the range of T; we wish to find all those values of $\mathbf{x}$ such that

$$\mathbf{y} = T(\mathbf{x})$$

i.e., the entire pre-image of y under T. We will denote this preimage by $T^{-1}(\mathbf{y})$ even though we do not assume T is invertible. We hope this does not lead to undue confusion. It is common practice.

Definition 1.9. Suppose $T \in L[\mathbf{X}, \mathbf{Y}]$, then the set $\{\mathbf{x} | T(\mathbf{x}) = \Theta\}$ is called the kernel or nullspace of T and is denoted by either $\ker(T)$ or $\mathcal{N}(T)$.

Remark. Note that the kernel of the operator $T_d(f) = \frac{df}{dx}$ in Exercise 1.5.(1) consists of exactly the constant functions in $F([a, b])$.

Proposition 1.2. *If $T \in L[\mathbf{X}, \mathbf{Y}]$, then $\ker(T)$ is a vector subspace of $\mathbf{X}$.*

Proof of Proposition 1.2. *Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $\ker(T)$ and $\alpha \in \Phi$, then*

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) = \Theta + \Theta = \Theta \text{ and } T(\alpha\mathbf{x}_1) = \alpha T(\mathbf{x}_1) = \alpha\Theta = \Theta.$$

Thus by Theorem 1.2, $\ker(T)$ is a subspace. ∎

Let's return briefly to the matter of zero divisors.

Example 1.6.

Suppose $D = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, then $AD = \Theta = DA$ and neither A nor D is Θ. Moreover, $D\mathbf{x} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 - x_2 \\ 0 \end{smallmatrix}\right) \in \ker(A)$ and $A\mathbf{x} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_2 \\ x_2 \end{smallmatrix}\right) \in \ker(D)$, i.e., D "projects" $\mathbb{R}^2$ onto $\ker(A)$ and A "projects" $\mathbb{R}^2$ onto $\ker(D)$. A little thought shows that this characterizes the zero divisor situation. Ω

Proposition 1.3. *$AD = \Theta$ if and only if $A : \text{Range}(D) \to \Theta$ or equivalently if and only if $\text{Range}(D) \subset \ker(A)$.*

Proposition 1.4. *If $T \in L[\mathbf{X}, \mathbf{Y}]$ and $\mathbf{x}_p$ is such that $T(\mathbf{x}_p) = \mathbf{y}$, then for each $\mathbf{x}_0 \in \ker(T)$, $(\mathbf{x}_p + \mathbf{x}_0) \in T^{-1}(\mathbf{y})$.*

Proof of Proposition 1.4. $T(\mathbf{x}_p + \mathbf{x}_0) = \mathbf{y} + \Theta = \mathbf{y}$. ∎

A converse also holds, namely:

Proposition 1.5. *If $T(\mathbf{x}_p) = \mathbf{y}$ and $T(\mathbf{x}_q) = \mathbf{y}$, then $(\mathbf{x}_p - \mathbf{x}_q) \in \ker(T)$.*

Remark. Propositions 1.4 and 1.5 show that two functions in $F([a, b])$ have the same derivatives if and only if they differ by a constant function, i.e., their difference lies in the kernel of $\frac{d}{dx}$.

Definition 1.10. If $\mathbf{A}$ and $\mathbf{B}$ are subsets of a vector space $\mathbf{X}$, then

$$\mathbf{A} \oplus \mathbf{B} \overset{\text{def}}{=} \{\mathbf{z} \in \mathbf{X} \quad | \quad \mathbf{z} = \mathbf{a} + \mathbf{b}, \text{ where } \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{b} \in \mathbf{B}\}$$

With Definition 1.10 we may now sum up the results of Propositions 1.4 and 1.5.

Theorem 1.10. *If $T \in L[\mathbf{X}, \mathbf{Y}]$ and $\mathbf{y} \in \text{Range}(T)$, then $T^{-1}(\mathbf{y})$ is given by $\mathbf{x}_p \oplus \ker(T)$ where $\mathbf{x}_p$ is any (particular) solution to $\mathbf{y} = T(\mathbf{x})$.*

Theorem 1.11. *If $T \in L[\mathbf{X}, \mathbf{Y}]$ and T is one-to-one, so that T^{-1} exists as a function, not necessarily in $L[\mathbf{Y}, \mathbf{X}]$, then T^{-1} is <u>linear</u> from $\text{Range}(T)$ into $\mathbf{X}$.*

Proof of Theorem 1.11.

(1).

$$\begin{aligned} T^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= T^{-1}(T(\mathbf{x}_1) + T(\mathbf{x}_2)) = T^{-1}(T(\mathbf{x}_1 + \mathbf{x}_2)) = \mathbf{x}_1 + \mathbf{x}_2 \\ &= T^{-1}T(\mathbf{x}_1) + T^{-1}T(\mathbf{x}_2) = T^{-1}\mathbf{y}_1 + T^{-1}\mathbf{y}_2 \end{aligned}$$

(2).

$$\begin{aligned} T^{-1}(\alpha\mathbf{y}) &= T^{-1}\alpha T(\mathbf{x})) = T^{-1}(T(\alpha\mathbf{x}) = \alpha\mathbf{x} = \alpha T^{-1}(T(\mathbf{x})) \\ &= \alpha T^{-1}(\mathbf{y}). \end{aligned}$$

∎

Our next concern is to consider whether one can in fact solve

$\mathbf{y} = T(\mathbf{x})$, or $\mathbf{y} = A\mathbf{x}$ (if A exists)

in other words answer the question "Does a solution exist?"

Proposition 1.6. *If A is invertible (or T is invertible), then $\mathbf{y} = T\mathbf{x}$ (or $\mathbf{y} = A\mathbf{x}$) has exactly one solution for each $\mathbf{y}$ in $\mathbf{Y}$, namely $\mathbf{x} = T^{-1}\mathbf{y}$ (or $\mathbf{x} = A^{-1}\mathbf{y}$.)*

Remark. In view of Theorem 1.10, Proposition 1.6 suggests that $\ker(T) \equiv \ker(A) \equiv \{\Theta_{\mathbf{X}}\}$. This is in fact true.

Proposition 1.7. *If A^{-1} (or T^{-1}) exists, then $\ker(A) = \ker(T)) = \{\Theta\}$.*

Proof of Proposition 1.7. $A\mathbf{x} = \Theta \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\Theta = \Theta$ *so* $\mathbf{x} = \Theta$. ∎

Interestingly the converse is also true.

Proposition 1.8. *If* $\ker(T) = \{\Theta\}$*, then* T^{-1} *exists, as a function (but is not necessarily in* $L[\mathbf{X},\mathbf{Y}]$ *since* Range(T) *may not be all of* $\mathbf{Y}$*), i.e.,* T *is one-to-one.*

Proof of Proposition 1.8. *Suppose* $\ker(T) = \Theta$ *and that there exists two values* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *such that* $T(\mathbf{x}_1) = T(\mathbf{x}_2)$*, then* $T(\mathbf{x}_1 - \mathbf{x}_2) = \Theta$*, i.e.,* $(\mathbf{x}_1 - \mathbf{x}_2) \in \ker(T) \equiv \{\Theta\}$ *so that* $\mathbf{x}_1 = \mathbf{x}_2$*.* ∎

Remark. T^{-1} exists (as a function) if and only if $\ker(T) = \{\Theta\}$ in which case $\mathbf{y} = T\mathbf{x}$ has exactly one solution $\mathbf{x} = T^{-1}(\mathbf{y})$ *provided* $\mathbf{y}$ is in the range of T. In case $\mathbf{y}$ is not in Range(T), there is no solution.

Example 1.7.

Suppose $T \in L[\mathbb{R}^2, \mathbb{R}^3]$ such that $T(x_1, x_2) = (x_1, x_2, 0)$. Then the matrix which represents T is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now take $\mathbf{y} = (0, 0, 1)$ and $\mathbf{y}$ is not in the range of A (nor the range of T), T is one-to-one and $\ker(T) = \{(0,0)\} = \{\Theta\}$. Ω

Remark. In case T is not invertible and $\mathbf{y}$ is in the range of T, then $\mathbf{y} = T\mathbf{x}$ has infinitely many solutions, $\mathbf{x}_p \oplus \ker(T)$

Exercise 1.10.

(1). Find the matrix representations of the following linear transformations:

- (a) $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x_1, x_2) = (x_1, 0)$
- (b) $T : \mathbb{R}^2 \to \mathbb{R}^3, T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$
- (c) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x_1, x_2, x_3) = (4x_1, 2x_1 - 3x_2, x_1 + 4x_3)$
- (d) $T : \mathbb{R}^4 \to \mathbb{R}^5, T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$

(2). Find the matrix representation for $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps (x_1, x_2) into

- (a) its reflection through the origin.
- (b) its reflection about the line $y = -x$.
- (c) its projection onto the "y-axis", i.e., onto $(0, x_2)$.

(3). Suppose $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $S(x_1, x_2) = (x_1, 0)$ and $T(x_1, x_2) = (x_2, x_1)$. Do S and T commute?

(4). Construct another example like Example 1.5, i.e., construct R, S and T such that $RS = RT$, but $S \neq T$.

Proposition 1.9. *If* $T \in L[\mathbf{X}, \mathbf{Y}]$*, then* Range$(T)$ *is a vector subspace of* $\mathbf{Y}$.

Exercise 1.11.

Prove Proposition 1.9.

Let's now summarize the results of Propositions 1.6 through 1.8 and the remarks related thereto.

Theorem 1.12. *Given* $T \in L[\mathbf{X}, \mathbf{Y}]$ *and* $\mathbf{y} \in \mathbf{Y}$, *then* $\mathbf{y} = T\mathbf{x}$ *has a solution if and only if* $\mathbf{y}$ *is in the range of* T. *If* $\mathbf{y}$ *is in the range of* T, *then* $\mathbf{y} = T\mathbf{x}$ *has*

(a). exactly one solution if and only if $\ker(T) = \{\Theta\}$ *and*

(b). infinitely many solutions in case $\ker(T) \neq \{\Theta\}$.

Remark. If $\ker(T) = \{\Theta\}$, then T has a left inverse defined on the vector subspace Range(T) of $\mathbf{Y}$ (see Proposition 1.9 above.) This observation gives us the following Corollary to Theorem 1.11:

Corollary 1.13. *If* $T \in L[\mathbf{X}, \mathbf{Y}]$, *then* T *has an inverse* $T^{-1} \in L[\mathbf{Y}, \mathbf{X}]$ *if and only if* $\ker(T) = \{\Theta\}$ *and* $\text{Range}(T) \equiv \mathbf{Y}$.

Proof of Corollary 1.13. *Define* $T^{-1}(\mathbf{y})$ *to be the* unique $\mathbf{x}$ *such that* $\mathbf{y} = T\mathbf{x}$. ∎

While this is all quite exciting, the only case so far in which we can actually write a formal solution to $\mathbf{y} = T\mathbf{x}$ is the case of Proposition 1.6 and even in that case we have no way to obtain T^{-1} or A^{-1}. We will now specialize to $\mathbf{X} = \mathbf{Y} = \mathbb{R}^n$ and look for a method.

Remark. If $\ker(T) \not\equiv \{\Theta\}$, then infinitely many solutions exist and in general we require additional restrictions in order to deteremine a unique solution if that is desired. For example, if we require $T_d(f) = \dfrac{df}{dx} = x^2$, there are infinitely many solutions, namely $\frac{x^3}{3} + k$ where k is a constant function in the kernel of T_d. If we *also* wish our solution to have the value 6 when $x = 0$, then only $\frac{x^3}{3} + 6$ will suffice.

Chapter 2

Determinants

Throughout this chapter we consider only linear transformations which are in either $L[\mathbb{R}^n, \mathbb{R}^n]$ or $L[\mathbb{C}^n, \mathbb{C}^n]$. Their matrix representations are thus $n \times n$ matrices and our attention will be directed to these representing matrices since we may use the calculus we have developed to aid us in our computations.

Definition 2.1. An $n \times n$ matrix E is called an elementary matrix if and only if it can be obtained from I_n (the identity matrix on $\mathbb{R}^n$) by an elementary row operation, i.e., by one of the following:

(a). multiplying a row by a non-zero scalar

(b). interchanging two rows

(c). adding a non-zero multiple of one row to another row.

Example 2.1.

$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$ is Type 1, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is Type 2, $\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is Type 3. Ω

Note. Elementary row operations are also elementary column operations.

Theorem 2.1. *Suppose E is an elementary $m \times m$ matrix, A is an $m \times n$ matrix and B is an $n \times m$ matrix,*

(1). EA is the matrix obtained by performing on A the same elementary row operation that produced E from I_m.

(2). BE is the matrix obtained by performing on B the same elementary column operation that produces E from I_m.

Proof of Theorem 2.1. *Do several examples; you'll understand.* ∎

One of the main results of Theorem 2.1 is that elementary matrices are invertible. This will be quite useful in our search for an inverse of an $n \times n$ matrix A, that is, if it has an inverse. The following chart indicates how this comes about.

Row operation which produced E	Row operation which produced E^{-1}	Type
multiply row i by $c \neq 0$	multiply row by $\frac{1}{c}$	1
interchange rows i and j	interchange rows i and j	2
add c times row j to row i	add $-c$ times row j to row i	3

Theorem 2.2. *If A is an $n \times n$ matrix, the following statements are equivalent:*

(1). A is invertible

(2). $A\mathbf{x} = \Theta$ has only the trivial (i.e, $\mathbf{x} = \Theta$) solution and

(3). A is row equivalent to to I_n, i.e, there exists a sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that $(E_k E_{k-1} \cdots E_1)A = I_n$, or equivalently stated, $A^{-1} = E_k E_{k-1} \cdots E_1$.

Proof of Theorem 2.2.

(a)⇒(b) follows from Proposition 1.6.

(b)⇒(c) follows from Gauss-Jordan elimination and

(c)⇒(a) is essentially the definition of row equivalence.

∎

Remark. For an $n \times n$ matrix A, $\ker(A) = \{\Theta\}$ (statement (b) of Theorem 2.2) implies A^{-1} exists and since A^{-1} is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, this implies that Range(A) is all of $\mathbb{R}^n$. Recall Corollary 1.13.

In applying Theorem 2.2 to say an equation $AX = B$. where X and B are of appropriate sizes and we wish solutions X for several different values of B, we can use Gauss-Jordan to generate A^{-1} as follows: Write $A|I_n$ in juxtaposition and then perform elementary row operations on A and I_n simultaneously until A is reduced to I_n, then I_n is reduced to A^{-1} as observed in part (c) of Theorem 2.2.

Exercise 2.1.

(1). Use the method above to compute A^{-1} for the following matrices:

(a) $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 7 & 6 \end{pmatrix}$

(c) $A_3 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

(2). Use the results in Exercise 2.1.(1) above to solve the following equations by Proposition 1.6.

(a) $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) $\mathbf{y} = A_1\mathbf{x}$ where $\mathbf{y} = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

(c) $\mathbf{y} = A_1\mathbf{x}$ where $\mathbf{y} = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

(d) $\mathbf{y} = A_2\mathbf{x}$ where $\mathbf{y} = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

(e) $\mathbf{y} = A_1\mathbf{x}$ where $\mathbf{y} = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

(f) Compare the solutions in (a), (e) and (b). Explain.

(g) Solve $A_3\mathbf{x} = \Theta$, i.e., compute $\ker(A_3)$. Explain.

Definition 2.2. Suppose S is a finite set. An ordering of the elements of S is called a permutation of S.

Example 2.2.
If $S = \{1,2,3\}$, then the permutations of S are $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$. In general, if S has n elements, then there are $n!$ permuatations of S (there are n ways to select the first element, $(n-1)$ ways to select the second and so on and only 1 way to select the last.) Ω

Definition 2.3.

(a). Denote by Π_n the set of permutations of the integers $\{1,2,\dots,n\}$, then an element $\kappa = (k_1, k_2, \dots, k_n)$ of Π_n is said to have an *inversion* if there exists $k_i > k_j$ for some $i < j$, that is, a larger integer precedes a smaller one.

(b). A permutation is called even if its number of inversions is even and it is called odd otherwise. The oddness or evenness of a permutation is called its parity.

Example 2.3.
In Π_3 of Example 2.2, the permutations $(1,2,3), (2,3,1)$ and $(3,1,2)$ are even. The others are, of course, odd. Ω

Definition 2.4. For each permutation κ in Π_n,

$$\delta(\kappa) = \begin{cases} 0 & \text{if } K \text{ is even} \\ 1 & \text{if } K \text{ is odd} \end{cases}$$

Definition 2.5. Suppose $A = (a_{ij})$ is an $n \times n$ matrix. The determinant of A, (denoted $|A|$ or $\det(A)$) is given by

$$|A| \stackrel{\text{def}}{=} \sum_{\kappa \in \Pi_n} (-1)^{\delta(\kappa)} a_{1k_1} \cdot a_{2k_2} \cdot \ldots \cdot a_{nk_n}$$

where $\kappa = (k_1, k_2, \dots, k_n)$ and the sum is taken over the $n!$ permutations in Π_n.

Remark. If A is 1×1, i.e, $A = (a_{11})$, then $|A| = a_{11}$.

If A is 2×2, i.e., $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$\begin{aligned} |A| &= (-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

If A is 3×3, i.e., $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

Notice that if A is 4×4, then $\det(A)$ has $4! = 24$ terms to compute. This can (does) quickly get out of hand; we need a better way to cope than just following the definition.

Definition 2.6. Suppose $A = (a_{ij})$ is an $n \times n$ matrix, then

(a). for each element a_{ij} in A, the minor of a_{ij} (i^{th} row, j^{th} column) is the matrix of order $(n-1) \times (n-1)$ which remains after striking out (deleting) the i^{th} row and j^{th} column of A. We denote this by M_{ij}.

(b). The cofactor of a_{ij} is $A_{ij} \overset{\text{def}}{=} (-1)^{i+j}\,|M_{ij}|$.

Remark. Notice that in the 3×3 case in the last remark above

$$\begin{aligned} |A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\ &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}. \end{aligned}$$

This is usually called the cofactor expansion of A using the first row. Actually, we could have rearranged the terms to get a cofactor expansion by any other row or column; for example we can rearrange to get

$$\begin{aligned} |A| &= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31}) \\ &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \qquad (2^{nd} \text{ row}) \\ \text{or} \quad &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \qquad (3^{rd} \text{ row}) \\ \text{or} \quad &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \qquad (1^{st} \text{ column}) \qquad \text{and so on.} \end{aligned}$$

It is a tedious (but not difficult) argument to show that in general,

Theorem 2.3. *If A is any $n \times n$ matrix, then $|A|$ can be expressed as a cofactor expansion using any row or any column of A.*

It is a tedious (but not difficult) argument to show that in general,

$$\begin{aligned} |A| &= a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \qquad (i^{th} \text{ row}) \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} \qquad (j^{th} \text{ column}) \end{aligned}$$

Exercise 2.2.

(1). Show that if $\kappa = (k_1, k_2, k_3, \ldots, k_n) \in \Pi_n$ and $c = (k_2, k_1, k_3, \ldots, k_n)$, then κ and c have different parity.

(2). Show that if $\kappa = (k_1, k_2, k_3, \ldots, k_n)$ and $c = (k_3, k_2, k_1, \ldots, k_n)$, then κ and c have different parity.

(3). Show that if $\kappa = (k_1, k_2, k_3, \ldots, k_n)$ and any two elements are switched, then the parity is changed.

(4). Write down several examples in Π_5, compute the parity of each, switch two elements and compute the *new* parity.

Definition 2.7. If A is an $n \times n$ matrix, $A = (a_{ij})$, then the transpose A^T of A is the matrix (a_{ji}).

Example 2.4.

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \ldots & a_{nm} \end{pmatrix}. \qquad \Omega$$

Theorem 2.4. *If A is an $n \times n$ matrix, then $|A^T| = |A|$.*

Proof of Theorem 2.4. *By induction on n: It is certainly true if A is 1×1 since $A^T = A = a_{11}$.*

Now suppose the result holds for all $k \times k$ matrices and that A is $(k+1) \times (k+1)$. Expand $|A|$ by the first row of A, i.e.,

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| - \ldots \pm a_{1,k+1}|M_{1,k+1}|$$

but since each M_{ij} is $k \times k$, this is the same as

$$= a_{11}\left|M_{11}^T\right| - a_{12}\left|M_{12}^T\right| + a_{13}\left|M_{13}^T\right| - \ldots \pm a_{1,k+1}\left|M_{1,k+1}^T\right|$$

which is $|A^T|$ expanded by its first column. ∎

Definition 2.8. An $n \times n$ matrix A is said to be triangular if either

(a). $a_{ij} = 0$ for $i < j$ (lower triangular) or

(b). $a_{ij} = 0$ for $i > j$ (upper triangular)

Theorem 2.5. *If A is a triangular matrix, then* $\det(A)$ *is the product of the diagonal elements.*

Proof of Theorem 2.5. *If*

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{pmatrix}$$

and we expand $|A|$ by the first row we obtain

$$|A| = a_{11}\begin{vmatrix} a_{22} & 0 & \dots & \dots & 0 \\ a_{32} & a_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n2} & a_{n3} & \dots & \dots & 0 \end{vmatrix} \qquad \textit{Do it again!}$$

$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 & \dots & 0 \\ a_{43} & a_{44} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n3} & \dots & \dots & 0 \end{vmatrix} = \dots = a_{11}a_{22}\cdots a_{nn}.$$

∎

Remark. Our plan of attack on the problem of computing det(A) is to reduce the computation to the computation of the determinant of a triangular matrix A' where we know the relations between $|A|$ and $|A'|$. We will begin with A and perform elementary row operations on A, i.e., left multiply by elementary matrices, and reduce A to A'. With this in mind, we then need to know the relation between $|A|$ and $|E \cdot A|$ where E is an elementary matrix. Sound the trumpets! String your bows, draw your weapons, here she goes!

Theorem 2.6. *If A is an $n \times n$ matrix and E_{ij} is an elementary Type 2 row matrix which interchanges row i and row j, then $|E_{ij}A| = -1 \cdot |A|$.*

Proof of Theorem 2.6. *Assume row i and row j are switched by E_{ij}, then*

$$|A| = \sum_{\kappa \in \Pi_n} (-1)^{\delta(\kappa)}(a_{1k_1} \cdot a_{2k_2} \cdot \dots \cdot a_{ik_i} \cdot a_{jk_j} \cdot \dots \cdot a_{nk_n})$$

where $\kappa = (k_1, k_2, \dots, k_i, \dots, k_j, \dots, k_n)$ while

$$|E_{ij}A| = \sum_{k \in \Pi_n} (-1)^{\delta(\kappa^*)}(a_{1k_1} \cdot \dots \cdot a_{jk_j} \cdot a_{ik_i} \cdot \dots \cdot a_{nk_n})$$

where $\kappa^* = (k_1, k_2, \dots, k_j, \dots, k_i, \dots, k_n)$ i.e., κ^* is κ with k_i and k_j interchanged. By Exercise 2.2(3), the permutations κ^* and κ are of different parity and therefore by Definition 2.4, the definition of $\delta(\kappa)$,

$$(-1)^{\delta(\kappa)} = -(-1)^{\delta(\kappa^*)} \textit{ and therefore } |A| = -|E_{ij}A|.$$

∎

Corollary 2.7. $|E_{ij}| = |E_{ij} \cdot I| = -|I| = -1$ *and therefore* $|E_{ij}A| = -1 \cdot |A| = |E_{ij}| \cdot |A|$.

Corollary 2.8. *Same is true for interchanging columns.*

Proof of Corollary 2.8. *Apply Corollary 2.7 to A^T.* ∎

Theorem 2.9. *If a row of A is multiplied by a nonzero constant α, say*

$$E_i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \alpha & \vdots & 0 \\ 0 & 0 & \cdots & \cdots & \vdots & 1 \end{pmatrix},$$

then $|E_iA| = \alpha|A| = |E_i||A|$.

Proof of Theorem 2.9. *Expand* $|E_iA|$ *by the* i^{th} *row.* ∎

Remark. The Theorem is also true if $\alpha = 0$! **Proof**: Look at the expansion!

Corollary 2.10. *Same is true if "row" is replaced by "column" in the Theorem 2.9.*

Corollary 2.11. *If* A *has a row or column of zero elements, then* $|A| = 0$.

(We already knew that since we could compute $|A|$ by expanding by that row or column.)

Corollary 2.12. *If* E *is an elementary Type 1 or Type 2 row matrix, then* $|EA| = |E| \cdot |A| = |AE|$.

Theorem 2.13. *If a multiple of one row, (say the* i^{th}*) (or column) of* A *is added to another row (the* j^{th}*) (or column) by an elementary Type 3 row matrix* E*, then* $|EA| = |AE| = |A| = |E| \cdot |A|$.

Proof of Theorem 2.13. *Write*

$$EA = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} + \alpha a_{i1} & a_{j2} + \alpha a_{i2} & \cdots & a_{jn} + \alpha a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and expand this by the j^{th} *row. We obtain:*

$$\begin{aligned} |EA| &= (a_{j1} + \alpha a_{i1})A_{j1} + (a_{j2} + \alpha a_{i2})A_{j2} + \ldots + (a_{jn} + \alpha a_{in})A_{jn} \\ &= (a_{j1}A_{j1} + a_{j2}A_{j2} + \ldots + a_{kn}A_{kn}) + \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \\ &= |A| + \alpha|\tilde{A}| \end{aligned}$$

where $\tilde{A}$ *is like* A *except that* both *the* i^{th} *row and the* j^{th} *row are* $(a_{i1}, a_{i2}, \ldots, a_{in})$. *This means that*

$E_{ij} \cdot \tilde{A} = \tilde{A}$ *so that* $|E_{ij}\tilde{A}| = |\tilde{A}|$

and by Theorem 2.6, $|E_{ij}\tilde{A}| = -|\tilde{A}|$, *i.e.,* $-|\tilde{A}| = |E_{ij} \cdot \tilde{A}| = |\tilde{A}|$ *in which case* $|\tilde{A}| = 0$ *and thus*

$|EA| = |A| + \alpha|\tilde{A}| = |A|$. *To see that* $|E| = 1$, *again take* $A = I$. ∎

Corollary 2.14. *If* A *has two rows (or columns) alike, then* $|A| = 0$.

Corollary 2.15. *If* E *is any elementary matrix, then* $|EA| = |E| \cdot |A|$.

Theorem 2.16. *(a).* $|E_{ij}| = -1$ *where* E_{ij} *switched rows* i *and* j

(b). $|\alpha E_i| = \alpha$ *(*αE *multiplies row* i *by* α*)*

(c). $|E| = 1$ *where* E *adds a multiple of one row to another.*

Proof of Theorem 2.16. *Apply above theorems to $A = I$.* ∎

Corollary 2.17. *Elementary row operation matrices have nonzero determinants.*

Recall that in Theorem 2.2 we listed three equivalent statements, one of which was that A is invertible. We are now in a position to add a fourth statement to that list.

Theorem 2.18. *If A is an $n \times n$ matrix, then the following statements are equivalent:*

(a). A *is invertible,*

(b). $A\mathbf{x} = \Theta$ *has only the trivial solution,*

(c). A *is row equivalent to* I_n, *and*

(d). $\det(A) \neq 0$.

Proof of Theorem 2.18. *We have already shown that (a), (b) and (c) are equivalent. Suppose (c) holds, then there exist elementary matrices* $E_1, E_2, \ldots, E_m$ *such that*

$$E_m \cdot E_{m-1} \cdot \ldots \cdot E_2 \cdot E_1 A = I_n$$

therefore

$$1 = |I_n| = |E_m \cdot E_{m-1} \cdot \ldots \cdot E_1 A| = |E_m| \cdot |E_{m-1}| \cdot \ldots \cdot |A|$$

and the statement that the product is not zero is equivalent to the statement that $|A| \neq 0$ *and (d) thus is equivalent to (c).*

∎

Definition 2.9. A is called singular if A is not invertible.

Theorem 2.19. *If A and B are $n \times n$ matrices, then* $\det(AB) = [\det(A)][\det(B)]$.

Proof of Theorem 2.19. *If B is singular, then by Theorem 2.18, there exists* $\mathbf{x} \neq \Theta$ *such that* $B\mathbf{x} = \Theta$, *thus* $AB\mathbf{x} = A\Theta = \Theta$ *so that AB is also singular, thus if* $|B| = 0$ *then* $|AB| = 0$ *also, and* $0 = |AB| = |A| \cdot |B| (= |A| \cdot |0|)$ *so that the theorem holds.*

If B is not *singular, then B is row equivalent to I_n, i.e., there exist elementary matrices* $E_1, \ldots, E_m$ *such that*

$$E_m E_{m-1} \ldots E_1 B = I$$

and since each E_j is invertible,

$$\begin{aligned} B &= E_1^{-1} \cdot E_2^{-1} \cdot \ldots \cdot E_m^{-1} && \text{so that} \\ |B| &= |E_1^{-1}| \cdot \ldots \cdot |E_m^{-1}| && \text{and since} \\ AB &= AE_1^{-1} \cdot \ldots \cdot E_m^{-1} \\ |AB| &= |AE_1^{-1} \cdot \ldots \cdot E_m^{-1}| && \text{but each of } E_j^{-1} \text{ is an elementary matrix, thus} \\ |AB| &= |A| \cdot |E_1^{-1}| \cdot \ldots \cdot |E_m^{-1}| = |A| \cdot |B|. \end{aligned}$$

∎

Definition 2.10. Suppose $A = (a_{ij})$ is an $n \times n$ matrix, then the (classical) adjoint of A is defined to be

$$\text{Adj}(A) = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

where A_{ij} is the cofactor of a_{ij}, i.e., $A_{ij} = (-1)^{i+j}|M_{ij}|$.

Theorem 2.20. *If A is an $n \times n$ matrix then $A(\text{Adj}(A)) = (\text{Adj}(A))A = |A|I_n$.*

Exercise 2.3.
Prove Theorem 2.20.

Corollary 2.21. *A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)}(\text{Adj}(A))$.*

Exercise 2.4.
Prove Corollary 2.21.

Corollary 2.22. *Suppose $A = (a_{ij})$ is invertible, then $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} = (b_1, b_2, \dots, b_n)^T$ has as its only solution*

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|}(\text{Adj}(A))\mathbf{b} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{12} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} b_1A_{11} + b_2A_{21} + \dots + b_nA_{n1} \\ b_1A_{12} + b_2A_{22} + \dots + b_nA_{n2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_1A_{1n} + b_2A_{2n} + \dots + b_nA_{nn} \end{pmatrix}$$

$$\text{i.e., } x_j = \frac{b_1A_{1j} + b_2A_{2j} + \dots + b_nA_{nj}}{|A|}$$

$$x_j = \frac{\begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}}{|A|}$$

where $\mathbf{b}$ is in the j^{th} column of A instead of the regular column.

Remark. This last formula is commonly called Cramer's Rule.

Exercise 2.5.

(1). For each of the following matrices A, compute Adj(A), $|A|$ and A^{-1} if it exists:

$$A_1 = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 3 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix}.$$

(2). Find $\ker(A_1)$, $\ker(A_2)$, $\ker(A_3)$ and $\ker(A_4)$.

(3). Solve

(a) $A_1\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and

(b) $A_1\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then write the solution to

(c) $A_1\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Explain how you knew the answer.

(4). Solve $A_1\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ using Cramer's Rule. Explain how you know in advance, from Cramer's Rule, that $x_2 = x_3 = 0$.

(5). Show that if

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

has a solution, then $\binom{y_1}{y_2}$ is a linear combination of the columns $\binom{a_{11}}{a_{22}}$ and $\binom{a_{12}}{a_{22}}$ and that x_1 and x_2 are the coefficients which makes them add to $\binom{y_1}{y_2}$.

(6). Does the same result hold if A is $n \times n$?

(7). Does the same result hold if A is $m \times n$?

(8). What might a reasonable person call the set of all linear combinations of the columns of a matrix A? Would "George" be a good name? Why?

(9). Is this set a vector subspace of $\mathbb{R}^m$ if A is $(m \times n)$? (Look at Theorem 1.2!)

(10). Now, what would be a good name for it?

(11). Suppose we needed to solve an equation like

$$(x_1, x_2, \dots, x_m)\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (y_1, y_2, \dots, y_n)$$

then where must $(y_1, \dots, y_n)$ be? i.e., characterize the set of all possible y's, show that it is a vector space and give it an appropriate name.

Our methods (so far) for constructing solutions to equations of the form

$$A\mathbf{x} = \mathbf{y}$$

where A is an $n \times n$ matrix, involve computing determinants. This is very time consuming in that it involves large numbers of computations. Recall that $\det(A)$ involves $n!$ separate products each with n factors with the factor $(-1)^{\delta(\kappa)}$ and then all of these are added together. The computation of $\delta(\kappa)$ involves n^2 comparisons for each product so that the total number of computations is about

$$n!(n + n^2) = n(n+1)! + n.$$

Even the most industrious, enthusiastic, eager calculating person would soon tire of this and (even if they didn't) with so many computations the round off errors would soon render the results meaningless, especially if $|A|$ is near zero.

Suppose you matrix A is 10×10 (small by today's standards) and you data is accurate to 3 decimals (fantastic in the real world.) There are about 4×10^3 operations with roundoff of about 10^{-3}. This is substandard if say $|A| = 1$. Can you imagine publishing a technical paper in which you report your results as being

$$\mu = 1 \pm 400,000?$$

As it happens, there is a way out of this dilemma in most cases (actually *all* cases if one is careful.) The key is contained in the proof of Theorem 2.18 and is what you learned formally in high school algebra.

Suppose A is an $n \times n$ matrix. We may use elementary row operations $E_1, \dots, E_m$ to reduce A to row echelon form

$$E_m \cdots E_2 \cdot E_1 A = U$$

where U is an upper triangular matrix. Since each E_j is invertible, we may then write

$$A = E_1^{-1} \cdot E_2^{-1} \cdots E_m^{-1} U$$

Now let's examine E_j, E_j^{-1} and products of such E_j^{-1}'s to see what

$$E_1^{-1} \cdot E_2^{-1} \cdots E_m^{-1}$$

looks like.

Recall that there are 3 elementary matrix types and they have inverses which are of the same type as themselves. (See the chart preceding Theorem 2.2). Recall also that this reduction process is a downwards method, i.e., higher rows are used to simplify lower rows. Let's consider the shapes of the elementary matrices by types. Type 1 elementary matrices are diagonal hence they are both upper and lower triangular. Type 2 are neither upper nor lower triangular and Type 3 are lower triangular.

Exercise 2.6.
Show that if A and B are both lower triangular, then so is $A \cdot B$.

Now suppose we have used elementary row operation matrices to reduce A to upper triangular form:

$$E_m \cdot E_{m-1} \cdots E_1 A = U$$

and that none of the E_j's are Type 2. Then

$$A = \left(E_1^{-1} \cdots E_m^{-1}\right) U = LU$$

where L is lower triangular. Thus we have:

Theorem 2.23. *If A is an $n \times n$ matrix which can be reduced to upper triangular form without interchanging rows, then A can be factored*

$$A = LU$$

where L is lower triangular and U is upper triangular.

Now suppose we wish to solve

$$A\mathbf{x} = \mathbf{y}$$

where A has an LU factorization. Then

$$A\mathbf{x} = LU\mathbf{x} = L[U\mathbf{x}] = L\mathbf{v} = \mathbf{y}$$

$$\text{or} \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots y_n \end{pmatrix}$$

which we can solve, row at a time by substituting in the pervious values, e.g., $v_1 = \dfrac{y_1}{l_{11}}$ etc. (This is called forward substitution). Now that we have v in hand, we solve

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots v_n \end{pmatrix}$$

for x, starting with x_n and working our way back up. (This is called backward substitution.)

In case A requires an interchange of rows to put it into row echelon form, do that *first*, say $E_{ij}A = B$ where B is okay, then

$$A\mathbf{x} = \mathbf{y}$$

becomes

$$E_{ij}A\mathbf{x} = E_{ij}\mathbf{y} \text{ or } B\mathbf{x} = \tilde{\mathbf{y}}$$

where $\tilde{\mathbf{y}}$ is the original $\mathbf{y}$ with y_i and y_j interchanged. Now do the LU factorization, solve for $\tilde{\mathbf{y}}$, then since $E_{ij}^{-1} = E_{ij}$ we have that

$$\mathbf{y} = E_{ij}\tilde{\mathbf{y}}.$$

In case A requires several interchanges of rows, one must keep good accounts so that the inversions can be made in the correct order at the end. Computers can do this very well.

Exercise 2.7.

(1). Find an LU factorization for

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \text{ and then solve } A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

(2). Get a different LU decomposition for A.

(3). See if you can find an LU factorization for

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Can you prove that Bob Jones can't find one?

(4). Rearrange the A in Exercise 2.7(3) so that you can solve

$$A\mathbf{x} = \mathbf{y}$$

by an LU decomposition of the *new* equation, $\tilde{A}\mathbf{x} = \tilde{\mathbf{y}}$, where $\tilde{A} = E_1E_2E_3A$. How is $\tilde{\mathbf{y}}$ related to $\mathbf{y}$ in this case?

Chapter 3

Dot Products, Norms, Geometry, etc.

Dot Products

We have already seen that $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces. Our intent here is to find a way to use the vector space structure to help us gain some control in computational matters concerning our intuitive geometric sense in such spaces. Specifically, we want to find a way to use our ordinary 3 dimensional world intuition to help us solve problems of a geometric nature.

One of the most important and useful notions in this regard is the concept of a right angle, straight up, that way etc. and this comes mainly (mathematically, that is) by way of the Pythagorean Theorem. In $\mathbb{R}^2$, this theorem is extended to arbitrary triangles and becomes the Law of Cosines, that is, the Law of Cosines is the Pythagorean Theorem corrected for the absence of a right angle. Since these two theorems involve not only the notion of angle, but also the notion of distance, we will use the concepts as being familiar to the reader as used in the plane, that is in $\mathbb{R}^2$

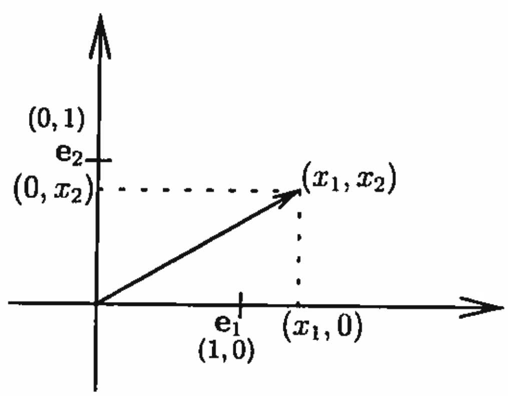

Figure 3.1:

Consider now in 3.1, the line segment from Θ to the point $\mathbf{x} = (x_1, x_2)$, complete with its arrow head indicating direction. Perhaps a hand with index finger extended would be more suggestive, but its much more bother to draw than the arrow head. We will continue to use the arrow head in what follows.

One can see in Figure 3.1 that if we construct a rectangular coordinate system with origin at Θ and such that the perpendicular projections of $\mathbf{x}$ onto the $\mathbf{X}_1$ and $\mathbf{X}_2$ axes fall at the points $(x_1, 0)$ and $(0, x_2)$, then $\mathbf{e}_1 = (1, 0)$ on the $\mathbf{X}_1$ axis and $\mathbf{e}_2 = (0, 1)$ on the $\mathbf{X}_2$ axis and the $\mathbf{X}_1$ and $\mathbf{X}_2$ axes are perpendicular.

The use of the line segment and arrow head is quite compatible with our tendency to be visual creatures and our insistence on being able to "see" things. This insistence does cause some discomfort and trouble in communication. The trouble is that we will wish to call things "vectors" which do not emanate from Θ. In Figure 3.2, we have *two* "vectors" which we will not wish to be considered as being "different" in our discussions, even though they clearly represent two different line segments.

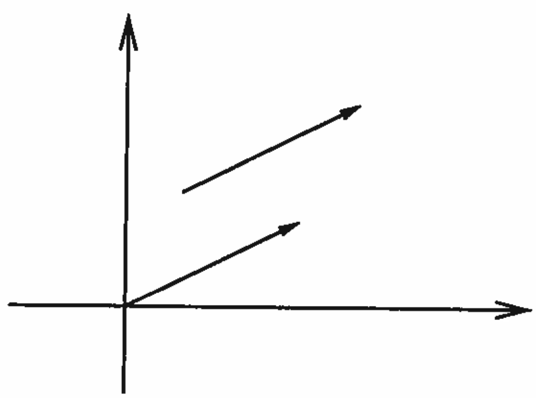

Figure 3.2:

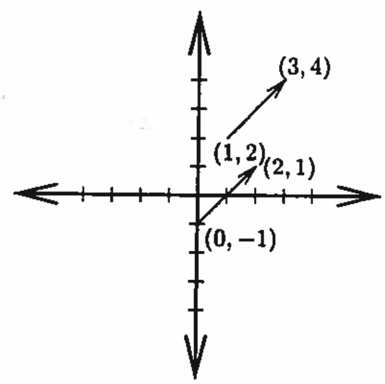

Figure 3.3:

We rationalize this behavior in the following way. A vector line segment has two end points, one with an arrow head and one without. The one without the arrow head is written first, the one with the arrow is written second as $\overrightarrow{[(a, b); (c, d)]}$ where the pairs (a, b) and (c, d) are the coordinates of the points in question. If we *translate* the coordinate system so that the origin is moved to (a, b), i.e., $(x_1 - a, x_2 - b) = (y_1, y_2)$ being the translation, then the "vector" has end points

$$\overrightarrow{[(0, 0); (c - a, d - b)]}$$

and emanates from the origin in the new coordinate sysetm. We think of two "vectors" as being *equivalent* if after such translations, we get the same representations.

Example 3.1.

Consider the "vectors"

$$\mathbf{A} = \overrightarrow{[1, 2]; [3, 4]} \text{ and } \mathbf{B} = \overrightarrow{[(0, -1); (2, 1)]}$$

Translating each (by subtracting its tail from its head), we get

$$\overrightarrow{[(0, 0); (2, 2)]} \text{ and } \overrightarrow{[(0, 0); (2, 2)]}$$

for each, thus the "vectors" $\mathbf{A}$ and $\mathbf{B}$ are equivalent in our visualization. Ω

Hereafter, we will not put quotation marks around the word vector, nor will we tell you whether we are discussing the algebraic object or our fantasized visualization of it. We will probably be thinking of both and so will you. Suppose now that we have two vectors in $\mathbb{R}^2$. Set $\mathbf{A} = (a_1, a_2)$ and $\mathbf{B} = (b_1, b_2)$, then $\mathbf{A} + \mathbf{B} = (a_1 + b_1, a_2 + b_2)$ and $\mathbf{A} - \mathbf{B} = (a_1 - b_1, a_2 - b_2)$, moreover, $2\mathbf{A} = (2a_1, 2a_2)$.

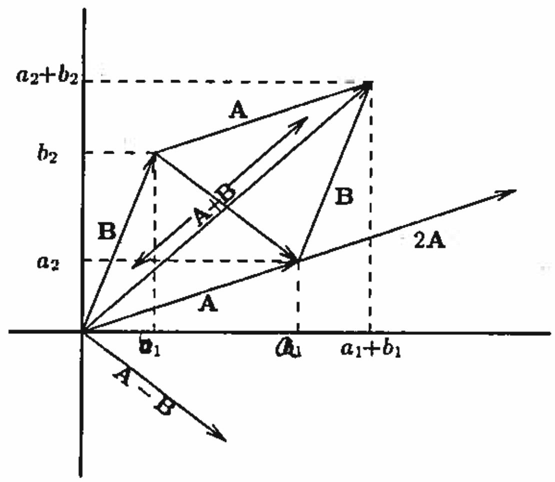

Figure 3.4:

Thus, visually $\mathbf{A} + \mathbf{B}$ is the main diagonal of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{A} - \mathbf{B}$ is the off diagonal with the arrowhead at $\mathbf{A}$. The vector $2\mathbf{A}$ is in the same directions as $\mathbf{A}$ but is twice as long.

The reader is advised that while we have illustrated these matters only in $\mathbb{R}^2$, similar discussions and visualizations are possible in $\mathbb{R}^3$ and higher. If these are new ideas for you, then you should do the next several exercises.

Exercise 3.1.

(1). Suppose $\mathbf{A} = (3, 2)$ and $\mathbf{B} = (2, -1)$. Draw these vectors in a coordinate system, compute and sketch $\mathbf{A} + \mathbf{B}, \mathbf{A} - \mathbf{B}, 3\mathbf{B}$ and $2\mathbf{A} - \mathbf{B}$, using the parallelogram method.

(2). Two vectors are said to be parallel if one is a non-zero multiple of the other. Show that $\mathbf{A} - \mathbf{B}$ in Exercise 3.1.(1) is parallel to $(-2, -6)$.

(3). If you have *three* vectors in $\mathbb{R}^3$, say $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, does $\mathbf{A} + \mathbf{B} + \mathbf{C}$ correspond to a diagonal of some sort as $\mathbf{A} + \mathbf{B}$ does in $\mathbb{R}^2$? Explain and make sketches to illustrate.

By length of a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ we will mean that as given in the Pythagorean Theorem. We will use double bars $||\mathbf{x}||$ to indicate this length and as pictured in figure 3.1,

$$||\mathbf{x}||^2 = x_1^2 + x_2^2, \text{ or } ||\mathbf{x}|| = (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

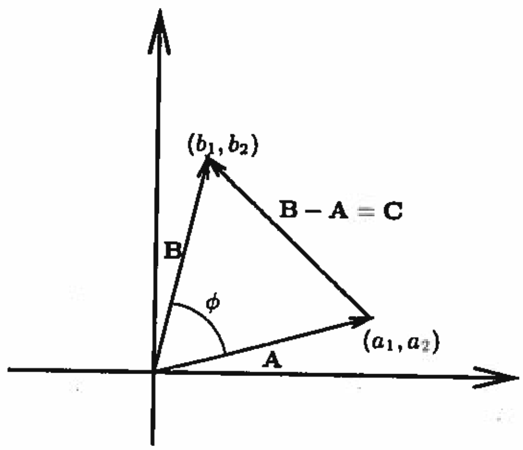

Figure 3.5:

With this notion of length, the law of cosines applied to the triangle in figure 3.5 gives

$\|\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\|\mathbf{A}\| \cdot \|\mathbf{B}\| \cos(\phi)$, that is , $(b_1 - a_1)^2 + (b_2 - a_2)^2 =$

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\|\mathbf{A}\| \cdot \|\mathbf{B}\| \cos(\phi)$$

and after squaring and canceling we get

$$-2a_1b_1 - 2a_2b_2 = -2\|\mathbf{A}\| \cdot \|\mathbf{B}\| \cos(\phi), \text{ or } a_1b_1 + a_2b_2 = \|\mathbf{A}\|\|\mathbf{B}\| \cos(\phi).$$

The same calculation carried out for a triangle formed by $\mathbf{A} = (a_1, a_2, \ldots, a_n)$; $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ gives

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = \|\mathbf{A}\| \cdot \|\mathbf{B}\| \cos(\phi),$$

whenever $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$ are defined so as to give us the lengths of $\mathbf{A}$ and $\mathbf{B}$ as $(a_1^2 + \ldots + a_n^2)^{\frac{1}{2}}$ and $(b_1^2 + \ldots + b_n^2)^{\frac{1}{2}}$ respectively.

Definition 3.1. If $\mathbf{A} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ are vectors in $\mathbb{R}^n$ then

$$(\mathbf{A}, \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

is called the inner product or dot product of $\mathbf{A}$ and $\mathbf{B}$.

If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbb{C}^n$, then we define

$$\mathbf{A} \cdot \mathbf{B} = a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots + a_n\bar{b}_n.$$

In both cases, we define

$$\|\mathbf{A}\|^2 = \mathbf{A} \cdot \mathbf{A}.$$

Remark. The complex conjugates are necessary in the $\mathbb{C}^n$ case in order to have $\|\mathbf{A}\| = (\mathbf{A}\bullet\mathbf{A})^{\frac{1}{2}}$ to be a non negative real number, a rather desirable property for length to possess.

Theorem 3.1. *Suppose* **A** *and* **B** *are in* $\mathbb{R}^n$ *(or* $\mathbb{C}^n$*), then*

(a). $\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A}$; $(\mathbf{A}\bullet\mathbf{B} = \overline{\mathbf{B}\bullet\mathbf{A}})$

(b). $\mathbf{A}\bullet(\mathbf{B}+\mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C}$; *(Same)*

(c). $k(\mathbf{A}\bullet\mathbf{B}) = (k\mathbf{A})\bullet\mathbf{B} = (\mathbf{A}\bullet k\mathbf{B})$; $(\mathbf{A}\bullet\bar{k}\mathbf{B})$

(d). $\mathbf{A}\bullet\mathbf{A} > 0$ *unless* $\mathbf{A} = \Theta$; *(Same)*

(e). $\mathbf{A}\bullet\mathbf{A} = 0$ *if and only if* $\mathbf{A} = \Theta$; *(Same).*

Exercise 3.2.
Verify Theorem 3.1.

Since $\mathbf{A}\bullet\mathbf{B} = \|\mathbf{A}\|\cdot\|\mathbf{B}\|\cos(\phi)$, we have that if neither **A** nor **B** is Θ, then $\|\mathbf{A}\|\cdot\|\mathbf{B}\| \neq 0$ and thus $\dfrac{\mathbf{A}}{\|\mathbf{A}\|}\bullet\dfrac{\mathbf{B}}{\|\mathbf{B}\|} = \cos(\phi)$ and thus $\cos(\phi) = 0$ if and only if $\phi = \frac{\pi}{2}$ or $\phi = -\frac{\pi}{2}$, in which case $\mathbf{A}\bullet\mathbf{B} = 0$, thus we have

Theorem 3.2.

$\mathbf{A}\bullet\mathbf{B} = 0$ *if and only if* $\mathbf{A} = \Theta$ *or* $\mathbf{B} = \Theta$ *or* $\mathbf{A} \perp \mathbf{B}$.

The formula $\dfrac{\mathbf{A}\bullet\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \cos(\phi)$ gives us a way to determine the angle between two vectors and this in turn gives us a way to compute projections of one vector upon another. This latter is a procedure in great use in mechanics.

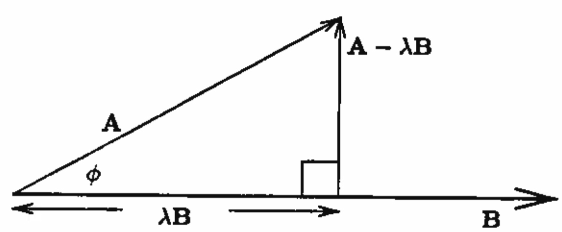

Figure 3.6:

The idea is to find λ so that the perpendicular (orthogonal) projection of **A** onto **B** is $\lambda\mathbf{B}$. This means that we need to find λ so that

$$
\begin{aligned}
&(\mathbf{A}-\lambda\mathbf{B}) \perp \mathbf{B} && \text{and that means}\\
&(\mathbf{A}-\lambda\mathbf{B})\bullet\mathbf{B} = 0 && \text{i.e.,}\\
&\mathbf{A}\bullet\mathbf{B} - \lambda(\mathbf{B}\bullet\mathbf{B}) = 0 && \text{or}\\
&\lambda = \frac{\mathbf{A}\bullet\mathbf{B}}{\mathbf{B}\bullet\mathbf{B}} = \frac{\|\mathbf{A}\|\|\mathbf{B}\|\cos(\phi)}{\|\mathbf{B}\|^2} = \frac{\|\mathbf{A}\|\cos(\phi)}{\|\mathbf{B}\|},
\end{aligned}
$$

Then the projection of **A** onto **B**, denoted $\text{proj}_{\mathbf{B}}(\mathbf{A})$, is given by

$$\text{proj}_{\mathbf{B}}(\mathbf{A}) = \lambda\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\right)\mathbf{B}.$$

Since $\|a\mathbf{A}\| = |a|\cdot\|\mathbf{A}\|$ holds for each $a \in \Phi$, we have that

$$\|\text{proj}_{\mathbf{B}}(\mathbf{A})\| = \frac{|\mathbf{A}\cdot\mathbf{B}|}{\|\mathbf{B}\|^2}\|\mathbf{B}\| = \frac{|\mathbf{A}\cdot\mathbf{B}|}{\|\mathbf{B}\|} = \|\mathbf{A}\|\cdot|\cos(\phi)|,$$

this last expression being what one would expect from Figure 3.6. Notice that the vector $\mathbf{A} - \lambda\mathbf{B}$ is perpendicular to **B** and it is

$$\mathbf{A} - \lambda\mathbf{B} = \mathbf{A} - \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\cdot\mathbf{B}$$

and is called the normal to **B**. This is, of course, an abuse of the word "The". There are many vectors which are normal or perpendicular to **B**.

Returning to Figure 3.6, if we denote the vector $\lambda\mathbf{B}$ by **u** and the normal vector $\mathbf{A} - \lambda\mathbf{B}$ by **v**, we see that $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} + \mathbf{v} = \mathbf{A}$. It is this type of use which we see in mechanics problems where **B** is horizontal and we desire the horizontal and vertical components of the "force" **A**, these are exactly **u** and **v** respectively.

Example 3.2.

Question: Suppose the force **F** is given by the vector $(4,3)$ in $\mathbb{R}^2$. What are the horizontal and vertical components of **F**?

Answer: The horizontal component is the projection onto $\mathbf{e}_1 = (1,0)$ and the vertical component is **F** minus the horizontal component. The horizontal component is $\lambda\mathbf{e}_1$ where $\lambda = \frac{\mathbf{F}\cdot\mathbf{e}_1}{\mathbf{e}_1\cdot\mathbf{e}_1} = \frac{(4,3)\cdot(1,0)}{(1,0)\cdot(1,0)} = \frac{4}{1}$ so $\lambda\mathbf{e}_1 = 4(1,0) = (4,0)$. The vertical component is $(4,3) - \lambda\mathbf{e}_1 = (4,3) - (4,0) = (0,3)$. Of course we already knew that, but it is reassuring that our methods worked.

Ω

Example 3.3.

A ball is rolling under the influence of gravity along a plane of slope m. What is the acceleration of the ball along the plane?

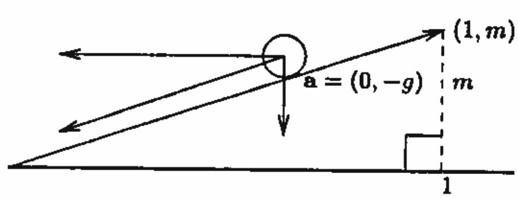

Figure 3.7:

The only force on the ball is due to the acceleration of gravity which has 0 horizontal component and $-g$ vertical component, thus $\mathbf{a} = (0, -g)$. We want the component of this vector along the plane's vector which is $\mathbf{B} = (1, m)$ (it has slope m). Thus

$$\lambda = \frac{(1,m)\cdot(0,-g)}{(1,m)\cdot(1,m)} = \frac{-gm}{1+m^2}$$

and $\lambda\mathbf{B} = \lambda(1,m) = \dfrac{-gm}{1+m^2}(1,m) = -gm\dfrac{(1,m)}{1+m^2}$.

One should notice that $\dfrac{(1,m)}{1+m^2}$ is a vector of length 1 and is directed along the plane on which the ball rests. The magnitude or length of the vector $\lambda\mathbf{B}$ is $\dfrac{gm}{(1+m^2)^{\frac{1}{2}}} = \dfrac{|-gm|}{(1+m^2)^{\frac{1}{2}}}$ and a physicist would describe the vector $\lambda\mathbf{B}$ as having magnitude $\dfrac{gm}{(1+m^2)^{\frac{1}{2}}}$ and direction "down" the plane, the word down referring to the minus sign in $\lambda\mathbf{B}$. Ω

This example suggests the need for some simplifying terminology.

Definition 3.2.

(1). The statement that $\mathbf{A}$ is a unit vector means that $\|\mathbf{A}\| = 1$.

(2). The phrase "in the direction of $\mathbf{A}$" refers to the unit vector $\dfrac{\mathbf{A}}{\|\mathbf{A}\|}$, thus when we refer to a direction, we really mean a unit vector.

Example 3.4.

The vector $(1,2,3)$ has magnitude $(14)^{\frac{1}{2}}$ and direction $\dfrac{1}{\sqrt{14}}(1,2,3) = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$. Ω

Remark. The numbers $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$ are frequently referred to as the direction cosines of the vector. We will see a motivation for this shortly.

To illustrate the utility of dot products and projections in geometric matters, consider a straight line in the plane given by

$$ax + by + c = 0,$$

and consider two points $\mathbf{A} = (x_1, y_1)$ and $\mathbf{B} = (x_2, y_2)$ on the line. We then have that

$$\begin{aligned} ax_2 + by_2 + c &= 0 \quad \text{and} \\ ax_1 + by_1 + c &= 0 \quad \text{so that} \\ a(x_2 - x_1) + b(y_2 - y_1) &= 0 \quad \text{or} \\ (a,b)\cdot(x_2 - x_1, y_2 - y_1) = (a,b)\cdot(\mathbf{B} - \mathbf{A}) &= 0 \end{aligned}$$

so that the vector $\mathbf{B} - \mathbf{A}$ which points along the line is orthogonal (perpendicular) to the vector (a,b). Stated the other way, the vector (a,b) is a normal to the line $ax + by + c = 0$.

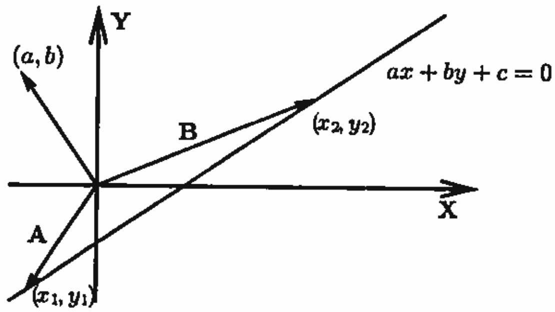

Figure 3.8:

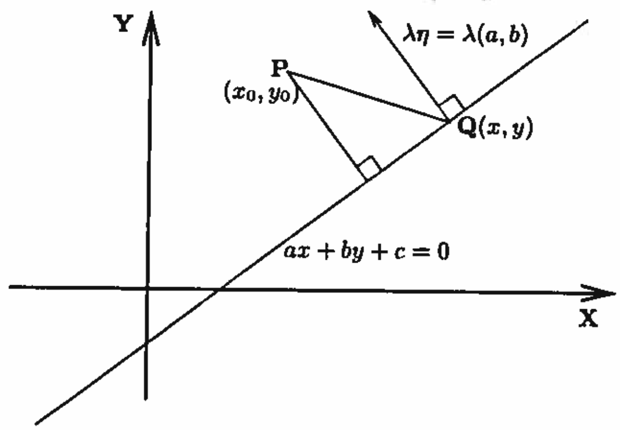

Figure 3.9:

Let's use this fact to derive a formula for the distance from a point to a line in the plane (shortest distance, that is.)

Our idea is to project **PQ** onto a normal η to get $\text{proj}_\eta(\mathbf{PQ})$ and then get its length. Recall $\eta = (a, b)$.

$$\left|\text{proj}_\eta(\mathbf{PQ})\right| = \frac{|\mathbf{PQ}\cdot\eta|}{\|\eta\|} = \frac{|a(x_0 - x) + b(y_0 - y)|}{(a^2+b^2)^{\frac{1}{2}}} = \frac{|ax_0 + by_0 \mp c|}{(a^2+b^2)^{\frac{1}{2}}}.$$

In an entirely analogous fashion, one can show that given a plane expressed by

$$ax + by + cz + d = 0$$

that the vector (a, b, c) is normal to the plane and then derive a distance formula for the distance from a point to a plane in $\mathbb{R}^3$.

Exercise 3.3.

(1). Find the shortest distance from the point $(3, 4)$ to the line $y = x$.

(2). Find the point on the line $y = x$ which is nearest the point $(3, 4)$.

(3). Generalize what you did in Exercise 3.3.(2) to show how to find the point on $ax + by + c = 0$ which is nearest the point $\mathbf{P} = (x_0, y_0)$ not on the line.

(4). Derive a formula for the shortest distance from a point $\mathbf{P} = (x_0, y_0, z_0)$ to a plane $ax + by + cz + d = 0$ which does not contain $\mathbf{P}_0$.

(5). Describe a method to locate the point on the plane in Exercise 3.3.(4) which is nearest $\mathbf{P}_0$.

Definition 3.3. Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are sets. By the Cartesian product of these sets, denoted by $\mathbf{X}_1 \otimes \mathbf{X}_2 \otimes \cdots \otimes \mathbf{X}_n = \prod_{i=1}^{n} \mathbf{X}_i$, we mean the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbf{X}_i$.

Example 3.5.

The usual Euclidean plane has as its objects $\mathbb{R} \otimes \mathbb{R}$. This is not the same as $\mathbb{R}^2$. The symbol $\mathbb{R}^2$ has the same objects as $\mathbb{R} \otimes \mathbb{R}$ but it also has superimposed upon it the usual algebraic structure, i.e., as a vector space $\mathbb{R}^2 = \{\mathbb{R} \otimes \mathbb{R}; \mathbb{R}, +, \cdot\}$ where $\Phi = \mathbb{R}$ and $+$ and $\cdot$ have their usual meanings. Ω

The dot product on a vector space maps the set $\mathbf{X} \otimes \mathbf{X}$ into the scalar field Φ, thus the "answer" is a scalar, not in general , a vector. A natural question arises: **Can one define a product which is vector valued and if so can one do it in such a way as to make it useful?** In case $\mathbf{X} = \mathbb{R}^3$, the answer is yes to both queries.

The usual way comes about from our knowledge of determinants. The notation is old and has become standard. That notation is:

$$\text{set}\quad \begin{array}{rclcl} \mathbf{i} & = & \mathbf{e}_1 & = & (1,0,0) \text{ (already we're in trouble.)} \\ \mathbf{j} & = & \mathbf{e}_2 & = & (0,1,0) \\ \mathbf{k} & = & \mathbf{e}_3 & = & (0,0,1). \end{array}$$

Definition 3.4. Suppose $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$.

$$\begin{aligned} A \times B &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &\stackrel{\text{def}}{=} \mathbf{i}(a_2 b_3 - b_2 a_3) - \mathbf{j}(a_1 b_3 - b_1 a_3) + \mathbf{k}(a_1 b_2 - a_2 b_1), \end{aligned}$$

where the determinant is a formal or symbolic way of thinking and the part following $\stackrel{\text{def}}{=}$ is the actual definition as a vector. The determinant notation is extremely useful and provides short proofs of most parts of the next theorem.

Theorem 3.3.

(a). $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

(b). $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$

(c). $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$

(d). $k(\mathbf{A}\times\mathbf{B}) = (k\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(k\mathbf{B})$

(e). $\mathbf{A}\times\Theta = \Theta\times\mathbf{A} = \Theta$

(f). $\mathbf{A}\times\mathbf{A} = \Theta$

(g). $\mathbf{A}\bullet(\mathbf{A}\times\mathbf{B}) = 0$

(h). $\mathbf{B}\bullet(\mathbf{A}\times\mathbf{B}) = 0$

(i). $$\begin{aligned}||\mathbf{A}\times\mathbf{B}||^2 &= ||\mathbf{A}||^2||\mathbf{B}||^2 - (\mathbf{A}\bullet\mathbf{B})^2 \text{ (Lagrange's Identity)}\\ &= ||\mathbf{A}||^2||\mathbf{B}||^2\sin^2(\phi)\end{aligned}$$ *where ϕ is the angle between* **A** *and* **B**.

Proof. The first 8 are immediate results from the properties of determinants and (i) is obtained by direct expansion of the left and right sides. The second expression in (i) follows from the fact that $\mathbf{A}\bullet\mathbf{B} = ||\mathbf{A}||||\mathbf{B}||\cos(\phi)$. ∎

Corollary 3.4. $||\mathbf{A}\times\mathbf{B}|| = ||\mathbf{A}||||\mathbf{B}||\sin(\phi)$.

Remark. (g) and (h) of Theorem 3.3 show that the vector $\mathbf{A}\times\mathbf{B}$ is orthogonal to both **A** and **B**. In $\mathbb{R}^3$ this says (among other things) that $\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane which contains the vectors **A** and **B**, i.e., $\mathbf{A}\times\mathbf{B}$ is a normal to that plane. Corollary 3.4 shows that the magnitude of the vector $\mathbf{A}\times\mathbf{B}$ is the same as the area of the parallelogram formed by edges **A** and **B**.

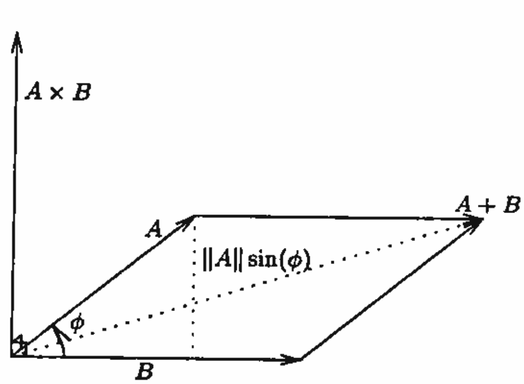

Figure 3.10:

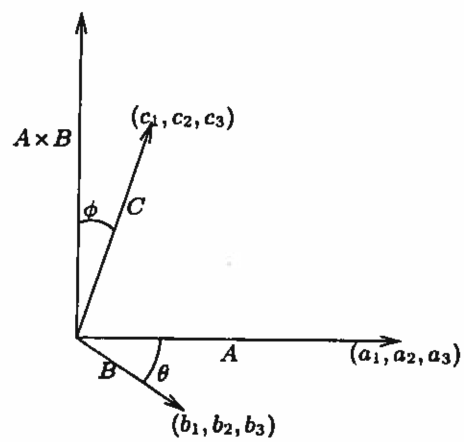

Figure 3.11:

Corollary 3.4 gives one cause to wonder whether one might use determinants to compute volumes and areas in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Let's look at this. Suppose we have three vectors in $\mathbb{R}^3$; $\mathbf{A} = (a_1, a_2, a_3)$ $\mathbf{B} = (b_1, b_2, b_3)$ and $\mathbf{C} = (c_1, c_2, c_3)$ and that they form the edges of a parallelepiped in $\mathbb{R}^3$, i.e., none lies in the plane generated by the other two.

$$\begin{aligned}\text{Then } \mathbf{C}\bullet(\mathbf{A}\times\mathbf{B}) &= \begin{vmatrix} c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= ||\mathbf{C}||\cdot||(\mathbf{A}\times\mathbf{B})||\cos(\phi) \text{ by the definition of dot product and thus}\\ &= [||\mathbf{C}||\cos(\phi)]\,||\mathbf{A}\times\mathbf{B}|| = ||\mathbf{C}||\cos(\phi)||\mathbf{A}||||\mathbf{B}||\sin(\theta) \text{ by Corollary 3.4}\end{aligned}$$

But, $||\mathbf{C}||\cos(\phi)$ is the height of the parallelepiped determined by $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ (or the negative of that height in case $\cos(\phi) < 0$), and $||\mathbf{A}||||\mathbf{B}||\cos(\theta)$ is the area of the base, thus

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \pm \text{Volume of the parallelepiped.}$$

Several questions immediately arise. Does this have an analogue in $\mathbb{R}^2$ which gives area? What does the $\pm$ sign mean?

Let's look first at the first question. How can one do this sort of thing in $\mathbb{R}^2$ and get area? Can one do it at all? What *is* the analogue? There's no cross product in $\mathbb{R}^2$. True, but there *is* a determinant. Suppose $\mathbf{A} = (a_1, a_2)$ and $\mathbf{B} = (b_1, b_2)$ form the edges of a parallelogram in $\mathbb{R}^2$.

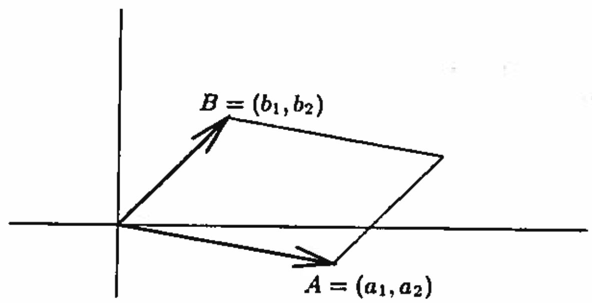

Figure 3.12:

Then $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = (a_1, a_2) \cdot (b_2, -b_1) = \mathbf{A} \cdot \bar{\mathbf{B}}$ where $\bar{\mathbf{B}} = (b_2, -b_1)$ is orthogonal to $\mathbf{B}$, i.e., $\bar{\mathbf{B}} \cdot \mathbf{B} = 0$, and moreover $||\bar{\mathbf{B}}|| = ||\mathbf{B}|| = (b_1^2 + b_2^2)^{\frac{1}{2}}$. Therefore,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \mathbf{A} \cdot \bar{\mathbf{B}} = ||\mathbf{A}||||\bar{\mathbf{B}}||\cos(\phi) = ||\mathbf{A}||||\mathbf{B}||\cos(\phi)$$

where ϕ is the angle between $\mathbf{A}$ and $\bar{\mathbf{B}}$ but $\bar{\mathbf{B}} \perp \mathbf{B}$ so that $\phi = \theta \pm \frac{\pi}{2}$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. Therefore $\cos(\phi) = \cos\left(\theta \pm \frac{\pi}{2}\right) = \mp \sin(\theta)$ and thus

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = ||\mathbf{A}||||\mathbf{B}||\cos(\phi) = \mp ||\mathbf{A}||||\mathbf{B}||\sin(\theta) = \pm \text{Area of parallelogram} \mathbf{BA}.$$

Notice that $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$ and similarly for the 3×3 determinants, thus the positiveness or negativeness of the number attained is determined by the *order* in which the vectors are "listed".

Example 3.6.

Notice that $\left|\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right| = 1$ and $\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right| = -1$, so that there's something "different" about the set $(\mathbf{e}_1, \mathbf{e}_2)$ and the set $(\mathbf{e}_2, \mathbf{e}_1)$ in $\mathbb{R}^2$.

Ω

Definition 3.5. Suppose that $\mathbf{v}_1 = (x_{11}, \dots, x_{1n}), \mathbf{v}_2 = (x_{21}, \dots, x_{2n}), \dots, \mathbf{v}_n = (x_{n1}, \dots, x_{nn})$ are vectors such that the matrix

$$V = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \text{ is non-singular,}$$

i.e., $|V| \neq 0$. We say that V the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is positively oriented if $|V| = \det(V) > 0$ and we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is negatively oriented if $\det(V) < 0$.

Exercise 3.4.

(1). Show that in $\mathbb{R}^n$, the set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is positively oriented and that the "volume" of the unit cube which they determine is 1.

(2). Show that in $\mathbb{R}^n$, the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_n\}$ is negatively oriented and the "cube" generated has "volume" -1.

(3). What happens to the orientation of a set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ if there is an interchange of two elements in the set? Why is that?

(4). Without computing the determinant, say what orientation the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4\}$ has in $\mathbb{R}^4$. What about $\{\mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$? What about $\{\mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2\}$? How did you know?

(5). Find the volume of the parallelepiped which has the origin and the points $\mathbf{p}_1 = (1, 1, 1); \mathbf{p}_2 = (0, 1, 2); \mathbf{p}_3 = (1, 0, 2)$ as vertices. What is the orientation of the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$?

Remark. Recall that Theorem 2.4 states that $|A| = |A^T|$, thus in Definition 3.5, it matters not whether we write the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ as columns or as rows for the matrix V, it does matter of course if we perform an inversion.

Exercise 3.5.
Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, while $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Remark. This is known as the "right-hand-rule." See Figure 3.13. The thought being if you curl the fingers of your ***right hand*** in the direction from $\mathbf{A}$ over to $\mathbf{B}$, then your right thumb points in the direction of $\mathbf{A} \times \mathbf{B}$.

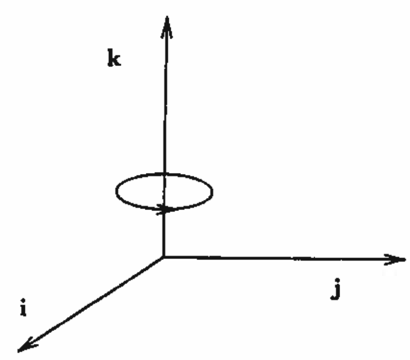

Figure 3.13:

Remark. If we had defined $\mathbf{A} \sim \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$, then we would have $\mathbf{i} \sim \mathbf{j} = -\mathbf{k}$; $\mathbf{j} \sim \mathbf{k} = -\mathbf{i}$ and $\mathbf{k} \sim \mathbf{i} = -\mathbf{j}$.

Exercise 3.6.
Show that with the vector product $\mathbf{A} \sim \mathbf{B}$ rather than $\mathbf{A} \times \mathbf{B}$, one actually has a left-hand-rule.

Equations for lines, planes and related topics

Suppose we wish an equation for a line in $\mathbb{R}^n$ which passes through a point $\mathbf{P}_0$ and a point $\mathbf{Q}_0$. The vector

$$\mathbf{X} = \mathbf{P}_0 + t(\mathbf{Q}_0 - \mathbf{P}_0)$$

consists of $\mathbf{P}_0$ with a multiple of $(\mathbf{Q}_0 - \mathbf{P}_0)$ added to it. In particular, in $\mathbb{R}^3$, say $\mathbf{P}_0 = (x_0, y_0, z_0)$ and

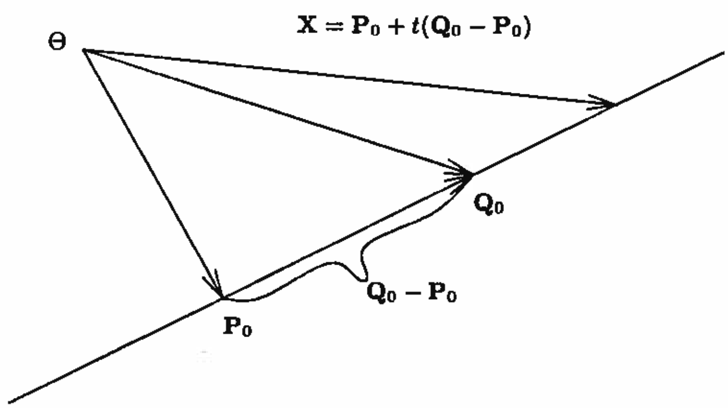

Figure 3.14:

$\mathbf{Q}_0 = (x_1, y_1, z_1)$, then

$$\begin{aligned} \mathbf{X} = (x, y, z) &= \mathbf{P}_0 + t(\mathbf{Q}_0 - \mathbf{P}_0) \\ &= (x_0, y_0, z_0) + t(x_1 - x_0, y_1 - y_0, z_1 - z_0) \end{aligned}$$

or (in parametric form)

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \\ z &= z_0 + t(z_1 - z_0). \end{aligned}$$

Solving each equation for t, gives

$$t = \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

which is called the symmetric form.

Clearly t plays a multiple slope role in the parametric form and this is emphasized in the symmetric form. An alternative way of thinking about this is that we have a line through $\mathbf{P}_0$ "in the direction of $\mathbf{Q}_0 - \mathbf{P}_0$." Recall that the actual direction is given by $\frac{\mathbf{Q}_0 - \mathbf{P}_0}{\|\mathbf{Q}_0 - \mathbf{P}_0\|}$. See Definition 3.2.

Let's look at this more carefully, e.g., visually.

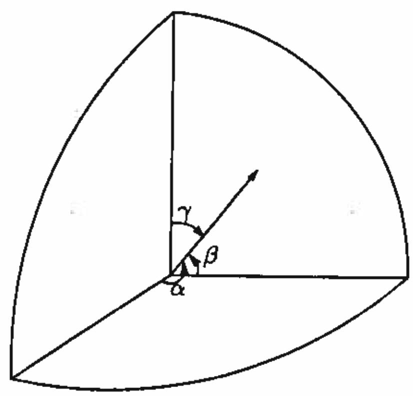

Figure 3.15:

Suppose $\mathbf{A} = (a_1, a_2, a_3)$, then $\frac{\mathbf{A}}{\|\mathbf{A}\|} = \left(\frac{a_1}{\|\mathbf{A}\|}, \frac{a_2}{\|\mathbf{A}\|}, \frac{a_3}{\|\mathbf{A}\|}\right)$ is a point on the sphere of radius 1 centered at the origin. Moreover, $\cos(\alpha) = \frac{a_1}{\|\mathbf{A}\|}$, $\cos(\beta) = \frac{a_2}{\|\mathbf{A}\|}$ and $\cos(\gamma) = \frac{a_3}{\|\mathbf{A}\|}$. This is the reason the components of a unit vector are called the direction cosines of the vector as mentioned after Example 3.4.

If we had desired to write an equation for a line through $\mathbf{P}_0 = (x_0, y_0, z_0)$ in the direction $\mathbf{A} = (a_1, a_2, a_3)$, i.e., $\|\mathbf{A}\| = 1$, then we would have written

$$\begin{aligned} \mathbf{X} &= \mathbf{P}_0 + t\mathbf{A}, \text{ or} \\ (x, y, z) &= (x_0, y_0, z_0) + t(a_1, a_2, a_3) \end{aligned}$$

$$\text{or } \begin{cases} x &= x_0 + ta_1 \\ y &= y_0 + ta_2 \\ z &= z_0 + ta_3 \end{cases}$$

$$\text{or } t = \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

where a_1, a_2 and a_3 are the direction cosines of $\mathbf{A}$ and $a_1^2 + a_2^2 + a_3^2 = 1$.

Now let's suppose we want a plane which contains $\mathbf{P}_0 = (x_0, y_0, z_0)$ and has normal $\eta = (a, b, c)$. We need that all vectors $\mathbf{X}$ in the plane have the property that $\eta \perp (\mathbf{X} - \mathbf{P}_0)$, that is

$$\eta \cdot (\mathbf{X} - \mathbf{P}_0) = 0$$

$$\text{or } \eta \cdot \mathbf{X} - \eta \cdot \mathbf{P}_0 = 0$$

$$\text{or } \eta \cdot \mathbf{X} = \eta \cdot \mathbf{P}_0 \text{ holds.}$$

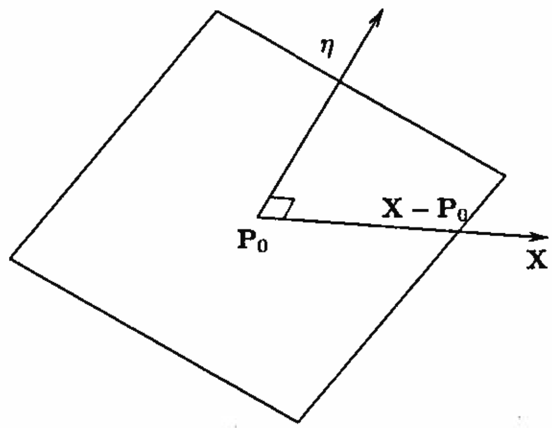

Figure 3.16:

Writing these in detail,

$$(a,b,c)\cdot(x-x_0, y-y_0, z-z_0) = 0$$
$$\text{or } a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
$$\text{or } ax+by+cz = ax_0+by_0+cz_0 = d$$
$$\text{or } ax+by+cz-d=0.$$

Exercise 3.7.

(1). In $\mathbb{R}^3$, write equations for a line through $\mathbf{P}_0 = (1,1,1)$ and $\mathbf{Q}_0 = (1,2,3)$, in vector form, in parametric form and in symmetric form.

(2). Find the distance from the origin to the line in Exercise 3.7.(1).

(3). Find the point on the line in Exercise 3.7.(1) which is nearest the origin.

(4). Find the direction cosines for the vector $\mathbf{Q}_0$ in Exercise 3.7.(1) and then find the radian measure of the direction angles α, β and γ.

(5). Write an equation for the plane through $\mathbf{P}_0 = (1,1,1)$ with normal $(0,1,2)$, i.e., the line in Exercise 3.7.(1) is normal to this plane.

(6). Draw the plane in Exercise 3.7.(5) and the line in Exercise 3.7.(1) in the same figure.

Now let's suppose that our given data for the plane is three points, rather than a point and a normal. Denote the points **A**, **B**, **C**. There are several ways to attack the problem. One possibility is to attempt to compute a normal the plane in question, then use that normal and either **A** or **B** or **C** for $\mathbf{P}_0$ in the previous case. For example, $(\mathbf{A}-\mathbf{C})$ and $(\mathbf{B}-\mathbf{C})$ are both vectors in the plane we seek and therefore, by Theorem 3.3, $(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C}) = \eta$ is orthogonal to both of these vectors and thus a normal to the plane which
C

they determine. We can therefore, use

$$\begin{aligned}
\mathbf{X}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})] &= \mathbf{A}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})] \\
\text{or } \mathbf{X}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})] &= \mathbf{B}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})] \\
\text{or } \mathbf{X}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})] &= \mathbf{C}\cdot[(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{C})].
\end{aligned}$$

Another procedure would be to solve

$$\left.\begin{aligned}
\eta\cdot(\mathbf{C}-\mathbf{A}) &= 0 \\
\eta\cdot(\mathbf{C}-\mathbf{B}) &= 0
\end{aligned}\right\}$$

for η (there are infinitely many solutions) and then write $\mathbf{X}\cdot\eta = \mathbf{C}\cdot\eta$ (or $\mathbf{A}\cdot\eta$ or $\mathbf{B}\cdot\eta$).

We can of course apply this method in $\mathbb{R}^n$ to find what is called an "affine subspace" through $\mathbf{P}_0$ which is normal to η, that is

$$\mathbf{X}\cdot\eta = \mathbf{P}_0\cdot\eta$$

provided η and $\mathbf{P}_0$ are known. One could not, of course, use cross product methods unless $n = 3$. One *could*, given n points $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, consider the equations

$$\eta\cdot(\mathbf{P}_n - \mathbf{P}_i) = 0 \quad i = 1, 2, \ldots, n-1,$$

solve for η and then write

$$\mathbf{X}\cdot\eta = \mathbf{P}_0\cdot\eta$$

for the affine subspace through $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$.

Exercise 3.8.

(1). Find an equation for the plane through the three points $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$.

(2). Find a normal to the plane in Exercise 3.8.(1).

(3). Find an equation for the plane through the three points $(1,0,0)$, $(1,1,0)$ and $(1,1,1)$.

(4). Find a normal to the plane in Exercise 3.8.(3).

Let's suppose we have, in $\mathbb{R}^3$, a plane $\mathcal{P}$ given by $ax + by + cz + d = 0$ i.e., $(x,y,z)\cdot(a,b,c) = -d$, where $\eta = (a,b,c)$ and that we have a point $\mathbf{P}_0 = (x_0, y_0, z_0)$ not on the plane $\mathcal{P}$. We want to find the point $\mathbf{Q}$ on $\mathcal{P}$ which is nearest $\mathbf{P}_0$ and then find $\|\mathbf{P}_0 - \mathbf{Q}\|$, i.e., how far $\mathbf{P}_0$ is from the plane $\mathcal{P}$.

Intuitively, we know that $\mathbf{P}_0 - \mathbf{Q}$ should serve as a normal to the plane $\mathcal{P}$. Let's write an equation for the line through $\mathbf{P}_0$ which is normal to $\mathcal{P}$, i.e., the line through $\mathbf{P}_0$ in the direction $\eta = (a,b,c)$. That line is given by $\mathbf{X} = \mathbf{P}_0 + t\eta = (x_0, y_0, z_0) + t(a,b,c)$.

Let's now find where this line intersects $\mathcal{P}$, i.e., find a value of t, say t_0 which puts $\mathbf{X}$ on both the line and the plane, i.e.,

$$\mathbf{X}\cdot\eta = (\mathbf{P}_0 + t\eta)\cdot\eta = \mathbf{P}_0\cdot\eta + t\eta\cdot\eta = \mathbf{P}_0\cdot\eta + t_0\|\eta\|^2 = -d$$

where $\eta = (a, b, c)$. This last equation tells us that

$$\begin{aligned} t_0 &= \frac{-d - \mathbf{P}_0 \cdot \eta}{\|\eta\|^2} = -\frac{\mathbf{P}_0 \cdot \eta + d}{\|\eta\|^2} \\ &= -\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2} \end{aligned}$$

and thus $\mathbf{X} = \mathbf{Q} = \mathbf{P}_0 + t\eta = \mathbf{P}_0 - \left(\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2}\right)(a, b, c)$ gives us the point $\mathbf{Q}$ and then

$$\|\mathbf{Q} - \mathbf{P}_0\| = \left|\frac{ax_0 + by_0 + cz_0 + d}{\|\eta\|^2}\right| \cdot \|\eta\| = \frac{|ax_0 + by_0 + cz_0 + d|}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}.$$

It is of some interest that this method works just fine in $\mathbb{R}^n$ (since it does not depend upon a cross product) and can locate the point $\mathbf{Q}$ in an affine subspace $\mathbf{S}$ given by

$$\mathbf{X} \cdot \eta = d$$

which is nearest a point $\mathbf{P}_0$ not in $\mathbf{S}$ and then one computes $\|\mathbf{Q} - \mathbf{P}_0\|$, the distance from $\mathbf{S}$ to $\mathbf{P}_0$. The technique is the same. Consider the line l through $\mathbf{P}_0$ in the direction η, i.e.,

$$\mathbf{X} = \mathbf{P}_0 + t\eta$$

and determine $t = t_0$ which puts $\mathbf{X}$ in $\mathbf{S}$, i.e., $d = \mathbf{X} \cdot \eta = (\mathbf{P}_0 + t_0\eta) \cdot \eta = \mathbf{P}_0 \cdot \eta + t_0\|\eta\|^2$ so $t_0 = \dfrac{d - \mathbf{P}_0 \cdot \eta}{\|\eta\|^2}$; then $\mathbf{Q} = \mathbf{P}_0 + t_0\eta = \mathbf{P}_0 + \dfrac{d - \mathbf{P}_0\eta}{\|\eta\|^2}\eta$ and $\|\mathbf{Q} - \mathbf{P}_0\| = \dfrac{|d - \mathbf{P}_0 \cdot \eta|}{\|\eta\|}$.

Exercise 3.9.

(1). Find the point $\mathbf{Q}$ on the plane $\mathbf{P}$ given by $3x + 2y + z = 4$ which is nearest to the origin (i.e., $\mathbf{P}_0 = \Theta = (0, 0, 0)$) and then compute $\|\mathbf{Q} - \mathbf{P}_0\|$, the distance of $\mathbf{P}$ from Θ.

(2). Find the point $\mathbf{Q}$ on the plane given by $x + y + z = -1$ which is nearest the point $\mathbf{P}_0 = (1, 1, 1)$ and compute $\|\mathbf{Q} - \mathbf{P}_0\|$.

Remark. The technique just illustrated is used quite often to find the best approximation to a point $\mathbf{P}_0$ from some subspace or affine subspace which does not contain $\mathbf{P}_0$. One must, of course, have a dot (or inner) product available.

Exercise 3.10.

(1). Show that $\mathbf{A} = (2, -1, 1)$; $\mathbf{B} = (3, 2, -1)$ and $\mathbf{C} = (7, 0, -2)$ are vertices of a right triangle.

(2). Suppose $\mathbf{x} = (k, 1)$ and $\mathbf{y} = (4, 3)$. Find a value for k so that $\mathbf{x} \perp \mathbf{y}$. Find a value of k such that the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\frac{\pi}{4}$. Find a value of k so that the angle is $\frac{\pi}{6}$.

(3). Establish the identity:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

This is sometimes called the "parallelogram law". Explain why this is done.

(4). Suppose $\mathbf{x} \perp \mathbf{y}_1$, and $\mathbf{x} \perp \mathbf{y}_2$ and show that for all scalars α and β, $\mathbf{x} \perp (\alpha\mathbf{y}_1 + \beta\mathbf{y}_2)$.(This idea was used in the discussion preceeding Exercises 3.8.)

(5). Suppose $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{y} = (1, 1, -1)$. Find all vectors $\mathbf{v}$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{y}$.

(6). Show that if $\mathbf{A} = (a_1, a_2, a_3)$; $\mathbf{B} = (b_1, b_2, b_3)$; and $\mathbf{C} = (c_1, c_2, c_3)$, then

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(7). Suppose that **A**, **B**, **C** are as in Exercise 3.10.(4). Show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

Hint: Do it first for $\mathbf{C} = \mathbf{e}_1, \mathbf{C} = \mathbf{e}_2$ and $\mathbf{C} = \mathbf{e}_3$ then use linearity.

Chapter 4

A Return to Calculus

Much of what you learned to do in the calculus of functions of a single variable (single independent variable, single dependent variable) could be done because you knew how to compute using the variables, i.e., in $y = f(x)$, one has an algebraic structure on the x's and an algebraic structure on the y's. One of the major reasons for the development of vector spaces is the need for such algebraic structures in the study of functions of several variables, i.e., $f : \mathbb{R}^n \to \mathbb{R}^m$ or

$$y = f(x)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. One needs to be able to do computations, even if in less complete algebraic structures than were available in the simpler cases.

For example, in computing a derivative of a function f at a point x_0, one considered a Fermat difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \epsilon(x, x_0)$$

where $\epsilon(x, x_0) \to 0$ as $(x - x_0) \to 0$. In the vector case this is not possible since one can't divide by a vector. We can, however, rewrite this expression as

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \epsilon(x, x_0)(x - x_0)$$

and avoid the problem of division. Recall that $f'(x_0)$ is a number in the $\mathbb{R}^1 \to \mathbb{R}^1$ case and thus $f'(x_0) \in L[\mathbb{R}^1, \mathbb{R}^1]$. This is what we will do.

Definition 4.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ (or more generally, $f : \mathbf{X} \to \mathbf{Y}$). The statement that f has a Frechet differential (or derivative) A at x_0 means that $A \in L[\mathbb{R}^n, \mathbb{R}^m]$ (or $L[\mathbf{X}, \mathbf{Y}]$) and there exists a function $\Phi(x, x_0)$ such that

$$f(x) - f(x_0) = A(x - x_0) + \Phi(x, x_0)||x - x_0||$$

where $\Phi(x, x_0) \to \Theta$ as $x \to x_0$.

Note. In case $f : \mathbb{R}^1 \to \mathbb{R}^1$, then A is a number and this is the usual derivative. If $f : \mathbb{R}^n \to \mathbb{R}^1$, then A is $1 \times n$ and is usually called the gradient of f.

Our first task is to gain more explicit information about the derivative A. We will usually denote it $Df(x_0)$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, then $A = Df(x_0)$ is an $m \times n$ matrix. We know that the columns are given by $\{A(\mathbf{e_1}), A(\mathbf{e_2}), \dots, A(\mathbf{e_n})\}$ and thus we try to arrange things to obtain these. What we need is to have $(x - x_0) = \mathbf{e}_1 \cdot \epsilon$ where ϵ is a yet to be determined constant (scalar) which we *can* divide out. Suppose $x_0 = (x_{01}, x_{02}, \dots, x_{0n})$, then we choose $x = (\epsilon + x_{01}, x_{02}, \dots, x_{0n})$ and we have

$$\begin{aligned} f(x) - f(x_0) &= f(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) - f(x_{01}, x_{02}, \dots, x_{0n}) \\ &= A(x - x_0) + \Phi(x, x_0)\|x - x_0\| = A(\epsilon, 0, 0, \dots, 0) + \Phi(x, x_0)\,|\epsilon|\,. \end{aligned}$$

Simplifying by dividing by ϵ, we have

$$\begin{aligned} \frac{f(x) - f(x_0)}{\epsilon} &= \frac{f(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) - f(x_{01}, x_{02}, \dots, x_{0n})}{(\epsilon + x_{01}) - x_{01}} \\ &= A\mathbf{e}_1 + \Phi(x, x_0)\frac{|\epsilon|}{\epsilon}. \end{aligned}$$

Now as $\|x - x_0\| = |\epsilon| =\!\!\to 0$, $\frac{|\epsilon|}{\epsilon}$ remains 1 or -1 while $\Phi(x, x_0) \to 0$, thus the right side becomes $A(\mathbf{e_1})$ which is the first column of A while the left side is, since $f(x) = (y \in \mathbb{R}^m) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

$$\frac{\begin{pmatrix} f_1(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) \\ f_2(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) \\ \cdots \\ f_m(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) \end{pmatrix} - \begin{pmatrix} f_1(x_{01}, x_{02}, \dots, x_{0n}) \\ f_2(x_{01}, x_{02}, \dots, x_{0n}) \\ \cdots \\ f_m(x_{01}, x_{02}, \dots, x_{0n}) \end{pmatrix}}{(\epsilon + x_{01}) - x_{01}} = \begin{pmatrix} \frac{f_1(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) - f_1(\cancel{\epsilon +} x_{01}, x_{02}, \dots, x_{0n})}{(\epsilon + x_{01}) - x_{01}} \\ \frac{f_2(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) - f_2(\cancel{\epsilon +} x_{01}, x_{02}, \dots, x_{0n})}{(\epsilon + x_{01}) - x_{01}} \\ \cdots \\ \frac{f_m(\epsilon + x_{01}, x_{02}, \dots, x_{0n}) - f_m(\cancel{\epsilon +} x_{01}, x_{02}, \dots, x_{0n})}{(\epsilon + x_{01}) - x_{01}} \end{pmatrix}$$

$$\to \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}\right|_{x=x_0} \quad \text{as } \epsilon \to 0.$$

The i^{th} column of A is computed by the same device and we see that

$$Df(x_0) = A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}\right|_{x=x_0}$$

and, if $m > 1$ this is called the Jacobian matrix, while if $m = 1$, we have $Df(x_0) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$ which is as mentioned earlier called the gradient of f. Sometimes the Jacobian is denoted $J\left(\frac{f_1, f_2, \dots, f_m}{x_1, x_2, \dots, x_n}\right)\Big|_{x=x_0}$ and the gradient is denoted $\nabla f|_{x=x_0}$. The geometric significance of ∇f will be studied shortly.

The usual relationships between the differentiation operation and the algebraic properties of the function spaces hold. They are more complicated to prove than in the scalar cases (because one can not divide), but do still follow in a rather straight forward manner.

Theorem 4.1. *If f and g each map* $\mathbf{X}$ *into* $\mathbf{Y}$ *and k is a scalar, then* $D(f+g) = Df + Dg$ *and* $D(kf) = kDf$.

Proof of Theorem 4.1. *Straight forward computation.* ∎

Exercise 4.1.

(1). Suppose $g(x_1, x_2, x_3) = x_1^2 - 2x_1x_3 + \cos(x_2)$. Compute Dg.

(2). Suppose $f = \begin{pmatrix} f_1(x_1,x_2,x_3,x_4) \\ f_2(x_1,x_2,x_3,x_4) \\ f_3(x_1,x_2,x_3,x_4) \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2x_4 + x_3^2 \\ x_1^2 + x_2 - x_3 \\ x_2^2 - x_4^2 \end{pmatrix}$. Compute Df.

(3). Suppose $g : \mathbb{R}^m \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^p$, both being differentiable. Compute a formula for $D(f(g))$, i.e., obtain the form of the Chain Rule.

Remark. There are now various forms of products possible. The next exercises address some of these cases.

(4). Suppose $g : \mathbb{R}^n \to \mathbb{R}^1$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. Compute $D[g(x)f(x)]$.

(5). Suppose $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. Compute $D[g(x) \cdot f(x)]$.

(6). Suppose $g : \mathbb{R}^n \to \mathbb{R}^3$ and $f : \mathbb{R}^n \to \mathbb{R}^3$. Compute $D[g(x) \times f(x)]$.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^1$ and consider the set (x, y, z) such that $f(x, y, z) = k$ (a constant). Such a set is called a level set or surface. Consider a point $\mathbf{P}_0$ on the surface and a curve $C = (x(t), y(t), z(t))$ lying in the surface, i.e., $f(x(t), y(t), z(t)) = k$, and containing the point $\mathbf{P}_0$. The function $f(C(t)) = k$ maps $\mathbb{R}^1 \to \mathbb{R}^1$, but is the composition of $f : \mathbb{R}^3 \to \mathbb{R}^1$ and $C : \mathbb{R}^1 \to \mathbb{R}^3$. the chain rule (Exercise 4.1.(3)) tells us that

$$\frac{df(C(t))}{dt} = (Df)(DC)$$

where Df is a 1×3 matrix, namely $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \nabla f$ and DC is a 3×1 matrix, namely $\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$ so that

$$\frac{df(C(t))}{dt} = (Df)(DC) = (f_x, f_y, f_z)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

(This is sometimes called the directional derivative of f in the direction of C at the point $\mathbf{P}_0$.) but since $f(C(t)) \equiv k$ a constant, $\frac{df(C(t))}{dt} = 0$, i.e., $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (x'(t), y'(t), z'(t)) = 0$. This tells us that the vector ∇f is perpendicular to the vector $(x'(t), y'(t), z'(t))$ which is tangent to the curve $C(t)$. However, the curve $C(t)$ was arbitrarily selected, so that ∇f is orthogonal to the tangent plane to the surface at $\mathbf{P}_0$. This is just another way of indicating that ∇f is normal to the surface at $\mathbf{P}_0$. This gives us a way to write equations for tangent planes to surfaces at given points if the surface is described by

$$f(x, y, z) = k$$

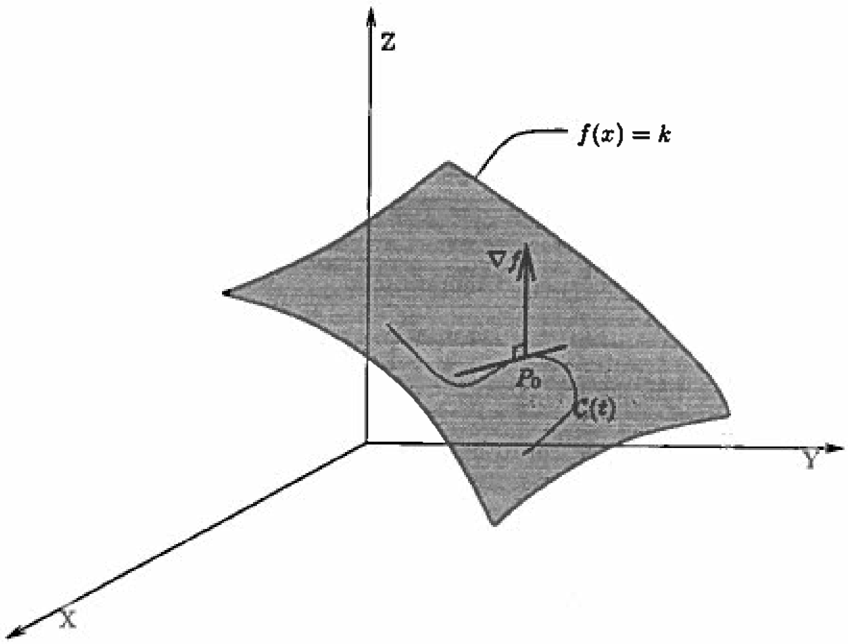

Figure 4.1:

Recall that a plane through $\mathbf{P}_0$ can be described as all points $\mathbf{X}$ such that

$$\mathbf{X} \cdot \eta = \mathbf{P}_0 \cdot \eta$$

and in this case $\eta = \nabla f|_{\mathbf{P}_0}$, i.e.,

$$(x,y,z) \cdot (f_x, f_y, f_z)|_{\mathbf{P}_0} = (x_0, y_0, z_0) \cdot (f_x, f_y, f_z)|_{\mathbf{P}_0} \text{ or } (x - x_0) f_x|_{\mathbf{P}_0} + (y - y_0) f_y|_{\mathbf{P}_0} + (z - z_0) f_z|_{\mathbf{P}_0} = 0.$$

As it happens, this aspect of ∇f has other interesting and useful properties. Returning to the directional derivative of f in the direction of the curve C at $\mathbf{P}_0$, namely

$$\begin{aligned} \frac{df(C(t))}{dt} &= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t) = \nabla f \cdot C'(t) \\ &= \|\nabla f\| \cdot \|C'(t)\| \cos(\phi) \end{aligned}$$

where ϕ is the angle between ∇f and the tangent direction of the curve C. Let's assume that $\|C'(t)\| = 1$ (we could alter the parameterization of the curve C and get this anyway), then

$$\frac{df}{dt} = \|\nabla f\| \cos(\phi)$$

and this is *maximum* when $\cos(\phi) = 1$, i.e., when $\phi = 0$; it is minimum when $\cos(\phi) = -1$, i.e., when $\phi = \pi$. UPSHOT: The direction of maximum rate of change of the function f is the direction of ∇f and the maximum rate of change is $\|\nabla f\|$; the minimum rate of change is $-\|\nabla f\|$ and is in the opposite direction.

Example 4.1.

Suppose $f(x,y,z) = x^2 - 3xy + z^2$ and $\mathbf{P}_0 = (1,1,1)$. Then $f(1,1,1) = -1$ and the level surface S of f which contains $\mathbf{P}_0$ is $x^2 - 3xy + z^2 = -1$. $\nabla f = (2x - 3y, -3x, 2z)$ and $\nabla f|_{\mathbf{P}_0} = (-1,-3,2)$ which is normal to S at $\mathbf{P}_0$ and the tangent plane at $\mathbf{P}_0$ is then

$$(x-1)(-1) + (y-1)(-3) + (z-1)(2) = 0.$$

The maximum rate of change of f at $\mathbf{P}_0$ is $\| \nabla f|_{\mathbf{P}_0} \| = \sqrt{14}$ in the direction $\frac{\nabla f}{\sqrt{14}}$ and the minimum rate of change of f at $\mathbf{P}_0$ is $-\| \nabla f|_{\mathbf{P}_0} \| = -\sqrt{14}$ in the direction $\frac{-\nabla f}{\sqrt{14}}$. Ω

We have thus seen that the ideas developed so far are useful for certain types of minimax problems. Let's look at yet another example of this type.

Suppose we are asked to find the extreme values of a function

$$u = f(x,y,z)$$

subject to the condition that the points (x,y,z) being considered are on the surface S:

$$g(x,y,z) = k \text{ (a constant).}$$

Assume that we have located such a point $\mathbf{P}_0$ and that it isn't a boundary point of the surface S, then at $\mathbf{P}_0$, ∇g is orthogonal to the surface S and along any curve $C = (x(t), y(t), z(t))$ on the surface S which contains $\mathbf{P}_0$, $\left.\frac{df(C(t))}{dt}\right|_{\mathbf{P}_0} = 0$ because we are at an extreme value of f, i.e., $\frac{df}{dt} = (\nabla f) \cdot (x', y', z') = 0$ and therefore ∇f is also normal to the surface S at $\mathbf{P}_0$, since the curve C is arbitrary.

This tells us that both ∇g and ∇f are normals to the surface S at $\mathbf{P}_0$ and therefore there must be some number λ such that

$$\nabla f|_{\mathbf{P}_0} = \lambda \nabla g|_{\mathbf{P}_0}, \text{i.e., } \left.\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right|_{\mathbf{P}_0} = \lambda \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) \text{ or } \begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \text{ and} \\ g(x,y,z) = k \end{cases}$$

must all hold at any extreme values of f on the surface S. This gives us four equations in the four unknowns x, y, z, λ which we need to solve in order to get the coordinates of the potential extreme values points.

In case we have, say *two* constraints, then we consider curves C which lie in *both* surfaces $\begin{cases} g_1(x,y,z) = k_1 \\ g_2(x,y,z) = k_2 \end{cases}$ and have ∇f orthogonal to the tangents to all such curves (but then so are ∇g_1, and ∇g_2) so that ∇f must lie in the vector subspace spanned by ∇g_1 and ∇g_2 (recall Exercise 3.10.(4)), that is, there are constants λ_1 and λ_2 such that

$$\begin{cases} \nabla f|_{\mathbf{P}_0} = \lambda_1 \nabla g_1|_{\mathbf{P}_0} + \lambda_2 \nabla g_2|_{\mathbf{P}_0} \text{ and} \\ g_1 = k_1 \\ g_2 = k_2 \end{cases}$$

and we again have as many equations as unknowns. The same argument works for any (integer) number of independent variables and constraints. We thus see that it is rather a naturally occuring idea to want (or need) to write one vector as a linear combination of other known vectors. We will pursue this further in the next chapter.

Example 4.2.
Find the extreme values of

$$u = f(x, y, z) = 2x - 3y + 4z$$

at points on the unit sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$. $\nabla f = (2, -3, 4)$ and $\nabla g = (2x, 2y, 2z)$ and thus

$$\begin{aligned} \nabla f &= \lambda \nabla g \text{ becomes} \\ 2 &= \lambda 2x \\ -3 &= \lambda 2y \\ 4 &= \lambda 2z. \end{aligned}$$

Squaring each side of these three equations and adding, we obtain

$$29 = \lambda^2 4(x^2 + y^2 + z^2) = \lambda^2 4 \text{ and therefore } \lambda = \frac{\pm\sqrt{29}}{2}.$$

If $\lambda = \frac{\sqrt{29}}{2}$, then we have $\mathbf{P}_0 = (x_0, y_0, z_0) = \frac{(2,-3,4)}{\sqrt{29}}$.
If $\lambda = \frac{-\sqrt{29}}{2}$, then we have $\mathbf{P}_1 = (x_1, y_1, z_1) = \frac{-(2,-3,4)}{\sqrt{29}}$.
There $u(\mathbf{P}_0) = f(x_0, y_0, z_0) = \sqrt{29}$ and $u(\mathbf{P}_1) = f(x_1, y_1, z_1) = -\sqrt{29}$.
Since these are the only candidates for extreme values, $u(\mathbf{P}_0)$ is the maximum value and $u(\mathbf{P}_1)$ is the minimum value.
Ω

Exercise 4.2.

(1). Find the points on the unit sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$ where $u = f(x, y, z) = x^2 + 2y^2 + z^2$ acheives maximum values and minimum values and find those values. CAUTION: There are infinitely many critical points!

(2). Find the points on the unit sphere where $u = f(x, y, z) = 2y - 3z + 4x$ achieves its extreme values and find those values.

(3). A manufacturer has total capital 1 (measured in millions of dollars) and expenses $x, y,$ and z, also measured in millions of dollars. It has been determined that the company's profits are given by $u = f(x, y, z) = x^3 + 3xy + 2z$. How should the 1 million dollars be allocated in order to maximize profits and what is the maximum profit?

NOMENCLATURE: The method just described is usually called the method of Lagrange multipliers, the function f whose extreme values are sought is called the utility function, the g's are called constraints and the λ's are called the Lagrange multipliers.

Chapter 5

Bases

Definition 5.1. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are vectors in some vector space $\mathbf{X}$. The symbols

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ or } [\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}]$$

denotes the set of all linear combinations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Theorem 5.1. Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a vector subspace of* $\mathbf{X}$.

Proof of Theorem 5.1. Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is closed under addition, and scalar multiplication, thus Theorem 1.3 completes the argument.* ∎

In as much as the combinations involve a fair amount of computation, it is natural to wonder whether one could reduce the work by dealing with fewer vectors, i.e., is it possible that one could throw out some of the **v**'s and have the same subspace for Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$?

If one of the vectors, say $\mathbf{v}_2$, is a linear combination of the other vectors, we could just throw that one out and still have enough to get the job done.

Definition 5.2. A set of non-Θ vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a linear space X are called *linearly dependent* if and only if one of them is a linear combination of the others.

Theorem 5.2. *A set* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is linearly dependent if and only if there exist constants* $c_1, c_2, \ldots, c_n$ *such that* not all *of the* c_i*'s are zero and*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \Theta.$$

Proof of Theorem 5.2. *Suppose* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *are dependent, then one of the* **v***'s, say* $\mathbf{v}_2$*, is a linear combination of the others, i.e.,* $\mathbf{v}_2 = c_1\mathbf{v}_1 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$ *and thus* $c_1\mathbf{v}_1 - 1 \cdot \mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n = \Theta$ *and* $(-1) \neq 0$*. The converse is also trivial.* ∎

Definition 5.3. A set of non-Θ vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space $\mathbf{X}$ is called *linearly independent* if and only if it is not linearly dependent.

Corollary 5.3. *A set* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is linearly independent if and only if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \Theta$$

implies that all of the c_i *'s are zero.*

Definition 5.4.

(1). A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space $\mathbf{X}$ is called a basis for $\mathbf{X}$ if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent and $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is $\mathbf{X}$.

(2). A vector space $\mathbf{X}$ is said to be *finite* dimensional if and only if it has a basis which is a finite set of vectors. It is called infinite dimensional otherwise.

(3). The dimension of a vector space is the number of vectors in a basis for the space.

Remark. Part (3) of Definition 5.4 requires that we prove that

Theorem 5.4. *All finite bases for a given vector space have the same number of elements.*

Proof of Theorem 5.4. *Suppose* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots, \mathbf{x}_m\}$ *is another set of vectors in* $\mathbf{X}$, $m > n$, *then*

$$\begin{aligned}
\mathbf{x}_1 &= c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + \cdots + c_{1n}\mathbf{v}_n \\
\mathbf{x}_2 &= c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{2n}\mathbf{v}_n \\
&\vdots \\
\mathbf{x}_n &= c_{n1}\mathbf{v}_1 + c_{n2}\mathbf{v}_2 + \cdots + c_{nn}\mathbf{v}_n \\
&\vdots \\
\mathbf{x}_m &= c_{m1}\mathbf{v}_1 + c_{m2}\mathbf{v}_2 + \cdots + c_{mn}\mathbf{v}_n.
\end{aligned}$$

Now suppose that

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n + \cdots + k_m\mathbf{x}_m = \Theta,$$

i.e.,

$$k_1(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + \cdots + c_{1n}\mathbf{v}_n) + \cdots + k_m(c_{m1}\mathbf{v}_1 + c_{m2}\mathbf{v}_2 + \cdots + c_{mn}\mathbf{v}_n) = \Theta$$

or

$$(k_1c_{11} + k_2c_{21} + \cdots + k_mc_{m1})\mathbf{v}_1 + \cdots + (k_1c_{1n} + k_2c_{2n} + \cdots + k_mc_{mn})\mathbf{v}_n = \Theta$$

and since the $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *set is linearly independent, Corollary 5.3 says that the coefficients are all zero, i.e.,*

$$\begin{aligned}
k_1c_{11} + k_2c_{21} + \cdots + k_mc_{m1} &= 0 \\
k_1c_{12} + k_2c_{22} + \cdots + k_mc_{m2} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
k_1c_{1n} + k_2c_{2n} + \cdots + k_mc_{mn} &= 0
\end{aligned}
\quad \text{or} \quad
\begin{pmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{pmatrix}
\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \\ \vdots \\ k_m \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $m > n$, *the matrix of* c_{ij} *'s is singular and thus the kernel is not identically* Θ, *i.e., the matrix-vector equation has a non-trivial solution and thus the set* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots, \mathbf{x}_m\}$ *is linearly dependent.* ∎

Corollary 5.5. *If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m\}$ each span the same subspace, the larger set is linearly dependent.*

Corollary 5.6. *Any two bases of a finite dimensional vector space have the same number of elements.*

Corollary 5.7. *The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are of dimension n.*

Proof of Corollary 5.7. *Count the elements of the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$.* ∎

Corollary 5.8. *If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent subset of a vector space $\mathbf{X}$ of dimension n, then it is a basis for $\mathbf{X}$.*

Corollary 5.9. *If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set which spans an n-dimensional vector space $\mathbf{X}$, then it is a basis for $\mathbf{X}$.*

Corollary 5.10. *If $r < n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a linearly independent set in an n-dimensional vector space $\mathbf{X}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ can be extended to a basis for $\mathbf{X}$.*

Lemma 5.1. *If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_h\}$ is a set of linearly independent vectors and $\mathbf{v} \notin \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r, \mathbf{v}\}$ is linearly independent.*

Proof of Lemma 5.1. *Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r, \mathbf{v}\}$ is linearly dependent. Then there exist $\alpha_0 \mathbf{v} + c_1 \mathbf{x}_1 + \cdots + c_r \mathbf{x}_r = \Theta$. If $\alpha_0 \neq 0$, then $\mathbf{v} \in \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ which is false, thus $\alpha_0 = 0$ and therefore $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_r \mathbf{x}_r = \Theta$, the c_i's are not all zero, thus $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ is linearly dependent, also false. Thus it is* not *true that $\{\mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ is linearly dependent.* ∎

Proof of Corollary 5.10. $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ *is not all of $\mathbf{X}$ since $r < n$. Choose a vector $\mathbf{v}$ in $\mathbf{X}$ not in the span of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r, \mathbf{v}\}$ is linearly independent. Repeat the procedure until n vectors are collected.* ∎

Definition 5.5.

(1). Suppose A is an $m \times n$ matrix. The vectors which form the rows of A are called row vectors of A and the columns are called column vectors of A.

(2). The subspace spanned by the row vectors of A is called the row space of A and the subspace spanned by the columns of A is called the column space of A. The row space is in $\mathbb{R}^n$, the column space is in $\mathbb{R}^m$.

Remark. This definition reflects back on Exercises 2.5.(5)-(11). We will now expand upon your discoveries.

Theorem 5.11. *Given a matrix A, elementary row operations give rise to a new matrix with the same row space as the matrix A. Similar remarks hold for columns,* mutatis mutandis

Proof of Theorem 5.11. *Just do it.* ∎

Theorem 5.12. *If A is an $m \times n$ matrix, then the row space and the column space have the same dimension.*

Proof of Theorem 5.12.

$$\text{Write } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and suppose that } \begin{array}{rcl} \mathbf{b}_1 & = & (b_{11}, b_{12}, \dots, b_{1n}) \\ \mathbf{b}_2 & = & (b_{21}, b_{22}, \dots, b_{2n}) \\ \vdots & \vdots & \dots\dots\dots\dots \\ \mathbf{b}_k & = & (b_{k1}, b_{k2}, \dots, b_{kn}) \end{array}$$

is a basis for the row space of A (of dimension k), then the rows of A can be written

$$\begin{aligned} r_1 &= (a_{11}, a_{12}, \dots, a_{1n}) = c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 + \cdots + c_{1k}\mathbf{b}_k \\ &= c_{11}(\mathbf{b}_{11}, \mathbf{b}_{12}, \dots, \mathbf{b}_{1n}) + c_{12}(\mathbf{b}_{21}, \mathbf{b}_{22}, \dots, \mathbf{b}_{2n}) + \cdots + c_{1k}(\mathbf{b}_{k1}, \mathbf{b}_{k2}, \dots, \mathbf{b}_{kn}) \end{aligned}$$

so that

$$\begin{aligned} a_{11} &= c_{11}\mathbf{b}_{11} + c_{12}\mathbf{b}_{21} + \cdots + c_{1k}\mathbf{b}_{k1} \\ a_{12} &= c_{11}\mathbf{b}_{12} + c_{12}\mathbf{b}_{22} + \cdots + c_{1k}\mathbf{b}_{k2} \\ \vdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n} &= c_{11}\mathbf{b}_{1n} + c_{12}\mathbf{b}_{2n} + \cdots + c_{1k}\mathbf{b}_{kn} \end{aligned}$$

and thus, from column i and the first row of A we may write $a_{1i} = c_{11}\mathbf{b}_{1i} + c_{12}\mathbf{b}_{2i} + \cdots + c_{1k}\mathbf{b}_{ki}$ in general we have from column i of A, going down one row at a time

$$\begin{aligned} a_{1i} &= c_{11}\mathbf{b}_{1i} + c_{12}\mathbf{b}_{2i} + \cdots + c_{1k}\mathbf{b}_{ki} \\ a_{2i} &= c_{21}\mathbf{b}_{1i} + c_{22}\mathbf{b}_{2i} + \cdots + c_{2k}\mathbf{b}_{ki} \\ \vdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{ni} &= c_{n1}\mathbf{b}_{1i} + c_{n2}\mathbf{b}_{2i} + \cdots + c_{nk}\mathbf{b}_{ki} \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = \mathbf{b}_{1i}\begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{pmatrix} + \mathbf{b}_{2i}\begin{pmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{pmatrix} + \cdots + \mathbf{b}_{ki}\begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{nk} \end{pmatrix}$$

$$= \mathbf{b}_{1i}\mathbf{C}_1 + \mathbf{b}_{2i}\mathbf{C}_2 + \cdots + \mathbf{b}_{ki}\mathbf{C}_k$$

and thus the vectors $\{\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k\}$ form a set which spans the column space of A, therefore the dimension of the column space is less than or equal k, the dimension of the row space.

The same argument, starting with columns, gives the reverse inequality and thus we have that the row space and the column space have the same dimension. ∎

Definition 5.6. The dimension of the row (column) space of a matrix A is called the rank of A.

Remark. It follows from Theorem 5.11 that if A is reduced to row echelon form, then the non-zero rows of the reduced matrix span the row space and thus Theorem 5.12 shows that the number of non-zero rows is the rank of A.

We can now add to our list of equivalent statements about the invertibility of an $n \times n$ matrix A. See Theorem 2.18 and Corollary 2.22.

Theorem 5.13. *If A is an $n \times n$ matrix, then the following statements are equivalent:*

(1). A is invertible

(2). $A\mathbf{x} = \Theta$ has only the zero solution ($\ker(A) \equiv \{\Theta\}$)

(3). A is row equivalent to I_n

(4). $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n (\mathbb{C}^n)$

(5). A has rank n

(6). $\det(A) \neq 0$

(7). The row vectors of A are linearly independent

(8). The column vectors of A are linearly independent.

Now suppose A is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m\,(\mathbb{C}^m)$ and consider the equation $A\mathbf{x} = \mathbf{b}$,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

which we can rewrite as

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

thus we may obtain a solution if and only if the vector $\mathbf{b}$ is in the column space of the matrix A.

Theorem 5.14. *The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A.*

In the same manner we can show that the equation $\mathbf{x}^T A = \mathbf{b}^T$ has a solution if and only if $\mathbf{b}^T$ is in the row space of A.

Finding a New or Better Basis

It is often the case that we have one basis for a vector space, but find that another may suit our purposes better. The word better here almost always has computational overtones and frequently is realized only after some algebraic or geometric insight is obtained. In order to make these remarks have more meaning, we need to look more closely at where we have already been in order to have some notion as to where we might go next. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ which we have used thus far has properties, not yet probed, which have contributed much to our past successes. Let's probe. Theorem 3.1 serves as a motivation.

Definition 5.7. Suppose $\mathbf{X}$ is a (real or) complex vector space. An inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{X}$ is a (real or) complex valued function on $\mathbf{X} \times \mathbf{X}$ such that

(1). $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

(2). $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

(3). $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$

(4). $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $x = \Theta$.

Note.

(i). $\langle \mathbf{x}, \Theta \rangle = \langle \mathbf{x}, 0 \cdot \mathbf{x} \rangle = \overline{\langle 0 \cdot \mathbf{x}, \mathbf{x} \rangle} = \overline{0 \langle \mathbf{x}, \mathbf{x} \rangle} = 0$

(ii). $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \overline{\langle \alpha\mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha \langle \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha} \cdot \overline{\langle \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$

(iii). $\langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle = \overline{\langle \mathbf{y}+\mathbf{z}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.

Thus while the inner product is linear in the first variable, $\langle \alpha\mathbf{x}+\beta\mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$, it is conjugate linear in the second variable, i.e.,

$$\langle \mathbf{x}, \alpha\mathbf{y}+\beta\mathbf{z} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle + \overline{\beta} \langle \mathbf{x}, \mathbf{z} \rangle .$$

In case $\mathbf{X}$ is a real vector space, the conjugates may be ignored.

(iv). The dot product on $\mathbb{R}^n$ and on $\mathbb{C}^n$ is an inner product.

(v). Suppose $\mathbf{X}$ is the set of complex valued continuous functions on the real interval $[a,b]$, with scalar field $\mathbb{C}$ and the usual operations. $\langle f, g \rangle = \int_a^b f(x)\overline{g}(x)dx$ is an inner product on $\mathbf{X}$. If $\rho(x) > 0$ on $[a,b]$, then $\langle f, g \rangle_\rho = \int_a^b f(x)\overline{g}(x)\rho(x)dx$ is also an inner product on $\mathbf{X}$. The function ρ is called a weight function for the inner product.

Definition 5.8. A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space $\mathbf{X}$ which has an inner product $\langle \cdot, \cdot \rangle$ is called an orthonormal system in $\mathbf{X}$ if and only if

(1). $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ (normalized) and

(2). $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$ (orthogonal).

This is usually written as $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$ where δ_{ij} is called the Kronecker delta function and has the value 1 if $i = j$ and the value 0 if $i \neq j$. We abbreviate this by saying the system is an ONS.

Exercise 5.1. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an ONS in both $\mathbb{R}^n$ and $\mathbb{C}^n$.

Theorem 5.15. *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an ONS and* $\mathbf{x} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, *then*

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i .$$

Proof of Theorem 5.15. *Write* $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, *then* $\langle \mathbf{x}, \mathbf{v}_i \rangle = c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \cdots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle$ *and all terms on the right are zero save the* i^{th}, *so* $\langle \mathbf{x}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i$. ∎

Definition 5.9. The coefficients $\langle \mathbf{x}, \mathbf{v}_i \rangle$ are called the Fourier coefficients of x relative to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Exercise 5.2.

(1). Show that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then for the ONS $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, $\langle \mathbf{x}, \mathbf{e}_i \rangle = x_i$.

(2). Suppose $\mathbf{X}$ is the vector space of continuous real (or complex) valued functions on $[0, 2\pi]$ with $\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g}(x)dx$. Show that $\{\sin(x), \sin(2x), \ldots\} = \mathbf{B}_1$ is an ONS in $\mathbf{X}$.

(3). Show that $\mathbf{B}_2 = \{1, \cos(x), \cos(2x), \ldots\}$ is an ONS in the same $\mathbf{X}$.

(4). Show that $\mathbf{B}_1 \cup \mathbf{B}_2 = \{1, \cos(x), \sin(x), \cos(2x), \ldots\}$ is an ONS in the same $\mathbf{X}$, i.e.,

$$\int_0^{2\pi} \sin(mx)\cos(nx)dx = 0 \text{ for all } m \text{ and } n.$$

(5). Use the results of Theorem 5.13 to show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ forms a basis in $\mathbb{R}^n$ ($\mathbb{C}^n$) if and only if the matrix A whose columns are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is invertible, i.e., $\det(A) \neq 0$.

(6). Use the results of Theorem 5.13 again to show that if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ ($\mathbb{C}^n$), then it can be changed, reversibly, into the system $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ which is an ONS; i.e., every basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ ($\mathbb{C}^n$) can be transformed into an ONS on $\mathbb{R}^n$ ($\mathbb{C}^n$), indeed into the standard $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ basis.

The result in Exercise 5.2.(6) raises the question as to whether one can do a similar thing in every inner product space. To answer this question in the affirmative is the burden of our next efforts. The procedure we follow is usually called the Gram-Schmidt process. The idea is to begin with a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$, hopefully a basis and from this set construct a new set, say $\{\mathbf{x}_1, \mathbf{x}_2, \dots\}$ which is an ONS and which has the same span as $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$. The method is based upon the idea of "orthogonal projections" which played a prominent role in Chapter 3.

Theorem 5.16. *Suppose* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ *is an ONS in an inner product space* $\mathbf{X}$ *and* $\mathbf{Y} = \operatorname{Span}\{\mathbf{x}_i\}_{i=1}^n$. *Suppose* $\mathbf{u} \in \mathbf{X}$ *and* $u \notin \mathbf{Y}$, *then* u *may be written as* $\mathbf{u} = \mathbf{y}_1 + \mathbf{y}_2$ *where* $\mathbf{y}_1 \in \mathbf{Y}$ *and* $\mathbf{y}_2$ *is orthogonal to every vector in* $\mathbf{Y}$. *(This is written* $\mathbf{y}_2 \perp \mathbf{Y}$.*) Moreover,* $\mathbf{y}_1 = \sum_{i=1}^n \langle \mathbf{u}, \mathbf{x}_i \rangle \mathbf{x}_i \in \mathbf{Y}$ *and* $\mathbf{y}_2 = \mathbf{u} - \mathbf{y}_1$. *This representation is unique.*

Proof of Theorem 5.16. *It is clear that* $\mathbf{y}_1$ *is in* $\mathbf{Y}$ *since it is a linear combination of* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ *and that with* $\mathbf{y}_2 = \mathbf{u} - \mathbf{y}_1$, $\mathbf{u} = \mathbf{y}_1 + \mathbf{y}_2$. *At issue then is whether* $\mathbf{y}_2 = \mathbf{u} - \mathbf{y}_1$ *is orthogonal to* $\mathbf{Y}$. *Suppose* $\mathbf{y} = \sum_{j=1}^n c_j \mathbf{x}_j$ *and consider*

$$\begin{aligned}
\langle \mathbf{y}_2, \mathbf{y} \rangle &= \left\langle \mathbf{u} - \sum_{i=1}^n \langle \mathbf{u}, \mathbf{x}_i \rangle \mathbf{x}_i, \sum_{j=1}^n c_j \mathbf{x}_j \right\rangle \\
&= \left\langle \mathbf{u}, \sum_{j=1}^n c_j \mathbf{x}_j \right\rangle - \sum_{i=1}^n \sum_{j=1}^n \langle \mathbf{u}, \mathbf{x}_i \rangle \bar{c}_j \langle \mathbf{x}_i, \mathbf{x}_j \rangle \\
&= \left\langle \mathbf{u}, \sum_{j=1}^n c_j \mathbf{x}_j \right\rangle - \sum_{j=1}^n \langle \mathbf{u}, \mathbf{x}_j \rangle \bar{c}_j \\
&= \left\langle \mathbf{u}, \sum_{j=1}^n c_j \mathbf{x}_j \right\rangle - \left\langle \mathbf{u}, \sum_{j=1}^n c_j \mathbf{x}_j \right\rangle = 0.
\end{aligned}$$

Suppose there are two such decompositions, i.e., $\mathbf{u} = \mathbf{y}_1 + \mathbf{y}_2 = \mathbf{z}_1 + \mathbf{z}_2$ *where* $\mathbf{y}_1$ *and* $\mathbf{z}_1$ *are in* $\mathbf{Y}$ *and* $\mathbf{z}_2$ *and* $\mathbf{y}_2$ *are orthogonal to* $\mathbf{Y}$. *Then* $\mathbf{y}_1 - \mathbf{z}_1 = \mathbf{z}_2 - \mathbf{y}_2$ *and the left side is in* $\mathbf{Y}$ *and the right side is orthogonal to* $\mathbf{Y}$, *therefore* $(\mathbf{y}_1 - \mathbf{z}_1) \perp (\mathbf{z}_2 - \mathbf{y}_2)$, but, *since* $(\mathbf{y}_1 - \mathbf{z}_1) = (\mathbf{z}_2 - \mathbf{y}_2)$, *this says that the inner product* $\langle \mathbf{y}_1 - \mathbf{z}_1, \mathbf{y}_1 - \mathbf{z}_1 \rangle = 0$ *and this by Definition 5.7.(4) says that* $\mathbf{y}_1 - \mathbf{z}_1 = \Theta = \mathbf{z}_2 - \mathbf{y}_2$ *which implies that* $\mathbf{y}_1 = \mathbf{z}_1$ *and* $\mathbf{y}_2 = \mathbf{z}_2$. ∎

Definition 5.10. Suppose $\mathbf{M}$ is a set in an inner product space $\mathbf{X}$. The orthogonal complement $\mathbf{M}^\perp$ of $\mathbf{M}$ is the set of all vectors $\mathbf{x}$ such that $\langle \mathbf{x}, \mathbf{m} \rangle = 0$ for each vector $\mathbf{m} \in \mathbf{M}$.

Exercise 5.3. Show that $\mathbf{M}^\perp$ is a subspace of $\mathbf{X}$. *Hint:* Check Theorem 1.2.

Remark. Theorem 5.16 now says that every $\mathbf{u} \in \mathbf{X}$ has a unique decomposition $\mathbf{u} = \mathbf{y}_1 + \mathbf{y}_2$ where $\mathbf{y}_1 \in \mathbf{Y}$ and $\mathbf{y}_2 \in \mathbf{Y}^\perp$.

Definition 5.11. Suppose $\mathbf{X}$ is an inner product space, $\mathbf{Y} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ where $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an ONS, then in the unique decomposition $\mathbf{u} = \mathbf{y}_1 + \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathbf{Y}$ and $\mathbf{y}_2 \in \mathbf{Y}^{\perp}$, the vector $\mathbf{y}_1 = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{x}_i \rangle \mathbf{x}_i$ is called the orthogonal projection of $\mathbf{u}$ onto $\mathbf{Y}$. It is denoted $\text{proj}_{\mathbf{Y}}(\mathbf{u})$.

Remark.

(1). This should be compared with the formula $\text{proj}_B(A) = \frac{A \bullet B}{\|B\|^2} \cdot B = \langle A, B \rangle \frac{B}{\|B\|^2}$ where $\frac{B}{B \bullet B}$ is in the role of $\frac{\mathbf{x}_i}{\langle \mathbf{x}_i, \mathbf{x}_i \rangle}$ *except*, recall that $\langle \mathbf{x}_i, \mathbf{x}_i \rangle = 1$ in the above case since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an ONS.

(2). Reread the remark following Exercise 3.9.(2) on page 49. It would be very interesting, if true, that $\text{proj}_{\mathbf{Y}}(\mathbf{u})$ is the best approximation to $\mathbf{u}$ from the space $\mathbf{Y}$ in the inner product space $\mathbf{X}$. In order for this to have meaning, best approximation is to mean that the vector $(\mathbf{u} - \text{proj}_{\mathbf{Y}}(\mathbf{u})) = \mathbf{y}_2$ is of minimal size, whatever that means. We need of course to have some sort of "norm" on the inner product space $\mathbf{X}$ for this to be meaningful, where norm is some sort of generalization of $\|A\|^2 = A \bullet A$ in an inner product space.

Definition 5.12. Suppose $\mathbf{X}$ is a vector space. The statement that $\|\cdot\|$ is a norm on $\mathbf{X}$ means that $\|\cdot\|$ is a function from $\mathbf{X}$ into the non-negative reals such that

(1). $\|\mathbf{x}\| \geq 0$; $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \Theta$

(2). $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$ and

(3). $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Theorem 5.17. *Suppose* $\mathbf{X}$ *is an inner product space and* $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$, *then*

(1). $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|$

(2). $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

(3). $\|\mathbf{x}\| \leq \|\mathbf{x} + \lambda\mathbf{y}\|$ *for every* $\lambda \in \mathbb{C}$ *if and only if* $\mathbf{x} \perp \mathbf{y}$, *i.e.,* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Remark. The inequality (1) is called the Cauchy-Schwarz-Bunyakowski inequality. We will denote it simply C-S. Statement (2) says that $\|\cdot\|$ satisfies the triangle inequality (3) in Definition 5.12. Clearly $\|\alpha\mathbf{x}\|^2 = \langle \alpha\mathbf{x}, \alpha\mathbf{x} \rangle = |\alpha|^2 \langle \mathbf{x}, \mathbf{x} \rangle = |\alpha|^2 \|\mathbf{x}\|^2$ and thus (2) of Definition 5.12 also holds. Thus $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ defines a norm on the inner product space $\mathbf{X}$.

Proof of Theorem 5.17. *Suppose* $\lambda \in \mathbb{C}$ *and consider*

$$0 \leq \langle \mathbf{x} + \lambda\mathbf{y}, \mathbf{x} + \lambda\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \lambda\mathbf{y} \rangle + \langle \lambda\mathbf{y}, \mathbf{x} \rangle + \langle \lambda\mathbf{y}, \lambda\mathbf{y} \rangle$$

or

$$\begin{aligned} 0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + \overline{\lambda} \langle \mathbf{x}, \mathbf{y} \rangle + \lambda \overline{\langle \mathbf{x}, \mathbf{y} \rangle} + |\lambda|^2 + \|\mathbf{y}\|^2 \\ &= \|\mathbf{x}\|^2 + 2Re(\lambda \langle \mathbf{x}, \mathbf{y} \rangle) + |\lambda|^2 \|\mathbf{y}\|^2 \end{aligned}$$

set $\alpha = \langle \mathbf{x}, \mathbf{y} \rangle$ *then this becomes*

$$0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + Re(\lambda\alpha) + |\lambda|^2 \|\mathbf{y}\|^2. \tag{5.1}$$

Now assume $\mathbf{x} \perp \mathbf{y}$, *i.e.,* $\alpha = 0$, *then we have* $0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + |\lambda|^2 \|\mathbf{y}\|^2 \geq \|\mathbf{x}\|^2$ *so that (3) holds* if $\mathbf{y} \perp \mathbf{x}$. *Then part (1) becomes* $0 \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ *and is trivial.*

Note that if $\mathbf{y} = \Theta$, *then both sides of C-S are zero and (1) holds trivially. In this same case (2) becomes* $\|\mathbf{x}\| \leq \|\mathbf{x}\|$, *also trivial. Part (3) is the same.*

Now suppose $\mathbf{y} \neq \Theta$ *and* $\alpha \neq 0$. *Set* $\lambda = \dfrac{-\overline{\alpha}}{\|\mathbf{y}\|^2}$, *then (5.1) becomes*

$$0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2Re\left(\frac{-\alpha\overline{\alpha}}{\|\mathbf{y}\|^2}\right) + \left|\frac{-\overline{\alpha}}{\|\mathbf{y}\|^2}\right| \cdot \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\frac{|\alpha|^2}{\|\mathbf{y}\|^2} + \frac{|\alpha|^2}{\|\mathbf{y}\|^2}$$

$$\text{or } 0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \frac{|\alpha|^2}{\|\mathbf{y}\|^2}, \tag{5.2}$$

thus, $\dfrac{|\alpha|^2}{\|\mathbf{y}\|^2} \leq \|\mathbf{x}\|^2$ *or* $|\langle \mathbf{x}, \mathbf{y}\rangle|^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2$ *from which (1) follows at once. Using this in (5.1) gives*

$$\begin{aligned} 0 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2Re(\lambda\langle \mathbf{x}, \mathbf{y}\rangle) + |\lambda|^2 \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2|\lambda| \|\mathbf{x}\| \cdot \|\mathbf{y}\| + |\lambda|^2 \|\mathbf{y}\|^2 = [\|\mathbf{x}\| + |\lambda| \cdot \|\mathbf{y}\|]^2 \end{aligned}$$

from which (2) follows for $\lambda = 1$. *If we return to (5.2) we see that for* $\alpha \neq 0$, $\|\mathbf{x} + \lambda\mathbf{y}\|^2 > \|\mathbf{x}\|^2$ *and (3) fails, thus (3) holds only if* $\alpha = 0$, *i.e.,* $\mathbf{x} \perp \mathbf{y}$. ∎

Remark.

(1). For real vector spaces $\mathbf{X}$, (5.1) becomes $0 \leq \lambda^2\|\mathbf{y}\|^2 + 2\lambda\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{x}\|^2$ and this requires that the quadratic have non-positive discriminant. This is exactly what Theorem 5.17.(1) asserts.

(2). Theorem 5.17.(3) gives an affirmative response to the question asked concerning best approximation, but only for a very limitied case. Suppose our ONS contains only the vector $\mathbf{y}$ and then $\mathbf{Y} = \mathrm{Span}\{\mathbf{y}\}$. Suppose $\mathbf{x} \notin \mathbf{Y}$ and $\mathbf{u} = \mathbf{x} + \lambda\mathbf{y}$, then $\mathbf{u} - \lambda\mathbf{y} = \mathbf{x}$ and in order that $\mathbf{x}$ be of minimum norm among all such vectors $\{\mathbf{u} + \lambda\mathbf{y}; \lambda \in \mathbb{C}\}$, we must have $\mathbf{x} \perp \mathbf{y}$ and if $\mathbf{x} \perp \mathbf{y}$, then it *is* of minimum norm. The problem is that this answers the question only for the case that $\mathbf{Y}$ is one dimensional. The next theorem takes care of the arbitrary finite dimensional case.

Exercise 5.4.

(1). Show that if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ then $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. You will no doubt recognize this as the Pythagorean Theorem and the proof is easy.

(2). Show that in an inner product space $\mathbf{X}$ if $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, then $\mathbb{R}(\langle \mathbf{x}, \mathbf{y}\rangle) = 0$, thus in a real inner product space the Pythagorean Theorem holds exactly. Is it true in a complex inner product space?

(3). Develop (find) a formula for $\langle \mathbf{x}, \mathbf{y}\rangle$ in terms of $\|\cdot\|$.

Theorem 5.18. *(Projection Theorem). Suppose* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ *is an ONS in an inner product space* $\mathbf{X}$ *and* $\mathbf{Y} = \mathrm{Span}\{\mathbf{x}_i\}_{i=1}^n$. *If* $\mathbf{u} \in \mathbf{X}$ *and* $\mathbf{u} \notin \mathbf{Y}$, *then* $\mathrm{proj}_{\mathbf{Y}}(\mathbf{u})$ *is the point in* $\mathbf{Y}$ *which is nearest to the point* $\mathbf{u}$, *i.e.,*

$$\|\mathbf{u} - \mathrm{proj}_{\mathbf{Y}}(\mathbf{u})\| < \|\mathbf{u} - \mathbf{y}\|$$

for all $\mathbf{y} \in \mathbf{Y}$, $\mathbf{y} \neq \mathrm{proj}_{\mathbf{Y}}(\mathbf{u})$.

Proof of Theorem 5.18. *From Theorem 5.16, we know that* $\mathbf{u}$ *has a unique representation* $\mathbf{u} = \text{proj}_\mathbf{Y}(\mathbf{u}) + \mathbf{y}^\perp$ *where* $\mathbf{y}^\perp$; *i.e.,* $(\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) \in \mathbf{Y}^\perp$. *Now suppose* $\mathbf{y} \in \mathbf{Y}$ *(*anywhere *in* $\mathbf{Y}$*) then* $\mathbf{u} - \mathbf{y} = (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) + (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y})$, *then* $(\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) \in \mathbf{Y}^\perp$ *and* $(\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}) \in \mathbf{Y}$, *since the decomposition is unique.*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{y}\|^2 &= \langle \mathbf{u} - \mathbf{y}, \mathbf{u} - \mathbf{y} \rangle = \langle (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) + (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}), (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) + (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}) \rangle \\
&= \langle (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})), (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) \rangle + \langle (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})), (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}) \rangle \\
&\quad + \langle (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}), (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})) \rangle + \langle (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}), (\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}) \rangle .
\end{aligned}$$

The second and third terms on the right side are zero since $(\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}) \perp (\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u}))$, *thus we have*

$$\|\mathbf{u} - \mathbf{y}\|^2 = \|\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})\|^2 + \|\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}\|^2$$

Thus $\|\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})\|^2 \leq \|\mathbf{u} - \mathbf{y}\|^2$ *with* $=$ *holding only when* $\|\text{proj}_\mathbf{Y}(\mathbf{u}) - \mathbf{y}\| = 0$. *Thus* $\|\mathbf{u} - \text{proj}_\mathbf{Y}(\mathbf{u})\|$ *is minimum among all vectors of the form* $(\mathbf{u} - \mathbf{y})$. ∎

The Gram-Schmidt Process

Theorem 5.19. *Suppose* $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *is an ONS in an inner product space* $\mathbf{X}$, *then* $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *is linearly independent.*

Proof of Theorem 5.19. *Suppose* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \Theta$, *then by Theorem 5,* $c_i = \langle \Theta, \mathbf{v}_i \rangle = 0$, *thus by Corollary 5.3, the result follows.* ∎

Corollary 5.20. *If* $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *is an ONS, then it is a basis for* $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

Suppose now that we have a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ for a subspace $\mathbf{Y}$ of an inner product space $\mathbf{X}$. We desire to construct a new basis for $\mathbf{Y}$ which is an ONS because it is much easier to compute coefficients, near points, etc. with an ONS. In order to be certain that we do not stray outside the subspace $\mathbf{Y}$, we will use only linear combinations of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

Now since $\mathbf{v}_1$ is a perfectly nice vector but not necessarily normal, we will normalize it and use that vector

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

as the first element in our new ONS. Clearly $\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 1$. We need a new vector $\mathbf{u}_2$ which is such that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0, \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = 1$ and such that $\{\mathbf{u}_1, \mathbf{u}_2\}$ has the same span as $\{\mathbf{v}_1, \mathbf{v}_2\}$; this last part is so we don't stray. The Projection Theorems 5.16 and 5.18 tell us the way.

Project $\mathbf{v}_2$ onto $\text{Span}\{\mathbf{u}_1\}$ and we get

$$\begin{aligned}
\mathbf{v}_2 &= \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 + (\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \cdot \mathbf{u}_1) \\
&= \mathbf{y}_1 + \mathbf{y}_2
\end{aligned}$$

where $\mathbf{y}_1 \in \text{Span}\{\mathbf{u}_1\}$ and $\mathbf{y}_2 \in [\text{Span}\{\mathbf{u}_1\}]^\perp$. Now we normalize $\mathbf{y}_2$ and call that $\mathbf{u}_2$, i.e.,

$$\mathbf{u}_2 = \frac{(\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1)}{\|\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1\|}.$$

We now have $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ and $\langle \mathbf{u}_2, \mathbf{u}_2 \rangle = 1$. Continue the process by projecting $\mathbf{v}_3$ onto the $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ to get

$$\mathbf{u}_3 = \frac{\mathbf{v}_3 - [\langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2]}{\|\mathbf{v}_3 - [\langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2]\|}$$

and finally

$$\mathbf{u}_n = \frac{\mathbf{v}_n - \sum_{i=1}^{n-1} \langle \mathbf{v}_n, \mathbf{u}_i \rangle \mathbf{u}_i}{\|\mathbf{v}_n - \sum_{i=1}^{n-1} \langle \mathbf{v}_n, \mathbf{u}_i \rangle \mathbf{u}_i\|},$$

and the set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an ONS with the same span as $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, i.e., it is a basis for $\mathbf{Y}$ which is an ONS.

We now have the following result.

Theorem 5.21. *Every finite dimensional vector space with an inner product has an ONS for a basis.*

CAUTION: If in the Gram-Schmidt process we permute the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ before we start we, in general, do not get the same ONS we would have gotten if we had not permuted them. We lower our risk if we consider a basis as being ordered.

Exercise 5.5.

(1). Set $\mathbf{v}_1 = (1,0,1)$ and $\mathbf{v}_2 = (1,1,0)$. Set $\mathbf{Y} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Use the Gram-Schmidt process starting with $\mathbf{v}_1$ to obtain an ONS $\{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathbf{Y}$.

(2). Same setup as part (1) but start with $\mathbf{v}_2$ instead of $\mathbf{v}_1$ and obtain an ONS for $\mathbf{Y}$, say $\{\mathbf{x}_1, \mathbf{x}_2\}$.

(3). Compute the orthogonal projection of $\mathbf{e}_1$, i.e., $\text{proj}_{\mathbf{Y}}(\mathbf{e}_1)$ using $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(4). Compute $\text{proj}_{\mathbf{Y}}(\mathbf{e}_1)$ using $\{\mathbf{x}_1, \mathbf{x}_2\}$.

(5). Use the results of part (3) to compute a third vector $\mathbf{u}_3$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an ONS, hence a basis for $\mathbb{R}^3$.

(6). Use the results in part (4) to compute a third vector $\mathbf{x}_3$ so that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an ONS, hence a basis for $\mathbb{R}^3$.

(7). Explain why $\mathbf{u}_3 = \mathbf{x}_3$. Should you have expected this?

(8). What is the distance from $\mathbf{e}_1$ to $\mathbf{Y}$?

(9). Suppose $\mathbf{X}$ is $\mathbf{P}_2$ (second degree or less polynomials with $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$.) The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the set of polynomials $\{1, t, t^2\}$. Use the Gram-Schmidt process to construct an ONS.

(10). Same as Part (9) except $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$.

Our experience so far with bases has been mostly with $\mathbb{R}^n$ (and $\mathbb{C}^n$.) Most vector spaces actually reduce to these cases in practice, in fact, all finite dimensional spaces do. To see why, suppose we have a vector space $\mathbf{X}$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and let's see how to represent the space as $\mathbb{R}^n$ or $\mathbb{C}^n$.

Theorem 5.22. *If $\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbf{X}$, then there is exactly one way to write a given vector $\mathbf{x} \in \mathbf{X}$ in the form of a linear combination of these basis elements*

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Proof of Theorem 5.22. *Suppose* $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$, *then* $(c_1 - \alpha_1)\mathbf{v}_1 + (c_2 + \alpha_2)\mathbf{v}_2 + \cdots + (c_n - \alpha_n)\mathbf{v}_n = \Theta$ *and since* $\{\mathbf{v}_i\}_{i=1}^n$ *is a basis, it is a linearly independent set and therefore by Corollary 5.3,* $(c_1 - \alpha_1) = (c_2 - \alpha_2) = \cdots = (c_n - \alpha_n) = 0$. ∎

Remark. Since, given $\mathbf{X}$, a basis $\mathbf{B}$ and the uniqueness for each $\mathbf{x}$ of $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, if we think of the right side as $(c_1, c_2, \ldots, c_n)\bullet(v_1, v_2, \ldots, v_n)$, recognizing the slight abuse of the dot product notation, we see that we might consider the $1-1$ identification of the vector $\mathbf{x} \in \mathbf{X}$ with the vector $(c_1, c_2, \ldots, c_n)$ in Φ^n, i.e., $\mathbb{R}^n$ or $\mathbb{C}^n$ depending upon which scalar field $\mathbf{X}$ has.

Definition 5.13. If $\mathbf{X}$ is a vector space with basis $\mathbf{B} = \{\mathbf{v}_i\}_{i=1}^n$, then for each vector $\mathbf{x} \in \mathbf{X}$, the unique vector $\mathbf{x} \in \Phi^n$ such that $\mathbf{x} = \mathbf{c}\bullet\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ is called the coordinate vector of $\mathbf{x}$ relative to the basis $\mathbf{B}$. One of the standard notations for this vector is $(\mathbf{x})_\mathbf{B} = (c_1, c_2, \ldots, c_n)$. Some writers write $[\mathbf{x}]_\mathbf{B} = (c_1, c_2, \ldots, c_n)^T = (\mathbf{x})_\mathbf{B}^T$. Some writers use both notations (as we just did).

An immediate question is whether this $1-1$ identification of $\mathbf{X}$ with Φ^n preserves the algebraic structure and does an inner product, if one exists, translate into a dot product in Φ^n.

Theorem 5.23. *Suppose* $\mathbf{B} = \{\mathbf{v}_i\}_{i=1}^n$ *is a basis for a vector space* $\mathbf{X}$, $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i$ *and* $\mathbf{y} = \sum_{i=1}^n b_i\mathbf{v}_i$, *i.e.,* $(\mathbf{x})_\mathbf{B} = (a_1, a_2, \ldots, a_n)$ *and* $(\mathbf{y})_\mathbf{B} = (b_1, b_2, \ldots, b_n)$, *then* $(\mathbf{x}+\mathbf{y})_\mathbf{B} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ *and if* $k \in \Phi$, *then* $(k\mathbf{x})_\mathbf{B} = (ka_1, ka_2, \ldots, ka_n)$. *Furthermore, if* $\mathbf{B}$ *is an ONS in an inner product space* $\mathbf{X}$, *then*

$$\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{x})_\mathbf{B}\bullet(\mathbf{y})_\mathbf{B} = \sum_{i=1}^n a_i\overline{b_i}, \text{ and thus } \|\mathbf{x} - \mathbf{y}\|_\mathbf{X} = \left[\sum_{i=1}^n |a_i - b_i|^2\right]^{\frac{1}{2}} = \|(\mathbf{x})_\mathbf{B} - (\mathbf{y})_\mathbf{B}\|_{\Phi^n}.$$

Exercise 5.6. Prove Theorem 5.23.

Remark. We may therefore manipulate $\mathbf{X}$, relative to the basis $\mathbf{B}$ as though it were in fact Φ^n since this identifiction is a $1-1$ map. Notice that we could even use the identification map to re-norm the space $\mathbf{X}$ in such a way that $\mathbf{B}$ becomes an ONS by *defining*

$$\langle \mathbf{x}, \mathbf{y} \rangle \stackrel{\text{def}}{=} (\mathbf{x})_\mathbf{B}\bullet(\mathbf{y})_\mathbf{B},$$

Then since $(\mathbf{v}_i)_\mathbf{B} = \mathbf{e}_i \in \Phi^n$,

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle_\mathbf{X} = \mathbf{e}_i\bullet\mathbf{e}_j = \delta_{ij}.$$

This may, of course change the original notions of orthogonality in $\mathbf{X}$ if there were any such notions already in place.

We may also use these tricks to represent linear maps between finite dimensional vector spaces as matrices. These matrices will, of course, depend upon the choice of basis elements.

Example 5.1.

Let's carry out the above identification process for a specific case of a finite dimensional vector space, $\mathbf{P}_n$ the polynomials of degree not greater than n. Since each such polynomial p has a representation

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

if we chose as a basis $\{1, t, t^2, \ldots, t^n\} \equiv \mathbf{B}_{n+1}$, (one must prove that these are linearly independent!) then we may think of

$$p(t) = (a_0, a_1, a_2, \ldots, a_n)\bullet(1, t, t^2, \ldots, t^n)$$

and identify p with the point $(a_0, a_1, a_2, \ldots, a_n)$ in $\mathbb{C}^{n+1}$. Let's carry this identification to a little finer detail.

$$\begin{aligned} \text{If } (p(t))_{\mathbf{B}} &= (a_0, a_1, a_2, \ldots, a_n), \text{ then} \\ (t^i)_{\mathbf{B}} &= \mathbf{e}_{i+1} \in \mathbb{C}^{n+1}, \text{ i.e.,} \\ (1)_{\mathbf{B}} &= \mathbf{e}_1 \\ (t)_{\mathbf{B}} &= \mathbf{e}_2 \\ \vdots \quad &\vdots \quad \vdots \\ (t^n)_{\mathbf{B}} &= \mathbf{e}_{n+1}. \end{aligned}$$

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Example 5.2.

Choose $\mathbf{B}_{n+2} = \{1, t, t^2, \ldots, t^{n+1}\}$ as a basis in $\mathbf{P}_{n+1}$ and $\mathbf{B}_{n+1} = \{1, t, t^2, \ldots, t^n\}$ as a basis in $\mathbf{P}_n$. Suppose we have done this and made the identifications $\mathbf{P}_{n+1} \sim \mathbb{C}^{n+2}$ and $\mathbf{P}_n \sim \mathbb{C}^{n+1}$,

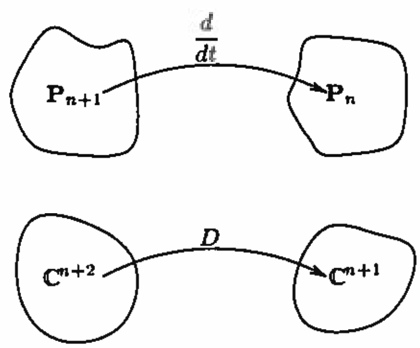

Figure 5.1:

then since we already have observed that $\frac{d}{dt}$ is a linear map of $\mathbf{P}_{n+1}$ into $\mathbf{P}_n$ (see Exercise 1.5), it follows from Theorem 5.23, that there is an induced linear map from $\mathbb{C}^{n+2}$ into $\mathbb{C}^{n+1}$, call it D, such that if $q(t) = b_0 + b_1 t + \ldots + b_{n+1} t^{n+1}$ so that $(q)_{\mathbf{B}_{n+2}} = (b_0, b_1, \ldots, b_{n+1}) \in \mathbb{C}^{n+2}$, then $\frac{d}{dt}(q(t)) = b_1 + 2b_2 t + \cdots + (n+1) b_{n+1} t^n$ and thus $\left(\frac{dq(t)}{dt}\right)_{\mathbf{B}_{n+1}} = (b_1, 2b_2, \ldots, (n+1)b_{n+1}) \in \mathbb{C}^{n+1}$ and D is described by

$$D(b_0, b_1, \ldots, b_{n+1}) = (b_1, 2b_2, \ldots, (n+1)b_{n+1}).$$

therefore D has a matrix representation which is $(n+2) \times (n+1)$ and can be written down if we compute

$$(D(\mathbf{e}_1), D(\mathbf{e}_2), \ldots, D(\mathbf{e}_{n+2})).$$

We saw in Example 5.1 that in $\mathbf{P}_m$ (any m) $(t^i)_{\mathbf{B}_{m+1}} = \mathbf{e}_{i+1}$, i.e., the polynomial $v_{i+1}(t) = t^i = 0 + 0 \cdot t + \cdots + 1 \cdot t^i + \cdots + 0 = (\mathbf{e}_{i+1}) \cdot (1, t, \ldots, t^{n+1})$ and since $\dfrac{d}{dt}(t^i) = it^{i-1} = (i\mathbf{e}_i) \cdot (1, t, \ldots, t^n)$, $D(\mathbf{e}^{i+1}) = (i)\mathbf{e}_i$

for $i = 1, 2, 3, \ldots$. Moreover since $\frac{d}{dt}1 = 0, D(\mathbf{e}_1) = \Theta$, therefore the matrix for D becomes

the matrix whose columns are $(D(\mathbf{e}_1), D(\mathbf{e}_2), \ldots, D(\mathbf{e}_{n+2})) = (\theta, 1 \cdot \mathbf{e}_1, 2 \cdot \mathbf{e}_2, 3 \cdot \mathbf{e}_3, \ldots, (n+1)\mathbf{e}_n)$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{pmatrix}.$$

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Exercise 5.7.

(1). Show that the set $\{1, t, \ldots, t^{n+1}\}$ is linearly independent and thus forms a basis for P_{n+1}. *Hint:* Use Corollary 5.3 and the fundamental theorem of algebra.

(2). Find the point in $\mathbb{C}^{(6)}$ whose coordinates are the same as those of $p(x) = 4 - 3x^2 + 5x^3 - 10x^5$ in $\mathbf{P}_5$, i.e., compute $(p(x))_{\mathbf{B}_6}$.

(3). Write the matrix D which represents $\frac{d}{dx} : P_5 \to \mathbf{P}_4$ and use it to compute the coordinates of the derivative of $p(x)$ in part (2). Verify the result by directly differentiating $p(x)$ and determining $\left(\frac{dp}{dx}\right)_{\mathbf{B}_5}$.

(4). Compute the matrix $\mathcal{I}$ which corresponds to the map $\mathcal{I}(p(x)) = \int_0^x p(t)dt$ (which maps $\mathbf{P}_n$ into $\mathbf{P}_{n+1}$), i.e., $\mathcal{I}$ maps $\mathbb{C}_{n+1}$ into $\mathbb{C}_{n+2}$ by mapping

$$(p(x))_{\mathbf{B}_{n+1}} \to (\mathcal{I}(p(x)))_{\mathbf{B}_{n+2}}.$$

(5). Compute the matrix products $\mathcal{I}D$ and $D\mathcal{I}$ and comment on the existence of inverses.

The last two examples and the exercises which follow them raise the natural question: Suppose we have a new, nicer basis, how does this change of basis change our coordinates in $\mathbb{C}^n$ and what does it do to our matrix representations for linear maps?

Let's check it out.

Suppose

$$\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ is a basis (old) for } \mathbf{X}$$
$$\text{and } \mathbf{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \text{ is another basis (new) for } \mathbf{X}.$$

Question: If we know, for a given vector $\mathbf{x} \in \mathbf{X}$, $(\mathbf{x})_\mathbf{B}$, can we find a way to simply write down $(\mathbf{x})_{\mathbf{B}'}$?

Let's try it. First let's express each of the vectors in $\mathbf{B}'$ in terms of the original basis $\mathbf{B}$:

$$\begin{aligned}
\mathbf{u}_1 &= p_{11}\mathbf{v}_1 + p_{12}\mathbf{v}_2 + \cdots + p_{1n}\mathbf{v}_n \\
\mathbf{u}_2 &= p_{21}\mathbf{v}_1 + p_{22}\mathbf{v}_2 + \cdots + p_{2n}\mathbf{v}_n \\
&\cdots \\
\mathbf{u}_n &= p_{n1}\mathbf{v}_1 + p_{n2}\mathbf{v}_2 + \cdots + p_{nn}\mathbf{v}_n
\end{aligned}$$

or $\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = P \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$. Now suppose

$$\begin{aligned}
(\mathbf{x})_{\mathbf{B}} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \text{ is known, then} \\
(\mathbf{x})_{\mathbf{B}'} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n \\
&= a_1(p_{11}\mathbf{v}_1 + p_{12}\mathbf{v}_2 + \cdots + p_{1n}\mathbf{v}_n) + \cdots + a_n(p_{n1}\mathbf{v}_1 + p_{n2}\mathbf{v}_2 + \cdots + p_{nn}\mathbf{v}_n) \\
&= (a_1p_{11} + a_2p_{21} + \cdots + a_np_{n1})\mathbf{v}_1 + \cdots + (a_1p_{n1} + a_2p_{n2} + \cdots + a_np_{nn})\mathbf{v}_n \\
&= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,
\end{aligned}$$

and since the representations are unique by Theorem 5.22, we have that

$$\begin{aligned}
p_{11}a_1 + p_{21}a_2 + \cdots + p_{n1}a_n &= c_1 \\
p_{12}a_1 + p_{22}a_2 + \cdots + p_{n2}a_n &= c_2 \\
\cdots\cdots\cdots\cdots\cdots & \\
p_{1n}a_1 + p_{2n}a_2 + \cdots + p_{nn}a_n &= c_n \\
\text{or } P^T[\mathbf{x}]_{\mathbf{B}'} &= [\mathbf{x}]_{\mathbf{B}}.
\end{aligned}$$

Now, again by Theorem 5.22 since $\mathbf{B}$ and $\mathbf{B}'$ are both bases and the representations exist uniquely, by Theorem 5.13 $\det[P^T] \neq 0$ and thus P^T is invertible *and*

$$[\mathbf{x}]_{\mathbf{B}'} = [P^T]^{-1}[\mathbf{x}]_{\mathbf{B}}.$$

Definition 5.14. The matrix P^T is called the transition matrix from basis $\mathbf{B}'$ to basis $\mathbf{B}$. Naturally $(P^T)^{-1}$ is the transition matrix from $\mathbf{B}$ to $\mathbf{B}'$.

By Theorem 5.23, the algebraic and inner product structures are preserved by the coordinate maps $\mathbf{x} \to (\mathbf{x})_{\mathbf{B}}$. Suppose both $\mathbf{B}$ and $\mathbf{B}'$ are ONS bases, then the vectors $\{(\mathbf{u}_i)_{\mathbf{B}}\}_{i=1}^n$ are an ONS in $\mathbb{C}^n$ and these are the rows of the matrix P and the columns of P^T. Therefore

$$P^TP = (((\mathbf{u}_i)_{\mathbf{B}}, (\mathbf{u}_j)_{\mathbf{B}})) = (\delta_{ij}) = I_n.$$

We have thus proved:

Theorem 5.24. *The transition matrix P from one ONS to another satisfies $P^{-1} = P^T$, i.e., its columns (and rows) form an ONS.*

Remark. Such a matrix is called an orthogonal matrix.

Now suppose that we have two finite dimensional vector spaces $\mathbf{X}$ and $\mathbf{Y}$ with bases $\mathbf{B} = \{\mathbf{x}_i\}_{i=1}^n$ and $\mathbf{V} = \{\mathbf{y}_j\}_{j=1}^m$ respectively and a linear transformation T from $\mathbf{X}$ to $\mathbf{Y}$. Under the basis $\mathbf{B}$, we may consider

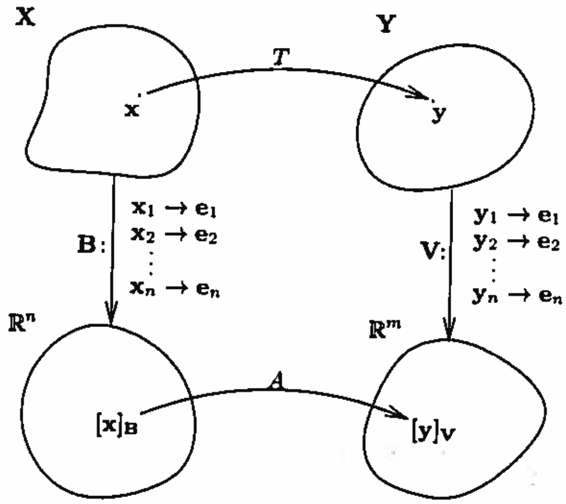

Figure 5.2:

$\mathbf{X}$ as $\mathbb{R}^n$ or $\mathbb{C}^n$ and under the basis $\mathbf{V}$ we may consider $\mathbf{Y}$ as $\mathbb{R}^m$ or $\mathbb{C}^m$ and therefore we may consider T as an $m \times n$ matrix A which maps

$$\mathbf{y} = T\mathbf{x}$$
$$\text{as } [\mathbf{y}]_{\mathbf{V}} = A[\mathbf{x}]_{\mathbf{B}}.$$

Now suppose we make a change of basis in either $\mathbf{X}$ or $\mathbf{Y}$ (perhaps both). How does this change the matrix representation A of our linear transformation T?

Let's do one change of basis at a time and check it out.

First, let's make a change of basis in $\mathbf{X}$, say from $\mathbf{B}$ to $\mathbf{B}'$ with transition matrix P^T, i.e.,

$$[\mathbf{x}]_{\mathbf{B}'} = (P^T)^{-1}[\mathbf{x}]_{\mathbf{B}},$$

then we'd need a new matrix B such that

$$B[\mathbf{x}]_{\mathbf{B}'} = A[\mathbf{x}]_{\mathbf{B}}$$

holds for all $\mathbf{x} \in \mathbf{X}$, i.e., $B[\mathbf{x}]_{\mathbf{B}'} = B[P^T]^{-1}[\mathbf{x}]_{\mathbf{B}}$ and if we do this for $[\mathbf{x}]_{\mathbf{B}} = \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ we have that (column by column)

$$B(P^T)^{-1} = A \text{ or } B = AP^T.$$

Let's check. Suppose $B = AP^T$, then

$$B[\mathbf{x}]_{\mathbf{B}'} = (AP^T)[\mathbf{x}]_{\mathbf{B}'} = A(P^T[\mathbf{x}]_{\mathbf{B}'}) = A[\mathbf{x}]_{\mathbf{B}}, \text{ done.}$$

Suppose instead that we made a change of variable in $\mathbf{Y}$, say from the basis $\mathbf{V}$ to $\mathbf{V}'$ with transition matrix Q^T, i.e., $[\mathbf{y}]_{\mathbf{V}'} = (Q^T)^{-1}[\mathbf{y}]_{\mathbf{V}}$ or $Q^T[\mathbf{y}]_{\mathbf{V}'} = [\mathbf{y}]_{\mathbf{V}}$. How does this affect A? Again, let's check.

$A[\mathbf{x}]_{\mathbf{B}} = [\mathbf{y}]_{\mathbf{V}} = Q^T[\mathbf{y}]_{\mathbf{V}'}$ so that $[Q^T]^{-1}A[\mathbf{x}]_{\mathbf{B}} = [\mathbf{y}]_{\mathbf{V}'}$ and therefore our new matrix is $[Q^T]^{-1}A$.

What if we did both? Easy; first the change of basis in $\mathbf{X}$ gives us AP^T and following this by the change in $\mathbf{Y}$ gives us $[Q^T]^{-1}AP^T$.

The challenge in this business is to find P and Q so that the resulting matrix for T is nice in some sense for the problem we want to solve, say $[Q^T]^{-1}AP^T$, is diagonal, so that it is then trivial to solve equations like $D\mathbf{x} = \mathbf{y}$ by just writing down the solution. In general we wish to make our manipulations easy and hopefully the physical or geometric interpretations of our results are also easy and straightforward.

In case $\mathbf{X} \equiv \mathbf{Y}$ and the same change of variables is made in the domain and range versions of the space, our new matrix for T becomes $(P^T)^{-1}AP^T$.

Definition 5.15. Two $n \times n$ matrices A and B are said to be similar if and only if there exists a non-singular matrix S such that

$$S^{-1}AS = B, (\text{or } A = SBS^{-1}).$$

In the event that this is true, we write $A \sim B$.

Exercise 5.8.

Show that if A, B, and C are $n \times n$ matrices, then

(i). $A \sim A$

(ii). if $A \sim B$, then $B \sim A$

(iii). if $A \sim B$, and $B \sim C$, then $A \sim C$.

Definition 5.16. Suppose $\mathbf{S}$ is a set. The statement that $\sim$ is an equivalence relation on the elements of $\mathbf{S}$ means that

(i). If $A \in \mathbf{S}$, then $A \sim A$ holds

(ii). If $A \sim B$, then also $B \sim A$ holds

(iii). If $A \sim B$, and $B \sim C$, then $A \sim C$.

Remark. Exercise 5.8 shows that similarity is an equivalence relation on the $n \times n$ matrices.

Definition 5.17. Suppose $\sim$ is an equivalence relation on a set $\mathbf{S}$ and $x \in \mathbf{S}$. By the symbol $\tilde{\mathbf{x}}$ we mean the set of all elements in $\mathbf{S}$ which are equivalent to x. The set $\tilde{\mathbf{x}}$ is called the equivalence class of x.

Exercise 5.9.

Suppose $\sim$ is an equivalence relation on a set $\mathbf{S}$ and that x and y are elements of $\mathbf{S}$. Show that the sets $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are either mutually exclusive or identical.

Remark. Exercise 5.9 shows that an equivalence relation partitions a set into disjoint sets, the individual sets having only elements which are indistinguishable under the given equivalence relation.

If we apply this idea to the case of $(n \times n)$ matrices with similarity as the equivalence relation, then we are identifying all matrix representations of a given linear transformation as an equivalence class. To see this, suppose that T maps $\mathbb{R}^n$ into $\mathbb{R}^n$ and A represents T relative to a set of basis elements $\mathbf{V}$, i.e., $\mathbf{y} = T\mathbf{x}$ becomes $[\mathbf{y}]_{\mathbf{V}} = A[\mathbf{x}]_{\mathbf{V}}$.

Then if we make a change of variables with transition matrix P^T to a basis $\mathbf{V}'$, i.e.,

$$\begin{aligned}[\mathbf{x}]_{\mathbf{V}'} &= (P^T)^{-1}[\mathbf{x}]_{\mathbf{V}}, \text{ then}\\ A[\mathbf{x}]_{\mathbf{V}} &= [\mathbf{y}]_{\mathbf{V}} \text{ becomes}\\ AP^T[\mathbf{x}]_{\mathbf{V}'} &= [\mathbf{y}]_{\mathbf{V}} \text{ and then}\\ (P^T)^{-1}AP^T[\mathbf{x}]_{\mathbf{V}'} &= (P^T)^{-1}[\mathbf{y}]_{\mathbf{V}'} \text{ or}\\ B[\mathbf{x}]_{\mathbf{V}'} &= [\mathbf{y}]_{\mathbf{V}'}\end{aligned}$$

where B is the matrix representing T relative to the basis $\mathbf{V}'$ and $\mathbf{A}$ and $\mathbf{B}$ are similar.

Theorem 5.25. *If A and B are similar $n \times n$ matrices, then they are different representations for the same linear transformation, the representations being relative to different sets of basis elements.*

CAUTION: One $n \times n$ matrix A can represent many different linear transformations on $\mathbf{X}$ if one keeps the matrix A fixed while changing the basis elements in $\mathbf{X}$.

Example 5.3.

Choose $A = \left(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right)$ in advance of any choice of $\mathbf{X}$ or its basis $\mathbf{V}$, and hold A fixed

Case One: $\mathbf{X} = \mathbf{P}_1, \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) = (1, t)$. Then $(p(t))_{\mathbf{V}} = (c_1 + c_2t)_{\mathbf{V}} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)_{\mathbf{V}} = (c_1, c_2)$ and $A\left(\begin{smallmatrix}c_1\\c_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right)\left(\begin{smallmatrix}c_1\\c_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}c_1+2c_2\\3c_1+4c_2\end{smallmatrix}\right)$ and $T(p) = (c_1 + 2c_2)\mathbf{v}_1 + (3c_1 + 4c_2)\mathbf{v}_2 = (c_1 + 2c_2) + (3c_1 + 4c_2)t$.

Case Two: $\mathbf{X} = \mathbf{P}_1, \mathbf{V}' = (\mathbf{v}_1', \mathbf{v}_2') = (1+t, t)$, then $(p(t))_{\mathbf{V}'} = (c_1 + c_2t)_{\mathbf{V}'} = (c_1\mathbf{v}_1 + (c_2 - c_1)\mathbf{v}_2)_{\mathbf{V}'} = (c_1, c_2 - c_1)$ and $A\left(\begin{smallmatrix}c_1\\c_2-c_1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right)\left(\begin{smallmatrix}c_1\\c_2-c_1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-c_1+2c_2\\-c_1+4c_2\end{smallmatrix}\right)$ and $(-c_1 + 2c_2)\mathbf{v}_1' + (-c_1 + 4c_2)\mathbf{v}_2' = (c_1 + 2c_2)(1 + t) + (-c_1 + 4c_2)t = (-c_1 + 2c_2) + (-2c_1 + 6c_2)t$ which is not the same as $T(p)$.

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Exercise 5.10.

(1). Show that the $n \times n$ Θ matrix always represents the Θ linear transformation on $\mathbf{X}$, no matter what ordered basis one chooses.

(2). Show that the $n \times n$ identity matrix always represent the identity transformation on $\mathbf{X}$, no matter what ordered basis one chooses.

(3). Find the transition matrix which changes the basis $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) = (1, t)$ into the basis $\mathbf{V}' = (\mathbf{v}_1', \mathbf{v}_2') = (1 + t, 1)$ in Example 5.3 above.

(4). Suppose $p(t) = 3 + 4t$; find $(p)_{\mathbf{V}}$ and $(p)_{\mathbf{V}'}$ as in Example 5.3.

(5). Find the transition matrix for the change of coordinates in P_2 which changes from $\{1, t, t^2\}$ to $\{1, 1 + t, 1 + t + t^2\}$.

(6). Find the transition matrix for the change of basis from $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the basis $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 2, 2)$ and $\mathbf{u}_3 = (1, 2, 3)$. Compute the $\mathbf{U}$ coordinates of the vector $(1, 0, 1)$; i.e., compute $(1, 0, 1)_{\mathbf{U}}$ and also $(0, 1, 0)_{\mathbf{U}}$.

(7). Suppose the matrix $A = \left(\begin{smallmatrix}1&0&1\\0&2&1\\1&2&0\end{smallmatrix}\right)$ represents the linear map L relative to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Find the matrix which represents L relative to the basis $\mathbf{U}$ in 5.10.(6) above.

Let's now summarize the things we have learned concerning bases and how they effect and affect the matrices which represent linear transformations.

Theorem 5.26. *Suppose* $\mathbf{B}$ *and* $\mathbf{B}'$ *are two ordered bases for a vector space* $\mathbf{X}$ *of dimension* n *and* T *is a linear transformation from* $\mathbf{X}$ *into* $\mathbf{X}$.

Suppose P^T *is the transition from* $\mathbf{B}'$ *into* $\mathbf{B}$ *and* A *is the matrix representing* T *with respect to* $\mathbf{B}$, *then the matrix which represents* T *with respect to* $\mathbf{B}'$ *is*

$$[P^T]^{-1}AP^T$$

and thus is a member of the equivalence class of all matrices similar to A.

Conversely, suppose the $n \times n$ *matrix* A *represents* T *with respect to the ordered basis* $\mathbf{B}$ *and that*

$$\mathcal{A} = S^{-1}AS,$$

then if we define a new ordered basis $\mathbf{B}'$ *by* $S^T\mathbf{B}$ *then* $\mathcal{A}$ *represents* T *relative to the basis* $\mathbf{B}'$.

Chapter 6

Eigenvalues and Eigenvectors

Equipped with the information of Chapter 5, we now want to to make a change of variable which will render our $n \times n$ matrix representation A of the linear transformation T as nice as it can possibly be in so far as solving

$$\mathbf{y} = A\mathbf{x}$$

is concerned, namely, we would *like* it to be diagonal after we have made the change of variable. What does this entail? Under what circumstances can it be done?

Suppose there exists a non-singular matrix S such that

$$S^{-1}AS = D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

then $AS = SD$. Let's write

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix}, \text{ then we have}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 s_{11} & \lambda_2 s_{12} & \cdots & \lambda_n s_{1n} \\ \lambda_1 s_{21} & \lambda_2 s_{22} & \cdots & \lambda_n s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 s_{n1} & \lambda_2 s_{n2} & \cdots & \lambda_n s_{nn} \end{pmatrix}$$

so that each column, say the j^{th}, satisfies a relation

$$A\begin{pmatrix} s_{1j} \\ s_{2j} \\ \vdots \\ s_{nj} \end{pmatrix} = \lambda_j \begin{pmatrix} s_{1j} \\ s_{2j} \\ \vdots \\ s_{nj} \end{pmatrix}.$$

With this motivation, we make our first definition.

Definition 6.1. Suppose A is an $n \times n$ matrix, (real or complex, we make no distinction). A scalar λ is called an eigenvalue of A if and only if there exists a non-Θ vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Such a vector $\mathbf{x}$ is called an eigenvector of A belonging to the eigenvalue λ. The space spanned by the collection of all eigenvectors corresponding to a given eigenvalue is called the eigensubspace corresponding to λ. We usually denote it by $\mathbf{E}_\lambda$.

Exercise 6.1.

(1). Suppose A is an $n \times n$ scalar matrix with eigenvalue λ. Show that $\mathbf{E}_\lambda$ is indeed a subspace of Φ^n.

(2). Show that if $\lambda_1 \neq \lambda_2$ and each is an eigenvalue of the $n \times n$ matrix A, then $\mathbf{E}_{\lambda_1} \cap \mathbf{E}_{\lambda_2} \equiv \{\Theta\}$.

Notice that $A\mathbf{x} = \lambda\mathbf{x}$ has a non-trivial solution $\mathbf{x}$ if and only if $(A - \lambda I)\mathbf{x} = \Theta$ has a non-Θ solution and this happens if and only if $(A - \lambda I)$ is non-singular and this occurs if and only if $\det(A - \lambda I) = 0$. This latter equation is called the characteristic equation of A and $\det(A - \lambda I)$ is called the characteristic polynomial of A.

In short, the eigenvalues of A are the roots of the characteristic polynomial of A and, for this reason, are frequently referred to as the characteristic roots of A or the characteristic values of A; sometimes one sees the word proper used instead of eigen or characteristic.

In summary, (recall Theorem 5.13):

Theorem 6.1. *For an $n \times n$ matrix A, the following statements are equivalent.*

(1). λ is an eigenvalue of A

(2). $(A - \lambda I)\mathbf{x} = \Theta$ has a non-trivial solution

(3). $\mathcal{N}(A - \lambda I) \not\equiv \Theta$

(4). $A - \lambda I$ is singular

(5). $\det(A - \lambda I) = 0$.

Remark. We could, no doubt, add to the list in Theorem 6.1 if we worked at it.

Now suppose $A = (a_{ij})$ is an $n \times n$ scalar matrix with characteristic polynomial $p(\lambda)$, i.e.,

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}.$$

It follows from the definition of determinant that the only term involving more than $(n-2)$ of the diagonal elements will be $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$, and when one expands this, the coefficient of λ^n is $(-1)^n$; thus if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A, then

$$p(\lambda) = (-1)^n(\lambda-\lambda_1)\cdot(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = (\lambda_1-\lambda)\cdot(\lambda_2-\lambda)\cdots(\lambda_n-\lambda).$$

Moreover, it follows from this and the definition of $p(\lambda)$ that the constant term in the polynomial, i.e., $p(0)$ is $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = p(0) = \det(A)$.

One can also see that the coefficient of $(-\lambda)^{n-1}$ is

$$\sum_{i=1}^{n} a_{ii} \text{ (trace of } A \text{ or Trace}(A))$$

which is, of course, the sum of the roots of the polynomial, i.e.,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n} a_{ii} = \text{Trace}(A).$$

Now let's return to the question of similarity transformations and ask: When can we make a change of variables and make A nice? Where by nice we mean diagonal, i.e., we want

$$S^{-1}AS = D, \text{ or } A = SDS^{-1}.$$

A matter of major importance in this is an extension of Exercise 6.1.(2), having to do with the distinctness of the eigensubspaces.

Theorem 6.2. *If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of an $n \times n$ matrix A with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$, then $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent.*

Proof of Theorem 6.2. *Denote by r the dimension of* Span$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *and suppose that $r < k$. Reorder the $\mathbf{x}$'s if necessary so that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r$ are linearly independent and thus* Span$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r, \mathbf{x}_{r+1}\}$ *must be linearly dependent by Theorem 5.4. Therefore there exists a non-trivial set $\{c_1, c_2, \dots, c_r, c_{r+1}\}$ of scalars such that*

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r = -c_{r+1}\mathbf{x}_{r+1} \neq \Theta \tag{6.1}$$

and thus not all of $\{c_1, c_2, \dots, c_r\}$ can be zero. Apply the matrix A to equation (6.1) and we get

$$c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_rA\mathbf{x}_r + c_{r+1}A\mathbf{x}_{r+1} = \Theta$$
$$\text{or } c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_r\lambda_r\mathbf{x}_r + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = \Theta.$$

Now subtract λ_{r+1} times (6.1) from this last equation and we obtain

$$c_1(\lambda_1-\lambda_{r+1})\mathbf{x}_1 + c_2(\lambda_2-\lambda_{r+1})\mathbf{x}_2 + \cdots + c_r(\lambda_r-\lambda_{r+1})\mathbf{x}_r = \Theta$$

which, since $(\lambda_i - \lambda_{r+1}) \neq 0$, shows that indeed $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ are linearly dependent. ∎

Definition 6.2. An $n \times n$ matrix A is said to be diagonalizable if and only if there exists a diagonal matrix in its equivalence class under similarity, i.e., there exist a diagonal matrix D and a non-singular matrix S such that

$$S^{-1}AS = D.$$

If this occurs, we say that "S diagonalizes A."

Theorem 6.3. *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

Proof of Theorem 6.3. *Suppose A has n linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct!) Let S be the matrix whose j^{th} column is $\mathbf{x}_j$, and the j^{th} column of AS is $A\mathbf{x}_j$ which is given by $A\mathbf{x}_j = \lambda_j \mathbf{x}_j$, i.e.,*

$$AS = (A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n) = (\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = SD.$$

therefore $S^{-1}AS = D$.

Conversely, suppose A is diagonalizable, then there exist a diagonal matrix D and a non-singular matrix S so that

$$AS = SD.$$

Call the columns of S, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ and the elements of the diagonal D, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Our equation now becomes

$$(A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n) = (\mathbf{x}_1\lambda_1, \mathbf{x}_2\lambda_2, \ldots, \mathbf{x}_n\lambda_n) \text{ or } A\mathbf{x}_i = \lambda_i\mathbf{x}_i.$$

Since S in non-singular, it is of rank n and the columns $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are linearly independent. ∎

Remark. (1). If A is diagonalizable, then the column vectors of the diagonalizing matrix S are eigenvectors of A and the diagonal matrix D has elements which are the corresponding eigenvalues.

(2). If A has n distinct eigenvalues then it is diagonalizable.

(3). The diagonalizing matrix is not unique since any multiple (non-zero) of an eigenvector is also an eigenvector and moreover we could reorder the columns of S and obtain a reordered diagonal matrix.

(4). If A is diagonalizable, then A can be factored into

$$A = SDS^{-1}.$$

This last remark may seem inane, but it happens to be very useful.

Exercise 6.2.

(1). Show that if A is SDS^{-1}, then $A^m = SD^mS^{-1}$ where

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{pmatrix}.$$

(2). Suppose $S^{-1}AS = D$ as in part (1) where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Show that, at least formally,

$$I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots + A^n\frac{t^n}{n!} + \cdots = S \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} S^{-1}.$$

Recall that in the case of complex space $\mathbb{C}^n$, the dot product becomes

$$\mathbf{a} \cdot \mathbf{b} = a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$$

and, if we write this as a matrix product, it becomes

$$\mathbf{a}\overline{\mathbf{b}}^T.$$

Some authors write $\mathbf{b}^H$ instead of $\overline{\mathbf{b}}^T$ where the H stands for Hermitian.

Definition 6.3. A matrix M is said to be Hermitian if and only if $M = M^H (\equiv \overline{M}^T)$.

Notice that for real matrices, $M^H = M^T$ and in this case, Hermitian means symmetric.

Hermitian matrices have many very nice properties. Physicists like them a lot for the following reason.

Theorem 6.4. *The eigenvalues of an Hermitian matrix are real and the corresponding eigenvectors of distinct eigenvalues are orthogonal, i.e., the eigenspaces of distinct eigenvalues are orthogonal.*

Proof of Theorem 6.4. *Suppose A is an Hermitian matrix; $(\lambda, \mathbf{x})$ is an eigen pair. If*

$$\alpha = \mathbf{x}^H A\mathbf{x}, \text{ then}$$
$$\overline{\alpha} = \alpha^H = (\mathbf{x}^H A\mathbf{x})^H = \mathbf{x}^H A^H \mathbf{x}^{HH} = \mathbf{x}^H A\mathbf{x} = \alpha$$

and thus α is real. It follows that $\alpha = \mathbf{x}^H A\mathbf{x} = \mathbf{x}^H \lambda\mathbf{x} = \lambda\|\mathbf{x}\|^2$ and therefore $\lambda = \dfrac{\alpha}{\|\mathbf{x}\|^2}$ is real.

If $(\lambda_1, \mathbf{x}_1)$ and $(\lambda_2, \mathbf{x}_2)$ are eigenpairs and $\lambda_1 \neq \lambda_2$ then $(A\mathbf{x}_1)^H\mathbf{x}_2 = \mathbf{x}_1^H A^H \mathbf{x}_2 = \mathbf{x}_1^H \lambda_2\mathbf{x}_2 = \lambda_2 \langle \mathbf{x}_2, \mathbf{x}_1 \rangle$ and $\lambda_1 \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = \langle \mathbf{x}_2, \lambda_1\mathbf{x}_1 \rangle$ and therefore, since $\langle \mathbf{x}_2, \lambda_1\mathbf{x}_1 \rangle = (A\mathbf{x}_1)^H\mathbf{x}_2$, we have $(\lambda_1 - \lambda_2)\langle \mathbf{x}_2, \mathbf{x}_1 \rangle = 0$ and since $\lambda_1 \neq \lambda_2$, we must have $\mathbf{x}_2 \perp \mathbf{x}_1$. ∎

Definition 6.4. An $n \times n$ matrix U is said to be unitary if and only if its columns form an ONS in $\mathbb{C}^n$.

Remark. Notice that U is unitary if and only if $U^H U = I$, (this is analogous to $u^2 = 1$ for complex numbers) and then, since the column vectors are orthogonal, U must be non-singular, of rank n and so on.

Suppose U is unitary, hence non-singular and $UU^H = I = U^H U$ and consider U as a "change of variables" on the space $\mathbf{X}$ (change of basis if you prefer). Then $\langle U\mathbf{x}, U\mathbf{x} \rangle \stackrel{\text{def}}{=} (U\mathbf{x}) \cdot (U\mathbf{x}) = (U\mathbf{x})^H (U\mathbf{x}) = (\mathbf{x}^H U^H)(U\mathbf{x}) = \mathbf{x}^H [U^H U]\mathbf{x} = \mathbf{x}^H\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle$, i.e., U preserves the inner product, hence all distances and all angles.

A real unitary matrix is called an orthogonal matrix. See Theorem 5.24 and the remark following.

Corollary 6.5. *If the eigenvalues of an Hermitian matrix A are distinct, then there exists a unitary matrix which diagonalizes A.*

Proof of Corollary 6.5. *Let $(\lambda_i, \mathbf{x}_i)$ be the eigen pairs for A and set $\mathbf{u}_i = \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|}$ and U be the matrix whose columns are the $\mathbf{u}_i$'s.*

∎

Theorem 6.6. *(Schur's Theorem) For each $n \times n$ matrix A, there exists a unitary matrix U such that $U^H AU$ is upper triangular.*

Proof of Theorem 6.6. *The proof is by induction on n.*

For $n = 1$, there's naught to do!

Assume the Theorem holds for all $k \times k$ matrices and assume A to be $(k+1) \times (k+1)$. Let λ_1 be an eigenvalue for A and $\mathbf{x}_1$ a corresponding eigenvector of norm 1. Now use Gram-Schmidt to construct $\{\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{k+1}\}$ so that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k+1}\}$ is a basis for $\mathbb{C}^{k+1}$. Denote by W the matrix whose i^{th} column is $\mathbf{x}_i$, then W is unitary.

The first column of $W^H AW$ will be $W^H A\mathbf{x}_1 = W^H \lambda_1 \mathbf{x}_1 = \lambda_1 W^H \mathbf{x}_1 = \lambda_1 \mathbf{e}_1$, this last equality because W is unitary. Therefore,

$$W^H AW = \begin{pmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \\ \vdots & M \\ 0 & \end{pmatrix}$$

where M is $k \times k$. By our induction hypothesis, there exists a unitary matrix V_1 which is $k \times k$ such that

$$V_1^H M V_1 = T_1$$

where T_1 is upper triangular. Set

$$V = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & V_1 \\ 0 & \end{pmatrix},$$

then V is unitary and

$$V^H W^H AWV = \begin{pmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \\ \vdots & V_1^H M V_1 \\ 0 & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \\ \vdots & T_1 \\ 0 & \end{pmatrix} \stackrel{def}{=} T$$

where T is upper triangular. Set $U = WV$, then U is unitary because

$$U^H U = (WV)^H WV = V^H W^H WV = I.$$

∎

This factorization is called the Schur decomposition of A.

In case A is Hermitian, then T will be diagonal.

Corollary 6.7. *If A is Hermitian, then there exists a unitary matrix U that diagonalizes A.*

Proof of Corollary 6.7. *There exists a unitary matrix U such that $U^H AU = T$ where T is upper triangular then*

$$T^H = [U^H AU]^H = U^H A^H U^{HH} = U^H AU = T$$

so that T is both upper and lower triangular, therefore diagonal. ∎

Remark. One should notice the remarkable likeness of the proof of Corollary 6.7 and the proof in Theorem 6.4 of the fact that $\overline{\alpha} = \alpha$.

Exercise 6.3.

(1). What does the above remark suggest?

(2). Calculate a unitary matrix which performs a Schur decomposition on the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Hint: The proof of Theorem 6.6 tells much more than the statement of the theorem.

Now let's summarize and reexamine what we have learned.

If A is Hermitian, then there exist a unitary U and a diagonal D such that

$$A = UDU^H,$$

the diagonal elements of D are eigenvalues of A and the columns of U are the corresponding eigenvectors which have been normalized. This says that A has n linearly independent eigenvectors (which generate an ONS.)

This is essentially an ideal situation for we can write, for each $\mathbf{x} \in \mathbf{X}$,

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{x}, \mathbf{u}_n \rangle \mathbf{u}_n \text{ and then}$$
$$A\mathbf{x} = \langle \mathbf{x}, \mathbf{u}_1 \rangle \lambda_1 \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \lambda_2 \mathbf{u}_2 + \cdots + \langle \mathbf{x}, \mathbf{u}_n \rangle \lambda_n \mathbf{u}_n$$

and calculations are rather easy.

As it happens, there are non-Hermitian matrices whose eigenvectors form a complete ONS. For example, skew symmetric and skew Hermitian matrices have this property, i.e., those matrices A such that $A^H = -A$.

In general, if A has a complete orthonormal set of eigenvectors and U is the matrix whose columns are these vectors, then the change of basis under U gives

$$A = UDU^H$$

where D is the diagonal of eigenvalues of A arranged according to the columns of U. Notice that these eigenvalues may be complex (and probably are) and therefore

$$D^H \neq D,$$
$$\text{thus } A^H = (UDU^H)^H = U^{HH} D^H U^H = UD^H U^H \neq A.$$
$$\text{However, } AA^H = (UDU^H)(UD^H U^H) = UDD^H U^H$$
$$\text{and } A^H A = (UDU^H)^H (UDU^H) = UD^H DU^H$$
$$\text{and } D^H D = DD^H = \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix}$$

so that $A^H A = AA^H$.

Definition 6.5. A matrix A is called normal if and only if $A^H A = AA^H$.

We have just shown above that the existence of a complete orthonormal set of eigenvectors implies that A is normal, in fact, we have proved the following theorem in one direction.

Theorem 6.8. *A matrix A is normal if and only if A has a complete (n of them) orthonormal set of eigenvectors.*

Proof of Theorem 6.8. *By Schur's Theorem (Theorem 6.6), there exists a unitary U and an upper triangular T such that*

$$T = U^H AU.$$

CLAIM 6.1. *T itself is normal if A is normal.*

Proof of CLAIM 6.1.

$$T^H T = (U^H A^H U)(U^H AU) = U^H A^H AU = U^H AA^H U = U^H AUU^H A^H U = TT^H.$$

∎

Now if we compare the diagonal elements of TT^H with those of $T^H T$ we see as follows:

$$\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{pmatrix} = \begin{pmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix}$$

so that along the diagonals we have:

$$|t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 = |t_{11}|^2$$

so that in the first row of T, all terms save t_{11} are zero. The second diagonal elements are

$$|t_{22}|^2 + |t_{23}|^2 + \cdots + |t_{2n}|^2 = |t_{12}|^2 + |t_{22}|^2 = 0 + |t_{22}|^2$$

and thus the second row is all zeroes, save t_{22}, etc. It follows that T is diagonal, and thus $A = UTU^H = UDU^H$ and the theorem then follows from the remarks preceding Definition 6.5. ∎