Homework for Math 6510 §1, Fall 2019

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Our texts this semester are

- Kevin Wortman's, *Math 6510 Notes,*

  `http://www.math.utah.edu/ wortman/6510.pdf`.
- John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013.

Please read the relevant sections in the texts as well as any cited reference. Each problem is due three class days after its assignment, and may be handed in no later than Monday, Dec. 9. Please write the question on your solutions.

1. [Aug. 19] **Euler Characteristic.** Draw a vector field on Σ_n that vanishes at only finitely many points of Σ_n. Let Δ be a triangulation of Σ_n Use your vector field to prove that $\chi(\Delta) = 2 - 2n$. [Wortman, *Math 6510 Notes,* p. 15.]

2. [Aug. 21] **Space of matrices of fixed rank.** Prove that the set of $m \times n$ real matrices of rank r is an analytic manifold of dimension $r(m + n - r)$. Hint: after multiplication by permutation matrices P and Q, such matrix has the form

$$PMQ = \begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{pmatrix} B & C \\ D & E \end{pmatrix} \end{array}$$

   where the $r \times r$ matrix B is nonsingular. Then prove that $\operatorname{rank} M = r$ if and only if $E - DB^{-1}C = 0$. [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 27.]

3. [Augh. 23.] **Smooth Maps.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

   (a) Power map: $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ given by $p_n(z) = z^n$ where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $n \in \mathbb{Z}$.

   (b) Antipodal map: $a : \mathbb{S}^n \to \mathbb{S}^n$ given by $a(x) = -x$.

   (c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ given by $F(w, z) = (z\bar{w} + w\bar{z},\, iw\bar{z} - iz\bar{w},\, z\bar{z} - w\bar{w})$ where $\mathbb{S}^3 = \{(w, z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}$.

   [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 48.]

4. [Aug. 26.] **Tangent Vectors.** Let (x, y) denote standaard coordinates of $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinate in $\mathbb{R}^2$ where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ in standard coordinates, and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_{\tilde{p}},$$

even though the coordinate functions x and $\tilde{x}$ are identically equal. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 65.]

5. [Aug. 28.] **Tangent Vector in Stereographic Coordinates.** Consider stereographic coordinate charts (see p. 23) for the sphere $\{(\mathbb{S}^2 - N, \varphi_1), (\mathbb{S}^2 - S, \varphi_2)\}$ where $N = (0, 0, 1)$, $S = (0, 0, -1)$ and

$$(y^1, y^2) = \varphi_1(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 - x^3}, \qquad (z^1, z^2) = \varphi_2(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 + x^3}.$$

If $p \in \mathbb{S}^2 - \{N, S\}$ and the vector at p in the φ_1 chart is $v^i \dfrac{\partial}{\partial y^i}$, what is it in the φ_2 chart? [Wortman, *Math 6510 Notes,* p. 34.]

6. [Aug. 30.] **Submersion of a Compact Manifold.** Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $f : M \to \mathbb{R}^k$ for any $k > 0$. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 96.]

7. [Sept. 4.] **Normal Form for Immersions.** Suppose $f : M^m \to N^n$ is a smooth immersion. Show that for all $p \in M$ there are local coordinates x for M at p and y for N at $f(p)$ so that in these coordinates f is given by

$$y(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0) \in \mathbb{R}^n.$$

[Wortman, *Math 6510 Notes,* p. 58.]

8. [Sept. 6.] **Transverse maps.** Let $f(x, y) = (e^y \cos x, e^y \sin x, e^{-y})$ and $\mathbb{S}^2(r)$ be the standard two sphere of radius r embedded in $\mathbb{R}^3$.

   (a) For which r is f trasverse to $\mathbb{S}^2(r)$?

   (b) For which r is $f^{-1}(\mathbb{S}^2(r))$ an embedded submanifold of $\mathbb{R}^3$?

   [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 148.]

9. [Sept. 9.] **Sphere is Simply Connected.** Prove that the sphere $\mathbb{S}^n$ is simply connected if $n > 1$. Let $p_0 \in \mathbb{S}^n$ be a fixed basepoint and $\gamma : \mathbb{S}^1 \to \mathbb{S}^n$ be a $\mathcal{C}^1$ map such that $\gamma(1) = p_0$. Show that there is a homotopy, a continuous map $H : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^n$ such that $H(x, 0) = \gamma(x)$ and $H(x, 1) = p_0$ for all $x \in \mathbb{S}^1$ and $H(p_0, t) = p_0$ for all $t \in [0, 1]$. Hint: if $n > 1$ show there is $y \in \mathbb{S}^n \backslash \gamma(\mathbb{S}^1)$ and use stereographic projection from y. [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 45.]

10. [Sept. 11.] **Tangent Space to the Pulled Back Manifold.** Let $f : M^m \to N^n$ be a smooth map of smooth manifolds. Let $Z^z \subset M^m$ and $Q^q \subset N^n$ be smooth embedded submanifolds. Show:

    (a) If $f(Z) \subset Q$ and $p \in Z$ then $df_p(T_p Z) \subset T_{f(p)}Q$.

    (b) If $y \in N$ is a regular value for f and $p \in f^{-1}(y)$ then $T_p(f^{-1}(y)) = (df_p)^{-1}(0)$.

    (c) If $f \pitchfork Q$ and $p \in f^{-1}(Q)$ then $T_p(f^{-1}(Q)) = (df_p)^{-1}(T_{f(p)}Q)$.

    [Wortman, *Math 6510 Notes,* p. 66.]

11. [Sept. 13.] **Stability of Maps.** Prove that the Stability Theorem is false on noncompact domains. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Define the family of maps $f_t : \mathbb{R} \to \mathbb{R}$ given by $f_t(x) = x\rho(tx)$. Verify that this is a counterexample to all six parts, immersion, submersion, local diffeomorphism, transverse to $Q \subset N$ (take $Q = \{0\}$), embedding and diffeomorphism of the Stability Theorem [Wortman p. 80]. [Guillemin & Pollack, *Differential Topology,* Prentice Hall 1974, p. 38 and John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 149.]

12. [Sept. 16.] **Veronese Surface.** Define a map $f : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \\ x^2 - y^2 \end{pmatrix}.$$

    Viewing $\mathbb{RP}^2 = \mathbb{S}^2/\sim$ where $p \sim -p$, show that f descends to a smooth embeddings $f : \mathbb{RP}^2 \to \mathbb{R}^4$, called the *Veronese Surface.*

    In fact, this implies that the map $g : \mathbb{R}^3 \to \mathbb{R}^5$ given by

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \\ \frac{1}{2}\left(x^2 - y^2\right) \\ \frac{1}{2\sqrt{3}}\left(x^2 + y^2 - 2z^2\right) \end{pmatrix}.$$

    is also an embedding. Moreover, g is *isometric* to $\mathbb{RP}^2$: if $\gamma : [0,1] \to \mathbb{RP}^2$ is a smooth curve, then $\text{length}(\gamma) = \text{length}(g \circ \gamma)$ where lengths are measured in the background Euclidean spaces.

    [H. Brandsma, Ask a Topologist, Embedding of projective plane into $\mathbb{R}^4$, June 3, 2003, (`http://at.yorku.ca/cgi-bin/bbqa`) ; Q. Han & J.X. Hong, *Isometric Embeddings of Riemannian Manifolds in Euclidean Spaces,* American Mathematical Society, 2006, p. 39.]

13. [Sept. 18.] **Riemnannian Metric.** A *Riemannian Metric* for a smooth manifold M is a positive definite quadratic form $\langle \cdot, \cdot \rangle_x$ on $T_x M$ defined for each $x \in M$ which varies smoothly from point to point: if $V(x)$ and $W(x)$ are any smooth vector fields on M then $x \mapsto \langle V(x), W(x) \rangle_x$ is a smooth function. Give two proofs that every smooth manifold M carries a Riemannian metric.

14. [Sept. 20.] **Vector Bundles.**

    (a) The *Möbius Band*, $\mathcal{MB}$ is the space $[0,1] \times \mathbb{R}/\sim$ where $(0,x) \sim (1,-x)$. Show that $\mathcal{MB}$ is a smooth vector bundle.

    (b) Suppose that $\pi : E \to M$ is a smooth vector bundle of rank r. Show that E is trivial if and only if M has a *smooth global frame, i.e.,* smooth global sections $\sigma_1(p), \ldots, \sigma_r(p)$ such that $\{\sigma_1(p), \ldots, \sigma_r(p)\}$ is a basis for $\pi^{-1}(p)$ for each p.

    (c) Use (b) to determine whether $T\mathbb{S}^1$, $\mathcal{MB}$ or $T\Sigma_g$ are trivial.

    [Wortman, *Math 6510 Notes,* pp. 8–15, 85–93.]

15. [Sept. 23.] **Whitney Sum of Vector Bundles.** Suppose E' and E'' are smooth vecor bundles over the manifold M of ranks k' and k'', resp. The *Whitney Sum* is the vector bundle $E' \oplus E''$ over M of rank $k' + k''$ whose total space is $E' \oplus E'' = \amalg_{p \in M} (\pi')^{-1}(p) \oplus (\pi'')^{-1}(p)$ with the obvious projection $\pi : E' \oplus E'' \to M$. Suppose $\{(U_i, \phi_i)\}$ is a system of coordinate charts small enough that $\Phi_i' : (\pi')^{-1}(U_i) \to U_i \times \mathbb{R}^{k'}$ trivializes $E' \to M$ with with $\Phi_i'(p, V') = (p, \eta_i'(p, V'))$ and $\Phi_i'' : (\pi'')^{-1}(U_i) \to U_i \times \mathbb{R}^{k''}$ trivializes $E'' \to M$ with $\Phi_i''(p, V'') = (p, \eta_i''(p, V'))$. Their transition functions are $\Phi_{ij}' : U_i \cap U_j \times \mathbb{R}^{k'} \to U_i \cap U_j \times \mathbb{R}^{k'}$ given by $\Phi_{ij}'(p, V') = (p, g_{ij}'(p)[V'])$ where $g_{ij}'(p) \in \mathrm{GL}_{k'}(\mathbb{R})$ and $\Phi_{ij}'' : U_i \cap U_j \times \mathbb{R}^{k''} \to U_i \cap U_j \times \mathbb{R}^{k''}$ given by $\Phi_{ij}''(p, V'') = (p, g_{ij}''(p)[V''])$ where $g_{ij}''(p) \in \mathrm{GL}_{k''}(\mathbb{R})$. The trivializations of $E' \oplus E''$ are $\Phi_i : \pi^{-1}(U_i) \to U_i \times (\mathbb{R}^{k'} \oplus \mathbb{R}^{k''})$ given by $\Phi_i'(p, (V', V'')) = (p, (\eta_i'(p, V), \eta_i''(p, V'')))$ with transition functions $\Phi_{ij} : U_i \cap U_j \times \mathbb{R}^{k'+k''} \to U_i \cap U_j \times \mathbb{R}^{k'+k''}$ given by $\Phi_{ij}'(p, (V', V'')) = (p, (g_{ij}'(p)[V'], g''(p)[V'']))$. The Whitney Sum of two nontrivial bundles may be trivial. Show that $\mathcal{MB} \oplus \mathcal{MB}$ is trivial where $\mathcal{MB}$ is the Möbius Band. [D. Barden and C. Thomas, *An Introduction to Differential Manifolds,* Imperial College Press 2003, p. 53.]

16. [Sept. 25.] **Cross Product Lie Algebra.**

    (a) Show that $\mathbb{R}^3$ with the cross product is a Lie Algebra.

    (b) Let $A \subset \Gamma(T\mathbb{R}^3)$ be the subspace spanned by $\{X, Y, Z\}$ where

    $$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

    Show that A is a Lie subalgebra of $\Gamma(T\mathbb{R}^3)$ and that it is isomorphic to $\mathbb{R}^3$ with the cross product. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 202.]

17. [Sept. 27.] **Linear Vector Fields.** Let A be a real $n \times n$ matrix and define the matrix exponential by the series
$$e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k.$$
which is convergent for all A.

   (a) Let $V(x) = Ax$ be a linear vector field on $\mathbb{R}^n$. Check that the flow generated by V is
   $$\phi_t(x_0) = e^{tA} x_0$$
   and that it is a one parameter group of diffeomorphisms.

   (b) If B is another real $n \times n$ matrix and $W(x) = Bx$ is its corresponding linear vector field, show that $[V, W]$ is a linear vector field and find its associated matrix. [Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover, 2016, p. 60.]

   (c) For these vector fields, compute the Lie derivative directly and check the formula
   $$\pounds_V W = [V, W].$$

18. [Sept. 30.] **Point-Moving Diffeomorphism.** Let $B \subset \mathbb{R}^n$ be the open unit ball centered at the origin and let $y \in B$. Define a vector field on B and its resulting flow to find a diffeomorphism $f : B \to B$ such that $f(0) = y$, f fixes a neighborhood of ∂B and f is homotopic to the identity on B. Conclude that if M is smooth connected manifold and $p, q \in M$ then there is a diffeomorphism $F : M \to M$ such that $F(p) = q$ and F is homotopic to the identity on M. [University of Utah Preliminary Examination in Geometry/Topology, January 2017.]

19. [Oct. 2.] **Foliation from Submersion.** Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth submersion. Show that the connected components of the nonempty level sets of F form a foliation of M. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 513.]

20. [Oct. 4.] **Curve in a Submanifold.** Suppose M is a smooth manifold, $S \subset M$ is a smooth submanifold, Δ_p a smooth involutive k-plane distribution on M and $\gamma : (a, b) \to M$ a smooth curve in M.

   (a) Show that if $\gamma((a, b)) \subset S$ and S is an embedded submanifold then $\dot\gamma(t) \in T_{\gamma(t)} S$ for all $t \in (a, b)$.

   (b) Find a counterexample to (a) if S is not embedded.

   (c) Show that if $\gamma((a, b)) \subset S$ and S is an integral manifold of Δ_p then $\dot\gamma(t) \in T_{\gamma(t)} S$ for all $t \in (a, b)$.

   (d) Show that if $\dot\gamma(t) \in \Delta_{\gamma(t)}$ for all $t \in (a, b)$ then $\gamma((a, b))$ is contained in a single leaf of the foliation defined by Δ_p.

   [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, pp. 124, 513.]

21. [Oct. 14.] **Two-plane Distribution on the Sphere.** Suppose that the three sphere $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$ is embedded into four space the standard way. Two independent vector fields on $\mathbb{S}^3$ are given by $U = (y, -x, w, -z)$, $V = (w, z, -y, -x)$. Determine whether it is possible to find a two dimensional submanifold $N \subset \mathbb{S}^3$ such that the vector fields U and V are tangent to N at every point of N, *i.e.,* $U(p), V(p) \in T_p N$ for every $p \in N$? [University of Utah Preliminary Examination in Geometry/Topology, August 2019.]

22. [Oct. 16.] **Left Invariant Vector Field.** Let $G = \mathbf{O}(n) = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^T A = I\}$.

   (a) Show that G is a smooth manifold.

   (b) Describe the matrices which form the tangent space $T_P G$ for any $P \in G$.

   (c) Given $V \in T_I G$ and $P \in G$, find $X_V(P)$ where X_V is the left invariant vector field on G such that $X_V(I) = V$. Show that your X_V is left invariant.

   (d) Let θ_t^X be the flow of the vector field $X = X_V$ for some $V \in T_I G$. Conpute $\theta_t^X(I)$ in case $G = \mathbf{O}(2)$ and $V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   [University of Utah Preliminary Examination in Geometry/Topology, August 2019.]

23. [Oct. 18.] **Lie Subgroups and Homomrphisms.**

   (a) Let G be a Lie subgroup of $\mathbf{GL}_n(\mathbb{R})$. Prove that if $A, B \in \mathfrak{g} \le \mathfrak{gl}_n(\mathbb{R})$ then $[A, B] = AB - BA$.

   (b) Using (a), prove that any Lie Group homomorphism $\varphi : \mathbf{SL}_2(\mathbb{R}) \to \mathbb{R}$ has $d\varphi_1 = 0$. [Wortman, *Math 6510 Notes,* p. 198.]

24. [Oct. 21.] **Abelian Lie Groups.** Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Prove that G is Abelian. [University of Utah Preliminary Examination in Geometry/Topology, January 2017. See Wortman, *Math 6510 Notes,* p. 212.]

25. [Oct. 23.] **Subgroups of** $\mathrm{SL}_2(\mathbb{R})$**.**

   (a) Show that $\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -4 \end{pmatrix}$ has no square root. Then deduce that exp does not map the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ onto $\mathrm{SL}_2(\mathbb{R})$. [R. Bishop & R. Crittenden, *Geometry of Manifolds,* Academic Press, 1964, p. 33.]

   (b) Show that $\mathrm{SL}_2(\mathbb{R})$ is connected. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, pp. 537, 563.]

   (c) Let $U = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$. Find all connected Lie subgroups of $\mathrm{SL}_2(\mathbb{R})$ containing U. [University of Utah Preliminary Examination in Geometry/Topology, May 2017. See Wortman, *Math 6510 Notes,* p. 212.]

26. [Oct. 25.] **Cartan's Lemma.** Let $\omega^1, \ldots, \omega^k$ be smooth one-forms on a smooth manifold M^n which are linearly independent at all points of M. Let $\alpha^1, \ldots, \alpha^k$ be smooth one-forms such that

$$\textstyle\sum_{i=1}^k \alpha^i \wedge \omega^i = 0.$$

   Show that there are uniquely determined smooth functions ${c^i}_j = {c^j}_i$ such that

$$\alpha^i = \textstyle\sum_{j=1}^k {c^i}_j \, \omega^j.$$

   [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, pp. 375.]

27. [Oct. 28.] **Invariant Formula for Exterior Derivative and Involutivity.** Let ω be a smooth one form and let X, Y be smooth vector fields on the smooth manifold M^m. Then the invariant formula for the exterior derivative is

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

(a) Show that the invariant description of $d\omega$ agrees with the coordinate description.

(b) A smooth k-plane distribution on M may be defined using one forms. For every $p \in M$ there is an open neighborhood U of p in M and $m-k$ linearly independent one forms ω^i on U such that

$$\Delta_q = \left\{X \in T_qM : \omega^{k+1}(X) = \omega^{k+2}(X) = \cdots = \omega^m(X) = 0\right\}, \qquad \text{for all } q \in U.$$

Show that Δ_p is involutive if and only if there are one forms $\omega^i{}_j$ such that on U,

$$d\omega^i = \sum_{j=k+1}^{m} \omega^i{}_j \wedge \omega^j, \qquad \text{for all } i = k+1, \ldots, m$$

(c) Determine whether the two plane distribution in $\mathbb{R}^3$ given by $\Delta_p = \ker \omega(p)$ is integrable, where $\omega = -y\,dx + x\,dy + dz$.

28. [Oct. 30.] **Orientability of the Tangent Bundle.** Let M be a smooth manifold. Show that the tangent bundle TM is orientable. [University of Utah Preliminary Examination in Geometry/Topology, Jan., 2016.]

29. [Nov. 1.] **Orientability of Boundaries.** Let M be an m-dimensional smooth manifold with boundary embedded in $\mathbb{R}^m$. Show that ∂M is orientable. Conclude that each Σ_g is orienbtable. [Wortman, *Math 6510 Notes,* p. 284.]

30. [Nov. 4.] **Integral of an Orientation Form.** Suppose that $U \in \mathbb{R}^n$ is an open set and $\psi_U \in \Omega^n(U)$ is an integrable, nonvanishing form that determines the orientation of U. Show that $\int_U \psi_U > 0$. Suppose that M^n is a smooth, compact, orientable manifold and $\omega \in \Omega^n(M)$ is an nonvanishing form that determines the orientation of M. Show that $\int_M \omega > 0$. [Wortman, *Math 6510 Notes,* p. 300.]

31. [Nov. 6.] **A Two Form on $\mathbb{S}^2$.** Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Let

$$\omega = \begin{cases} \dfrac{dy \wedge dz}{x}, & \text{if } x \neq 0; \\[2ex] \dfrac{dz \wedge dx}{y}, & \text{if } y \neq 0; \\[2ex] \dfrac{dx \wedge dy}{z}, & \text{if } z \neq 0. \end{cases}$$

(a) Show that ω is a well defined two form on $\mathbb{S}^2$

(b) Find $\displaystyle\int_{\mathbb{S}^2} \omega.$

(c) Is ω exact? Why?

[University of Utah Preliminary Examination in Geometry/Topology, May 2109.]

32. [Nov. 8.] **No Retraction.** Let M^m be a smooth compact oriented manifold with boundary.

    (a) Let $f : \partial M \to \partial M$ be smooth a diffeomorphism. Show that f has no smooth extension to $F : M \to \partial M$. [Hint: Let $i : \partial M \hookrightarrow M$ so $f = F \circ i$. Consider $\int_{\partial M} f^* \psi_{\partial M}$.]

    (b) Deduce that there is no smooth retraction of the Euclidean ball $F : \overline{B_1^n(0)} \to \mathbb{S}^{n-1}$ and conclude that any smooth map $g : \overline{B_1^n(0)} \to \overline{B_1^n(0)}$ has a fixed point.

    [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 186, and Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover 2016, p. 221.]

33. [Nov. 11.] **Divergence Theorem.** Let $M^3 \subset \mathbb{R}^3$ be a smooth, connected, compact imbedded submanifold with boundary. Let $V = v^i \frac{\partial}{\partial x^i}$ be a smooth vector field in the usual coordinates of $\mathbb{R}^3$.

    (a) Find $\omega \in \Omega^2(\mathbb{R}^3)$ such that

    $$d\omega = \operatorname{div}(V)\, dx^1 \wedge dx^2 \wedge dx^3, \qquad \text{where } \operatorname{div}(V) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i}.$$

    (b) For $p \in \partial M$, let (u^1, u^2) be local coordinates for ∂M near p. Express $i^*\omega\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)$ in terms of V, $i_*\left(\frac{\partial}{\partial u^1}\right)$ and $i_*\left(\frac{\partial}{\partial u^2}\right)$ where $i : \partial M \hookrightarrow M$.

    (c) Deduce the usual Divergence Theorem using Stokes Theorem where N is the outer unit normal and $d\sigma = \left| i_*\left(\frac{\partial}{\partial u^1}\right) \times i_*\left(\frac{\partial}{\partial u^2}\right) \right| du^1 \wedge du^2$ is the usual area form of ∂M

    $$\int_M \operatorname{div}(V)\, dx^1 \wedge dx^2 \wedge dx^3 = \int_{\partial M} V \cdot N \, d\sigma.$$

    [T. Aubin, *A Course in Differential Geometry,* Amer. Math. Soc., 2000, pp. 60–61.]

34. [Nov. 13.] **A One Form on the Punctured Plane.** For $(x, y) \in M = \mathbb{R}^2 - \{(0, 0)\}$ let

    $$\omega(x, y) = \frac{-y\, dx}{x^2 + y^2} + \frac{x\, dy}{x^2 + y^2}.$$

    (a) Find $\displaystyle\int_{C(r)} \omega$ where $C(r)$ is the circle of radius $r > 0$ centered at the origin with clockwise orientation.

    (b) Find a function $f \in \mathcal{C}^\infty(M)$ such that $df = \omega$ or explain why there is no such f.

    [University of Utah Preliminary Examination in Geometry/Topology, Jan. 2013.]

35. [Nov. 15.] **Poincaré Lemma.** Let $k \ge 1$ and $\omega \in \Omega^k(B_1^n(0))$. Show that $Q\, d\omega + d\, Q\omega = \omega$ and conclude that $H^k(\mathbb{R}^n) = \{0\}$ for all $k > 0$. The linear maps $Q : \Omega^k(B_1^n(0)) \to \Omega^{k-1}(B_1^n(0))$ are defined for all k and all multi-indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ for the form $\omega = g(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ by

    $$Q\omega = \left(\int_0^1 t^{k-1} g(tx)\, dt \right) \sum_{j=1}^{k} (-1)^{j-1} x^{i_j}\, dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots dx^{i_k},$$

    where "hat" means omit. [I. Singer and J. Thorpe, *Lectures on Elementary Topology and Geometry,* Springer, 1967, p. 130.]

36. [Nov. 18.] **Cohomology of a Punctured Manifold.** Let M be a connected smooth manifold of dimension $n \geq 3$. For any $x \in M$ and $0 \leq p \leq n-2$, prove that $H^p(M) \to H^p(M - \{x\})$ induced by inclusion $M - \{x\} \hookrightarrow M$ is an isomorphism. Prove that the same is true for $p = n-1$ if M is compact and orientable. Hint: use the Mayer-Vietoris Theorem. The cases $p = 0$, $p = 1$ and $p = n-1$ require special handling. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 465.]

37. [Nov. 20.] **Extending a Homotopy.** Let M and N be smooth manifolds, $I = [0,1]$ and $H : I \times M \to N$ be a smooth homotopy. Then there is a smooth map $\tilde{H} : \mathbb{R} \times M \to N$ such that $\tilde{H}(t,x) = H(t,x)$ for all $(t,x) \in I \times M$.

38. [Nov. 22.] **Connected Sum.** Let M and N be compact connected smooth manifolds of dimension n. Their *connected sum* $M \# N$ may be defined using two embeddings $f_M : \mathbb{R}^n \hookrightarrow M$ and $f_N : \mathbb{R}^n \hookrightarrow N$ and on

$$W = (M \backslash \{f_M(0)\}) \ \amalg \ (N \backslash \{f_N(0)\})$$

using the equivalence relation $f_M(x) \sim f_N(x/|x|^2)$. (Points not in the image of f_M or f_N are equivalent only to themselves.) Then $M \# N = W/\!\sim$ with the induced smooth structure.

   (a) Prove that $M \# N$ is orientable if and only only if both M and N are.

   (b) For M and N both orientable and $n > 1$, calculate the de Rham cohomology of $M \# N$ in terms of $H^*(M)$ and $H^*(N)$.

   [D. Barden and C. Thomas, *An Introduction to Differential Manifolds,* Imperial College Press 2003, p. 117 and John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, p. 465.]

39. [Nov. 25.] **Poincaré Lemma for Compactly Supported Cohomology.** Let M^n be a smooth manifold and $\pi : M \times \mathbb{R} \to M$ be projection. Show that $\pi_\# : H_c^{k+1}(M \times \mathbb{R}) \to H_c^k(M)$ is an isomorphism.

   (a) $\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M)$, integration along the fiber, is defined as follows. A compactly supported form on $M \times \mathbb{R}$ may be written

$$\omega = f(x,t)\, \pi^*\phi + g(x,t)\, \pi^*\psi \wedge dt$$

   where ϕ and ψ are forms on the base M, not necessarily of compact support, and f and g are functions on $M \times \mathbb{R}$ with compact support. Define

$$\pi_*\omega = \left(\int_{\mathbb{R}} g(x,t)\, dt \right) \psi$$

   Show $d\,\pi_* = \pi_*\, d$ so $\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M)$ is a cochain map.

   (b) Let $e \in \Omega_c^1(\mathbb{R})$ so that $\int_{\mathbb{R}} e = 1$. Define $e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(M \times \mathbb{R})$ by $\phi \mapsto (\pi^*\phi) \wedge e$. $d\,e_* = e_*\,d$ so this gives a cochain map $e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(M \times \mathbb{R})$ such that $\pi_* e_* = \mathrm{Id}$. Prove that $\omega - e_*\pi_*\omega = (-1)^{k-1}(Kd\omega - dK\omega)$ for $\omega \in \Omega_c^k(M \times \mathbb{R})$ where the cochain homotopy $K : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M \times \mathbb{R})$ is defined by

$$K(f\,\pi^*\phi + g\,\pi^*\psi \wedge dt) = \left(\int_{-\infty}^t g(x,s)\, ds \right) \pi^*\psi - \left(\int_{-\infty}^t e \right) \left(\int_{\mathbb{R}} g(x,s)\, ds \right) \pi^*\psi$$

   (c) Use this to show $H_c^n(\mathbb{R}^n) = \mathbb{R}$ and $H_c^k(\mathbb{R}^n) = \{0\}$ for $k \neq n$.

   [R. Bott and L. Tu, *Differential Forms in Algebraic Topology,* Springer, 1982, pp 37–40.]

40. [Nov. 27.] **Degree and Fundamental Theorem of Algebra.** Suppose $f(z) \in \mathbb{C}[z]$ satisfies $f(0) = 1$ and has no roots. Prove the Fundamental Theorem of Algebra by showing $f = 1$. Use $\mathbb{S}^1 \subset \mathbb{C}$ and the functions

$$F_1 : \mathbb{S}^1 \times I \to \mathbb{S}^1, \quad F_1(z,t) = \frac{f(tz)}{|f(tz)|} \qquad \text{and} \qquad F_2 : \mathbb{S}^1 \times I \to \mathbb{S}^1, \quad F_2(z,t) = \frac{t^n f\left(\frac{z}{t}\right)}{\left|t^n f\left(\frac{z}{t}\right)\right|}.$$

[Wortman, *Math 6510 Notes,* p. 349.]

41. [Dec. 2.] **Degree of Maps to Tori.**

    (a) Let $n \ge 2$ and $f : \mathbb{S}^n \to \mathbb{T}^n$ be smooth. Show $\deg f = 0$.

    (b) Let $p \ge 1$, $q \ge 1$ and $g : \mathbb{S}^{p+q} \to \mathbb{S}^p \times \mathbb{S}^q$ be smooth. Show $\deg g = 0$. [Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover 2016, p. 249.]

    (c) Let $h, i : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ be given by $i(z,w) = (z,w)$ and $h(z,w) = (w, \bar{z})$. Show $\deg h = \deg i$ but h is not homotopic to i. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, p. 466.]

42. [Dec. 4.] **Gauss Curvature.** For each planar metric, find the Gauss curvature $K(x,y)$.

    (a) A metric conformal to the Euclidean plane given by $ds^2 = e^{2u(x,y)}(dx^2 + dy^2)$, where $u(x,y)$ is a smooth function.

    (b) Use (a) for the Poincaré metric $ds^2 = \dfrac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$ in $x^2 + y^2 < 1$.

    (c) The Chebychev metric $ds^2 = dx^2 + 2\cos\varphi(x,y)\, dx\, dy + dy^2$ where $0 < \varphi(x,y) < \pi$ is a smooth function.

The last day to turn in any remaining homework is Monday, Dec. 9.