Homework for Math 6410 §1, Fall 2016

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Our main text this semester is Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012 ISBN 978-0-8218-8328-0. Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Monday, Dec. 12, whichever comes first.

1. [Aug. 22] **Falling Stone.**

   Consider the case of the stone dropped from height h. Denote by r the distance to the surface of the earth. The initial conditions are $r(0) = h$ and $r'(0) = 0$. The equation is

   $$\text{(exact model) } \ddot{r} = -\frac{\gamma M}{(R+r)^2}; \qquad \text{(approximate model) } \ddot{r} = -g.$$

   where $g = \gamma M/R^2$ and R, M are the radius and mass of the earth, respectively.

   (a) Transform both equations into first order systems.

   (b) Compute the solution of the approximate systemm for the given initial conditions. Compute the time it takes for the stone to hit the surface ($r = 0$).

   (c) Assume that the exact equation also has a unique solution corresponding to the given initial conditions. What can you say about the time it takes for the stone to hit the surface in comparison with the approximate model? Will it be longer or shorter? estimate the difference between the solutions of the exact equation and the approximate equation. [Hint: Do not compute the solution of the exact equation. Look for the minimum and maximum of the force.]

   (d) Look up the physical constants and give a numerical values for the case $h = 10$ meters.

   [1.1 from Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012]

2. [Aug. 24] **Compute a Phase Portrait using a Computer Algebra System.**

   This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the Maple worksheet from today's lecture

   `http://www.math.utah.edu/~treiberg/M6412eg1.mws`

   `http://www.math.utah.edu/~treiberg/M6412eg1.pdf`

   or my lab notes from Math 2280,

   `http://www.math.utah.edu/~treiberg/M2282L4.mws`.

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. Compute the Jacobian at rest points and explain why your system satisfies this. (Everyone in class should have a different ODE.) Using your favorite computer computer algebra system, *e.g.,* MAPLE or MATLAB, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include the stable and unstable curves at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any separatrices.

3. [Aug. 26] **Escape Times.**

   Show that each solution $\big(x(t), y(t)\big)$ of the initial value problem

$$\begin{cases} x' = & y + x^2 \\ y' = & x + y^2 \end{cases} \qquad\qquad \begin{cases} x(0) = & x_0 \\ y(0) = & y_0 \end{cases}$$

   with $x_0 > 0$ and $y_0 > 0$ cannot exist on an interval of the form $[0, \infty)$.

   [*cf.* Wilson, *Ordinary Differential Equations,* Addison-Wesley, 1971, p.255.]

4. [Aug. 29] **Ode on a Grecian Urn.**

   Let $f \in \mathcal{C}(\mathbf{R})$ and $x(t)$ a solution to

$$\dot{x} = f(x). \tag{1}$$

   (a) Show that if

$$x_1 = \lim_{t\to\infty} x(t)$$

   exists then

$$0 = \lim_{t\to\infty} \dot{x}(t).$$

   (b) Give a counterexample to show that (a) is false if x is just continuously differentiable but (1) does not hold.

   [1.27 from Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012]

5. [Aug. 31] **Periodic Solution.**

   Let $a \in \mathbf{R}$ and $g \in \mathcal{C}(\mathbf{R})$ such that $g(t) \ge 0$ and $g(t+1) = g(t)$ for all t. Find conditions on a and g so that there exist a periodic solution. When is it unique?

$$\dot{x} = ax + g(t).$$

   [1.36 from Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012]

6. [Sept. 2] **Delay Differential Equation.**

   The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \le 0$. Using the Contraction Mapping Principle, show the local existence of a solution to the delay differential equation.

   Theorem. Let $f \in \mathcal{C}(\mathbf{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbf{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|).$$

   Let $g \in \mathcal{C}(\mathbf{R})$ such that $g(t) \le t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbf{R}\big)$. Show that there is an $r > 0$ such that the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) & \text{for } 0 < t < r, \\ \quad x(t) = \varphi(t) & \text{for all } t \le 0 \end{cases}$$

   has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbf{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbf{R}\big)$.

   [*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

7. [Sept. 7] **Linear Equations are Benign.**

Let the $n \times n$ matrix $A(t)$ and the vector $b(t) \in \mathbf{R}^n$ be continuous functions for all $t \in \mathbf{R}$. Show that solutions of the inhomogeneous linear equation

$$\begin{cases} \dfrac{dx}{dt}(t) = A(t)x(t) + b(t) \\ x(t_0) = x_0 \end{cases}$$

are bounded on finite intervals so exist for all time.

8. [Sept. 9] **Alternative Proof of the Picard-Lindelöf Theorem.**

We set up the same as in the Picard-Lindelöf Theorem. Let $V \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and $(t_0, x_0) \in V$. Suppose $f(t,x) \in \mathcal{C}(V, \mathbf{R}^n)$ satisfies a local Lipschitz condition in x uniformly in t. Let $T, \delta > 0$ so that the closed cylinder $\Sigma = [t_0 - T, t_0 + T] \times \overline{B_\delta(x_0)} \subset V$. Let

$$M = \sup_{(t,x)\in\Sigma} |f(t,x)|$$

and L be the Lipschitz constant on Σ. Let $\epsilon = \min\{T, \delta/M\}$ and $I = [t_0 - \epsilon, t_0 + \epsilon]$. Let $X = \mathcal{C}(I, \mathbf{R}^n)$ be the Banach Space of continuous curves with sup-norm $\| \bullet \|$. Then $C = \{y \in X : |y(t) - x_0| \le \delta \text{ for all } t \in I\}$ is a closed subset. For $y \in C$ we defined the operator

$$K[y](t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

We have shown that $K : C \to C$. Put $y_0 = x_0$ and let $y_{n+1} = K(y_n)$ for all n. We have $y_n \in C$ for all n by induction.

(a) Prove

$$|y_{n+1}(t) - y_n(t)| \le \frac{ML^n|t - t_0|^{n+1}}{(n+1)!} \qquad \text{for all } n = 0, 1, 2, \dots \text{ and } t \in I.$$

(b) Using (a), show that there is a solution $\bar{y} \in \mathcal{C}^1(I, \mathbf{R}^n)$ of the initial value problem

$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t)\big), \qquad \text{for } t \in I; \\ x(t_0) &= x_0. \end{aligned}$$

Note that this improves our result using contraction mapping, exploiting the fact that the integral equation is of Volterra type.

9. [Sept. 12.] **Compare Solutions of Two Mathieu Equations.**

One solution for Problem 19 of the 2010 Math 6410 depended on comparing the solutions of the perturbed and unperturbed problems. Find a sharp estimate for the difference in values and derivatives at $T = \frac{2\pi}{3}$ of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\begin{aligned} \ddot{x} + x &= 0, & \qquad \ddot{y} + (1 + \epsilon \sin(3t))y &= 0, \\ x(0) &= u_0, & y(0) &= u_0, \\ \dot{x}(0) &= u_1; & \dot{y}(0) &= u_1. \end{aligned}$$

Let $y(t; u_0, u_1, \epsilon)$ solve the IVP. Use your estimate to show $|y(T; 1, 0, \epsilon) + \dot{y}(T; 0, 1, \epsilon)| < 2$ for small $|\epsilon|$.

10. [Sept. 14.] **Concrete Variational Equation.**

Let

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbf{R}^3$ of

$$\begin{aligned} \frac{dx}{dt} &= f\big(x(t)\big), \\ x(0) &= y. \end{aligned}$$

Find

$$\Phi(t, y) = D_2\,\varphi(t, y).$$

Show that it satisfies the variational equation

$$\begin{aligned} \frac{d\Phi}{dt} &= Df\big(\varphi(t, y)\big) \cdot \Phi(t, y), \\ \Phi(0) &= I. \end{aligned}$$

[Perko, p. 84.]

11. [Sept. 16] **Regular Perturbation Problem.**

Approximate the solution up to first order in ϵ.

$$\ddot{x} + x + \epsilon x^3 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0.$$

[2.17 from Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012]

12. [Sept. 19] **Differentiability of Fixed Points.**

Let $f(x, \epsilon) \in \mathcal{C}^1(\mathbf{R}^2, \mathbf{R})$ be a continuously differentiable contraction: there is a $\lambda \in (0, 1)$ so that

$$|f(x, \epsilon) - f(y, \epsilon)| \le \lambda |x - y| \qquad \text{for all } x, y, \epsilon.$$

Show that the unique fixed point $g(\epsilon)$ which satisfies $g(\epsilon) = f(g(\epsilon), \epsilon)$ is a continuously differentiable function of ϵ.

13. [Sept. 21.] **Existence via Schauder's Theorem.**

Give another proof of the Peano Existence Theorem using the Schauder Fixed Point Theorem.

Theorem. [Peano Existence Theorem] Let $D \subset \mathbf{R} \times \mathbf{R}^n$ be a domain and $f \in \mathcal{C}(D, \mathbf{R}^n)$. Then for any $(t_0, x_0) \in D$ there is $\epsilon > 0$ and a continuously differentiable function $x(t) : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbf{R}^n$ which solves the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= f(t, (x(t))), \qquad \text{for all } t \in [t_0 - \epsilon, t_0 + \epsilon]; \\ x(t_0) &= x_0. \end{aligned}$$

Theorem. [Schauder Fixed Point Theorem] Let $\mathcal{A}$ be a closed, bounded, convex subset of a Banach space $\mathcal{X}$ and $T : \mathcal{A} \to \mathcal{A}$ be a completely continuous function. Then T has a fixed point in $\mathcal{A}$.

A subset $\mathcal{K}$ of a Banach space is *compact* if any sequence in $\{\phi_i\}_{i=1,2,\ldots} \subset \mathcal{K}$ has a subsequence that converges to an element in $\mathcal{K}$. f is *compact* if for any bounded set $\mathcal{B} \subset \mathcal{X}$, the closure of the set $f(\mathcal{B})$ is compact. f is *completely continuous* if it is both compact and continuous. [*cf.* Hale, p. 14.]

14. [Sept. 23] **Nagumo's Uniqueness Theorem.**

Prove the theorem and show that it implies the uniqueness statement in the Picard Theorem.

Theorem. [Nagumo, 1926] Suppose $f \in \mathcal{C}(\mathbb{R}^2)$. Then the initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(t, y), \\ y(0) &= 0, \end{aligned}$$

has a solution by Peano's Existence Theorem. Assume in addition that

$$|f(t, y) - f(t, z)| \leq \frac{|y - z|}{|t|}$$

for all $t, y, z \in \mathbb{R}$ such that $t \neq 0$. Then the solution is unique.

15. [Sept. 26.] **Jordan Form.**

Find the generalized eigenvectors, the Jordan form and the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} 6 & 6 & 4 \\ -2 & -2 & -4 \\ 2 & 6 & 8 \end{pmatrix} \mathbf{y}.$$

16. [Sept. 28.] **Real Canonical Form.**

Let A be a real 2×2 matrix whose eigenvalues are $a \pm bi$ where $a, b \in \mathbb{R}$ such that $b \neq 0$. Show that there is a real matrix Q so that

$$Q^{-1} A Q = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Use this fact to solve the system

$$\begin{aligned} x' &= -13x - 10y \\ y' &= 20x + 15y \end{aligned}$$

17. [Sept. 30.] **Just Multiply by t.**

Consider the n-th order constant coefficient linear homogeneous scalar equation

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$$

where a_i are complex constants. Convert to a first order differential system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to the distinct eigenvalues μ_i and $0 \leq k < m_i$ where m_i is the algebraic multiplicity of μ_i. [*cf.*, Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.68.]

18. [Oct. 3.] **Variation of Parameters Formula.**

Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

19. [Oct. 5] **Application of Liouville's Theorem.**

Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases} x'' + \dfrac{1}{t}\,x' + x = 0 \\ x(0) = 1, \qquad x'(0) = 0 \end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t \to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

20. [Oct. 7.] **To Use Jordan Form or Not to Use Jordan Form.**

Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations.

(a) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a matrix B with distinct eigenvalues so that $\|A - B\| \le \epsilon$.

(b) Give three proofs of $\det(e^A) = e^{\operatorname{trace}(A)}$.

(c) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}. \tag{2}$$

Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.,* Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

(d) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (2) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

(e) [Optional.] Find all continuous scalar valued functions $f \in C(\mathcal{M}_{n\times n}(\mathbb{C}), \mathbb{C})$ so that

$$f(AB) = f(A)f(B) \qquad \text{for all } A, B.$$

You can probably find several different arguments on your own. [*ibid.*; or Kurosh, *Higher Algebra*, p. 334.]

21. [Oct. 17.] ***T*-Periodic Linear Equations.**

Consider the T-periodic non-autonomous linear differential equation

$$\dot{x} = A(t)\, x, \qquad x \in \mathbb{R}^n, \qquad A(t+T) = A(t).$$

Let $\Phi(t)$ be the fundamental matrix with $\Phi(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t+T) = \mu\chi(t)$, where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \dots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\rho_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)\, A_0\, x$, $x \in \mathbb{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

22. [Oct. 19.] **Blowup in a Periodic Linear Equation.**

Let $\phi(t)$ be a real, continuous, π-periodic function. Consider the scalar equation

$$\ddot{x} - (\cos^2 t)\dot{x} + \phi(t)\, x = 0.$$

Show that there is a real solution that tends to infinity as $t \to \infty$.

[*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 162.]

23. [Oct. 21.] **Boundedness of Solutions in Mathieu's Equation.**

Show that if $|\varepsilon|$ is small enough, then all solutions are bounded.

$$\ddot{x} + \big[1 + \varepsilon \sin 3t\big] x = 0.$$

[U. Utah PhD Preliminary Examination in Differential Equations, January 2004.]

24. [Oct. 24.] **Condition for Asymptotic Stability.**

Suppose that the zero solution to $\dot{x} = Ax$ is asymptotically stable. Let $g(t, x) \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy $g(t, 0) = 0$ and

$$|g(t, x)| \le h(t)|x|, \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(s)\, ds \le kt + r, \qquad \text{for all } t \ge 0.$$

Show that there is a constant $k_0(A) > 0$ such that if $k \le k_0$, then the zero solution of

$$\dot{x} = Ax + g(t, x)$$

is asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 243.]

25. [Oct. 26.] **Linearized Stability of Fixed Points.**

The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\begin{aligned} \dot{S} &= \mu K - \beta SI - \mu S \\ \dot{I} &= \beta SI - \gamma I - \mu I - \alpha I \\ \dot{R} &= \gamma I - \mu R \end{aligned}$$

(a) Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

(b) Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.

(c) Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

26. [Oct. 28.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

Let $T \in \mathcal{C}(\mathbf{R}^n, \mathbf{R}^n)$. Consider the difference equation

$$\begin{aligned} x_0 &= x, \\ x_{n+1} &= T(x_n). \end{aligned} \tag{3}$$

Writing $Tx := T(x)$, a solution sequence of (3) can be given as the n-th iterates $x_n = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. The *forward orbit* of a point x is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. A set

$H \subset \mathbf{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ ($H \subset T(H)$). H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. The solution $T^n x$ starting from a given point x is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (3).

(a) Let A be a real $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\|\cdot\|$ on $\mathbb{R}^n$ so that $\|Ax\| \le \gamma \|x\|$ for all $x \in \mathbb{R}^n$.

(b) Let $P \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbb{R}^n$

$$\begin{aligned} x_1 &= x, \\ x_{n+1} &= P(x_n). \end{aligned}$$

27. [Oct. 31.] **Liapunov Functions.**

Use a Liapunov Function to show that the zero solution is asymptotically stable

$$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$

Hint: A sneaky way is to show that this equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y - x^3 \\ \dot{y} &= -x + 2x^3 - 2y. \end{aligned}$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

28. [Nov. 2.] **Četaev's Theorem.**

Show that the zero solution is not stable.

$$\begin{aligned} \dot{x} &= x^3 + xy \\ \dot{y} &= -y + y^2 + xy - x^3. \end{aligned}$$

[J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 286.]

29. [Nov. 4.] **Stability of the Origin in the Lorenz System**

The famous chaotic equations of meteorologist E. N. Lorenz model convective (predominantly vertical) flow realized by a fluid that is warmed from below and cooled from above. For b, r and σ positive constants,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

(a) Show symmetry: if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$.

(b) The positive and negative axes are invariant sets.

(c) The origin is a critical point. If $0 < r < 1$ then the origin is a global attractor and the zero solution is asymptotically stable. [Hint: $V = x^2 + \sigma y^2 + \sigma z^2$.]

(d) If $r > 1$ the origin is unstable.

(e) The vector field is forward complete. There exists a compact positively invariant set (depending on b, r and σ) into which each forward trajectory enters but never leaves. [Hint: $V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$. Additional hints in W. Walter, *Ordinary Differential Equations,* Springer 1998, p. 330.]

30. [Nov. 7.] **Stationary Points of a Hamiltonian System.**

    Show that the system is Hamiltonian.

$$\dot{x} = (x^2 - 1)(3y^2 - 1)$$
$$\dot{y} = -2xy(y^2 - 1)$$

    Find the equilibrium points and classify them. Find the Hamiltonian. Using obvious exact solutions and the Hamiltonian property, draw a rough sketch of the phase diagram. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 79.]

31. [Nov 9.] **Brusselator System.**

    Show that there is a nonconstant periodic trajectory for the system

$$\dot{x} = 1 - 4x + x^2 y$$
$$\dot{y} = 3x - x^2 y$$

    [University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2004.]

32. [Nov. 11.] **Dulac's Criterion.**

    Prove the following theorem of Dulac.
    Theorem. Let $A \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(A, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(A, \mathbb{R})$. Show that if $\operatorname{div}(\rho f)$ is not identically zero and does not change signs in any open subset of A then the equation $x' = f(x)$ has at most one periodic solution in A.

    Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \neq 0$)

$$\dot{x} = y$$
$$\dot{y} = -x + \lambda(1 - x^2)y$$

    has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

33. [Nov. 14.] **Stable and Unstable Manifolds.**

    Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\dot{x} = -x$$
$$\dot{y} = -y + x^2$$
$$\dot{z} = z + y^2.$$

    [*cf.* Perko, *Differential Equations and Dynamical Systems,* Springer, 1991, p. 116–117.]

34. [Nov. 16.] **Periodic Orbit in Predator-Prey System.**

    A generalization of a predator-prey system given by Brauer and Castillo-Chavez is

$$\dot{x} = x\left(1 - \frac{x}{30} - \frac{y}{x + 10}\right),$$
$$\dot{y} = y\left(\frac{x}{x + 10} - \frac{1}{3}\right).$$

    (a) Show that the fixed points are $(0, 0)$, $(30, 0)$ and $(5, 12.5)$ and have saddle, saddle, source type, resp.

(b) Show that the region bounded by $0 \le x$, $0 \le y$ and $x + y \le 50$ is forward invariant.

(c) Show that there is no orbit Γ with $\alpha(\Gamma) = \{(30, 0)\}$ and $\omega(\Gamma) = \{(0, 0)\}$. Conclude that there is a nonconstant periodic orbit.

[R. C. Robinson, *An Introduction to Dynamical Systems Continuous and Discrete*, Pearson/Prentice Hall, 2004, p. 238.]

35. [Nov. 18.] **Center Manifold.**

Find a center manifold for the system

$$\begin{aligned}\dot{x} &= -xy\\ \dot{y} &= -y + x^2 - 2y^2\end{aligned}$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold. Determine the stability of the origin.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_k\, x^k$$

using the condition of invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi(0) = \psi'(0) = 0$. Find the first few terms of the expansion, guess the rest and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

36. [Nov. 21.] **Topological Conjugacy.**

Let

$$A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha > 0.$$

Explicitly construct the conjugacy found in the proof of Theorem 9.10. [*cf.,* Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.270.]

37. [Nov. 23.] **Hartman-Grobman Theorem.**

Find a homeomorphism h in a neighborhood of 0 that establishes an topological conjugacy between the flow of the differential system and the flow of the linearized system, *i.e.,* $h(\psi(t, x)) = e^{tA} h(x)$ where $A = Df(0)$ and $\psi(t, x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -y + xz,\\ \dot{z} &= z.\end{aligned}$$

[Teschl discusses a proof, but you can guess h from the solutions and verify.]

38. [Nov. 28.] **Homeomorphisms Close to the Identity.**

Suppose that $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism of the form

$$\varphi(x) = x + h(x)$$

where $h(x)$ is bounded. Show that the inverse homeomorphism

$$\varphi^{-1}(y) = y + j(y)$$

where j is again bounded with the same bound $\|j\| \le \|h\|$.

[9.14 from Teschl, *Ordinary Differential Equations and Dynamical Systems,* A.M.S., 2012]

39. [Nov. 30.] **Stability of a Periodic Orbit.**

$\gamma(t) = (\cos t, \sin t, 0)$ is a periodic solution to the system

$$\begin{aligned}\dot{x} &= y + x(1 - x^2 - y^2)\\ \dot{y} &= x + y(1 - x^2 + y^2)\\ \dot{z} &= -z,\end{aligned}$$

Is $\gamma(t)$ orbitally asymptotically stable? Choose a transverse section Σ. Determine the Poincaré map $\wp : \Sigma \to \Sigma$ and find its differential at the periodic orbit. In particular, find the variational equation, the fundamental matrix associated with the periodic orbit and compute the Floquet Multipliers. [*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 213.]

40. [Dec. 2.] **Level Set Periodic Orbit.**

Let $f \in \mathcal{C}^2(\mathbf{R}^2)$ be such that zero is a *regular value* ($\nabla f(x) \ne 0$ whenever $f(x) = 0$.) Then the zero level set

$$A = \left\{(x, y) \in \mathbf{R}^2 : f(x, y) = 0\right\}$$

consists of embedded $\mathcal{C}^2$ curves. Prove that each bounded component of A is an attracting hyperbolic limit cycle for the differential equation

$$\begin{aligned}\dot{x} &= -f_y - f\, f_x,\\ \dot{y} &= f_x - f\, f_y.\end{aligned}$$

Describe the phase portrait in case

$$f(x, y) = \left((x - \mu)^2 + y^2 - 1\right)\left(x^2 + y^2 - 9\right).$$

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 197.]

41. [Dec. 5.] **Perturbation of Stable Solution of van der Pol Equation.**

Suppose $g \in \mathcal{C}^2(\mathbb{R}^2)$ and $\mu > 0$. Show that there is an $\varepsilon_0 > 0$ such that for all ε satisfying $|\varepsilon| < \varepsilon_0$ there is a unique periodic solution of the perturbed van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \varepsilon g(x, \dot{x})$$

in the neighborhood of the unique nontrivial periodic solution of

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

Can you prove that this solution is orbitally asymptotically stable? [J. Hale, *Ordinary Differential Equations,* Dover 2009; repub. R. Krieger 1980; orig. J. Wiley 1969, p. 227.]

42. [Dec. 7.] **Extended Center Manifold.**

Find the value of μ for which there is a bifurcation at the origin for the system

$$\dot{x} = y - x - x^2$$
$$\dot{y} = \mu x - y - y^2.$$

(a) Find the evolution equation on the extended center manifold correct to third order.

(b) What is the nature of the bifurcation?

[Glendenning, *Stability, Instability and Chaos,* Cambridge U. Press, 1994, p. 246.]