Math 5470 § 1. **Second Midterm Exam** Name: Solutions
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1. *Consider the system in polar coordinates. Show that there are numbers $0 < r_1 < r_2$ so that $r_1 \le r \le r_2$ is a trapping region. Show that the system has a nontrivial periodic trajectory.*

$$\dot r = r(3 - 2r^2 + r^2 \sin^2 \theta)$$
$$\dot\theta = 1$$

Using the fact that $0 \le \sin^2\theta \le 1$ we find that

$$r(3-2r^2) \le \dot r = r(3 - 2r^2 + r^2\sin^2\theta) \le r(3-r^2)$$

It follows that if one takes any $0 < r_1 < \sqrt{\frac{3}{2}}$ then $\dot r > 0$ whenever $r = r_1$ and flow is outward through the circle $r = r_1$. If one takes any $\sqrt{3} < r_2$ then $\dot r < 0$ whenever $r = r_2$ so flow is inward through the circle $r = r_2$. Thus the annulus $r_1 \le r \le r_2$ is a trapping region. Also, observe that $\dot\theta \ne 0$ for all $r > 0$ so that there are no fixed points in the annulus. Since the vector field is smooth, we may apply the Poincaré-Bendixson Theorem, which asserts that any trajectory starting in the trapping region tends to a nonconstant limit cycle, which is a nontrivial periodic trajectory in the trapping annulus.

2. *Consider the system in polar coordinates. The system undergoes a bifurcation as the parameter $\mu > 0$ passes the critical value μ_c. Find the value and sketch the phase portraits for $\mu < \mu_c$, $\mu = \mu_c$ and $\mu > \mu_c$. What kind of bifurcation is this?*

$$\dot r = r(1-r^2)$$
$$\dot\theta = 1 + \mu\cos\theta$$

The zeros of the $\dot r = r(1-r^2)$ occur at $r = 0$ and $r = 1$. The origin is an unstable rest point and the circle $r = 1$ is an invariant set. Since $\dot r > 0$ in $0 < r < 1$ and $\dot r < 0$ if $1 < r$ we see that the $r = 1$ is an attractive rest point. Flows starting away from the origin or unit circle tends toward the unit circle. $\dot\theta = 1 + \mu\cos\theta \ge 1 - \mu$ for all θ so $\theta(t)$ is strictly increasing for $0 < \mu < 1$ and, except for the origin, the flow approaches the limit cycle $r = 1$. At $\mu = 1$, an infinite time bifurcation occurs: $\theta(t)$ stops increasing at one point where $1 + \cos\theta = 0$ or $\theta = \pi$. For $\mu > 1$ there are two rest points $\theta = \phi_\pm$ with $0 < \phi_- < \pi < \phi_+$ which solve

$$0 = 1 + \mu\cos\theta, \qquad \text{namely, } \phi_\pm = \arccos(1/\mu).$$

Then $\dot\theta < 0$ for $\phi_- < \theta < \phi_+$ and $\dot\theta > 0$ otherwise. Radially flow approaches the $r = 1$ circle but $\dot\theta < 0$ for initial angles $\phi_- < \theta_0 < \phi_+$ and positive otherwise. This makes $(r,\theta) = (1, \phi_-)$ a stable node and $(1, \phi_+)$ an unstable saddle.

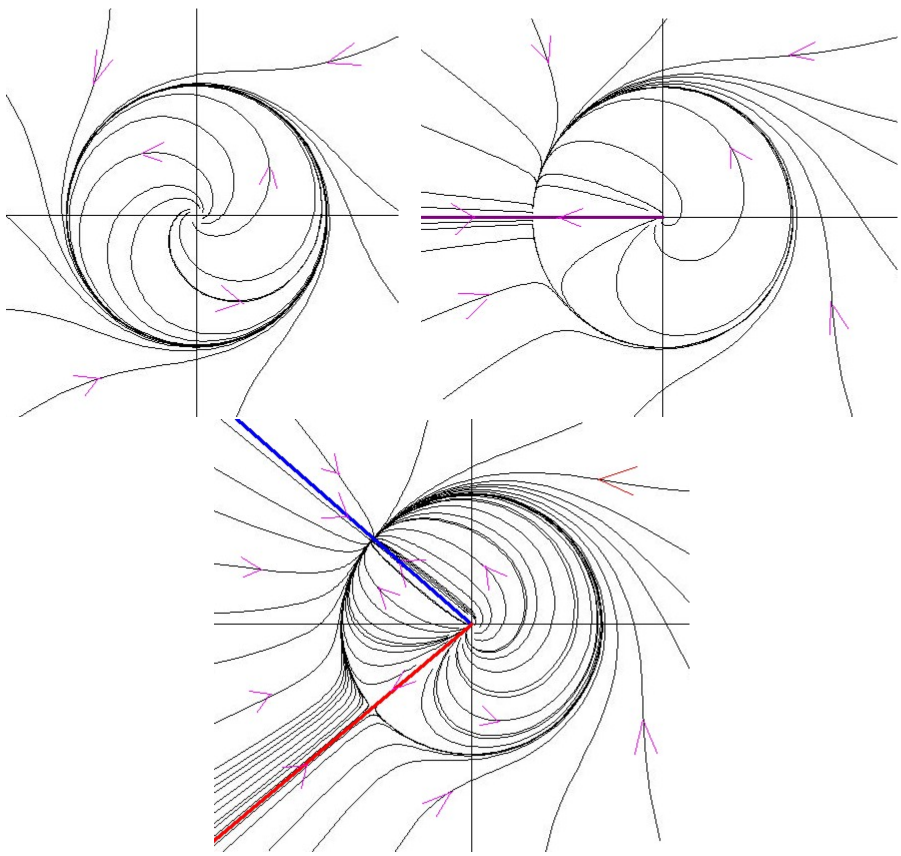

Figure 1: Plots with $\mu = .5, 1, 1.3$ using 3D-XPlorMath©

3. *Answer the following questions about periodic trajectories.*

(a) *Show that this equation has no nontrivial periodic solutions.*

$$\ddot{x} + x^2\,\dot{x} + x = 0.$$

Viewing this as a spring equation with nonnegative drag depending on velocity, one expects that energy decay under the flow. Written as a system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - xy^2\end{aligned}$$

Computing

$$\frac{d}{dt}E = \frac{d}{dt}\frac{x^2+y^2}{2} = x\,\dot{x} + y\,\dot{y} = xy + y(-x - xy^2) = -x^2y^2 \le 0.$$

If $(x,y) \neq (0,0)$, the flow does not stop at $x = 0$ because $\dot{x} = y \neq 0$ nor at $y = 0$ because then $\dot{y} = -x \neq 0$. Otherwise $\dot{E} < 0$ so the energy is strictly decreasing function of time. It follows that there cannot be nontrivial periodic trajectories because the energy cannot be periodic.

(b) *Show that this equation has a nontrivial periodic solution.*

$$\ddot{x} + x\,(\dot{x})^2 + x = 0.$$

Written as a system

$$\begin{aligned}\dot{x} &= y && = f(x,y)\\ \dot{y} &= -x - x^2y && = g(x,y)\end{aligned}$$

This system is reversible because $f(x,-y) = -f(x,y)$ is odd in y and $g(x,-y) = g(x,y)$ is even in y. Its only zero is when $0 = f(x,y) = y$ hence $y = 0$ and when $0 = g(x,0) = -x$ so $x = 0$ as well. Linearizing at zero we find

$$J(0,0) = \left.\begin{pmatrix} 0 & 1 \\ -1 - 2xy & -x^2 \end{pmatrix}\right|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is a matrix whose eigenvalues are $\pm i$. The origin is a center for the linearized equation. By the theorem about centers for reversible systems, because the vector field is smooth and because the reversible system has centers for the linearization at a rest point, then the nonlinear system, too, has centers at the rest point. In particular, the origin in this problem is surrounded by closed nontrivial trajectories.

4. *The system undergoes a bifurcation as the parameter μ passes the critical value μ_c. Find the value. Find the critical points and determine their stability. Sketch the phase portraits for $\mu < \mu_c$, $\mu = \mu_c$ and $\mu > \mu_c$. What kind of bifurcation is this?*

$$\begin{aligned} \dot{x} &= -x + y + y(\mu - y) &&= f(x, y) \\ \dot{y} &= y(\mu - y) &&= g(x, y) \end{aligned}$$

The rest points are at solutions of $0 = g(x, y) = y(\mu - y)$ which is at $y = 0$ or $y = \mu$. If $y = 0$ then $0 = f(x, 0) = -x$ so $x = 0$. If $y = \mu$ then $0 = f(x, \mu) = -x + \mu$ so $x = \mu$. Computing the Jacobian we find

$$J(x, y) = \begin{pmatrix} -1 & 1 + \mu - 2y \\ 0 & \mu - 2y \end{pmatrix}; \; J(0, 0) = \begin{pmatrix} -1 & 1 + \mu \\ 0 & \mu \end{pmatrix}; \; J(\mu, \mu) = \begin{pmatrix} -1 & 1 - \mu \\ 0 & -\mu \end{pmatrix}$$

At $(0, 0)$, the eigenvalues are -1 and μ with corresponding eigenvectors $\binom{1}{0}$ and $\binom{1}{1}$. At (μ, μ), the eigenvalues are -1 and $-\mu$ with corresponding eigenvectors $\binom{1}{0}$ and $\binom{1}{1}$. Thus for $\mu > 0$, the rest point $(0, 0)$ is a saddle and the rest point (μ, μ) is a stable node. The points swap roles if $\mu < 0$ when the rest point $(0, 0)$ is a stable node and the rest point (μ, μ) is a saddle. Thus a transcritical bifurcation occurs at $(0, 0)$ when $\mu = \mu_c = 0$.

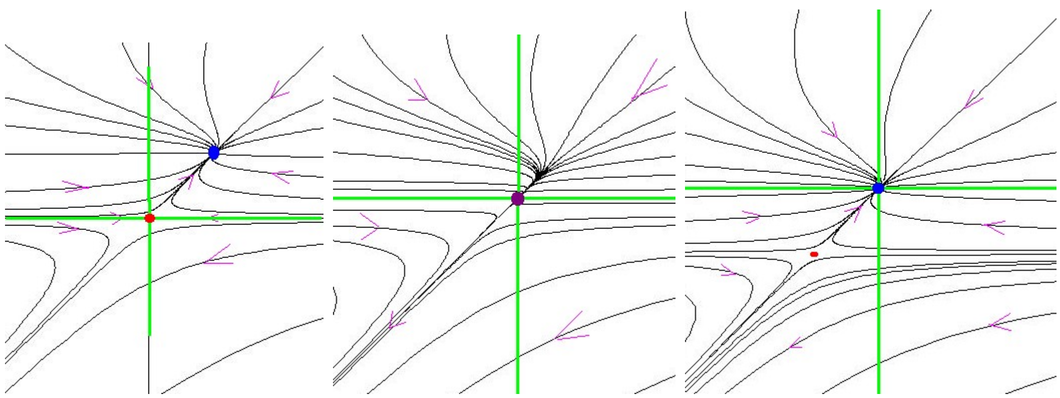

Figure 2: Plots with $\mu = .5, 0, -.5$ using 3D-XPlorMath©

5. *Consider the system*

$$\begin{aligned} \dot{x} &= \mu x - y + (\mu+1)x^2 - xy & &= f(x,y) \\ \dot{y} &= x + x^2 & &= g(x,y) \end{aligned}$$

(a) *The system undergoes a Hopf Bifurcation when the parameter μ passes a critical value μ_c. What is this critical value? What are the rest points? Can you tell if the bifurcation is subcritical or supercritical?*

The rest points satisfy $0 = g(x,y) = x(1+x)$ so $x = 0$ or $x = -1$. If $x = 0$ then $0 = f(0,y) = -y$ implies $y = 0$. If $x = -1$ at a rest point then $0 = f(-1,y) = -\mu - y + (\mu+1) + y = 1$ has no solution. Thus $(0,0)$ is the only rest point for all μ. Computing the Jacobian we find

$$J(x,y) = \begin{pmatrix} \mu + 2(1+\mu)x - y & -1-x \\ 1+2x & 0 \end{pmatrix}; \qquad J(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix}$$

Its characteristic equation is

$$(\mu - \lambda)(-\lambda) + 1 = \lambda^2 - \mu\lambda + 1 = 0$$

whose solutions are

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

For $|\mu| < 2$, as μ crosses $\mu_c = 0$, the eigenvalues are complex with real part $\mu/2$. Thus the Hopf bifurcation occurs at $\mu_c = 0$ when the conjugate eigenvalues cross the imaginary axis. If $-2 < \mu < 0$, the origin is a stable spiral if and if $0 < \mu < 2$, an unstable spiral. For $\mu < -2$, the origin is a stable node (both eigenvalues are negative) and if $\mu > 2$ and an unstable node (both eigenvalues are positive).

Consider the energy. For trajectories

$$\frac{d}{dt}E = x\dot{x} + y\dot{y} = x(\mu x - y + (\mu+1)x^2 - xy) + y(x + x^2) = \mu x^2 + (1+\mu)x^3$$

so at least if $\mu = -1$ the energy is strictly decreasing (the flow doesn't stop when $x = 0$ because then $\dot{x} = -y$) hence no periodic solutions. This suggests that periodic solutions occur when $\mu > 0$ when the origin is unstable. Thus the bifurcation is supercritical.

(b) *What is the index at the origin of the vector field in part (a)? Does it depend on μ? Why? [Hint: Does it help you to know that a Hopf Bifurcation occurs?]*

We know that a Hopf Bifurcation occurs, so that for some $\mu > 0$ there is a periodic solution C that surrounds the origin. Since there are no rest points other than the origin, the index of C gives the index at the origin. Because the vector field is tangent to C, $I_C = 1$, which is the index at the origin for such μ.

The index is the same for every μ. We know that the origin is the only rest point for any μ so the vector field never vanishes away form the origin, and we know that the vector field varies continuously as μ is varied. Hence the total angle change of the vector field on a fixed loop C around the origin varies continuously. But as the angle change is an integer multiple of 2π, it has to be constant. The index is confirmed for any μ such that $|\mu| > 2$ since the origin is a node or for any $0 < |\mu| < 2$ when the origin is a spiral, both of whose index is one.