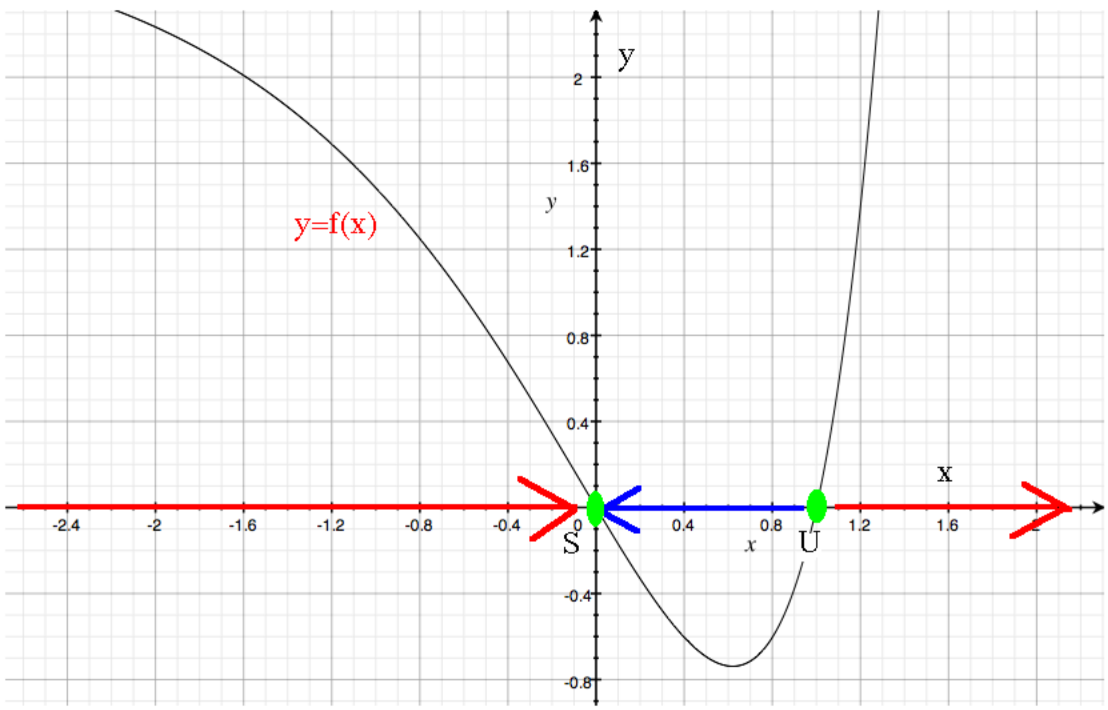
Math 5470 § 1. **First Midterm Exam** Name: Solutions
Treibergs Feb. 18, 2016

1. *Consider the equation on the line. Sketch the phase portrait. Find the rest points and use a graphical argument to determine their stability. For each of the rest points, use linearized stability to check your stability determination. Find the potential function $V(x)$. Sketch the potential function use it to check the stability of your rest points again.*

$$\dot{x} = (e^x - 1)(e^x - e) = f(x)$$

The zeros of $f(x)$ are $x = 0$ and $x = 1$. f is positive on $(-\infty, 0)$ and $(1, \infty)$ where flow is to the right. f is negative on $(0, 1)$ wher flow is to the left. Hence $x^* = 0$ is a stable rest point and $x^* = 1$ is an unstable rest point. The phase portrait is the x-axis. The graph $y = f(x)$ is superimposed.



Multiplying out and differentiating we find

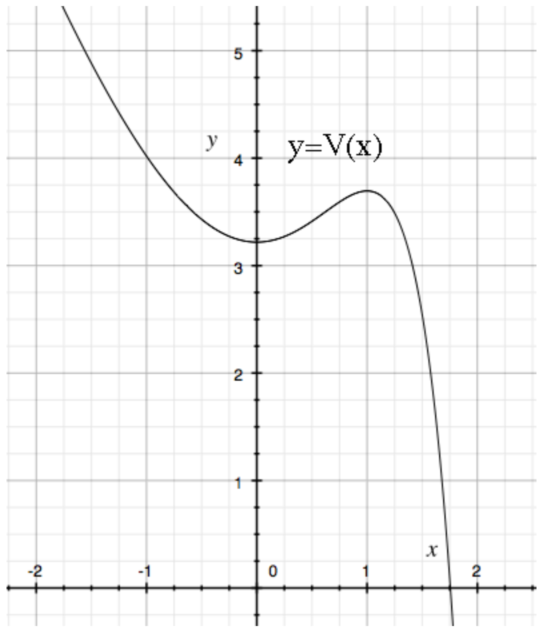
$$f(x) = e^{2x} - (1+e)e^x + e$$
$$f'(x) = 2e^{2x} - (1+e)e^x$$

Thus at the rest point $x^* = 0$, $f'(0) = 2 - 1 - e < 0$ so it is stable. At the rest point $x^* = 1$, $f'(1) = 2e^2 - e - e^2 = e(e-1) > 0$ so it is unstable.

The potential satisfies $V'(x) = -f(x)$ so

$$V(x) = -\frac{1}{2}e^{2x} + (1+e)e^x - ex + C$$

Choosing $C = 0$ we have $x = 0$ a local minimum of V and $x = 1$ a local maximum. After all $V' = 0$ at rest points and $V'' = -f'$ which is positive at $x = 0$ and negative at $x = 1$.



2. *Sketch the qualitatively different vector fields that occur as r is varied. Find and classify the bifurcation points. Sketch the bifurcation diagram.*

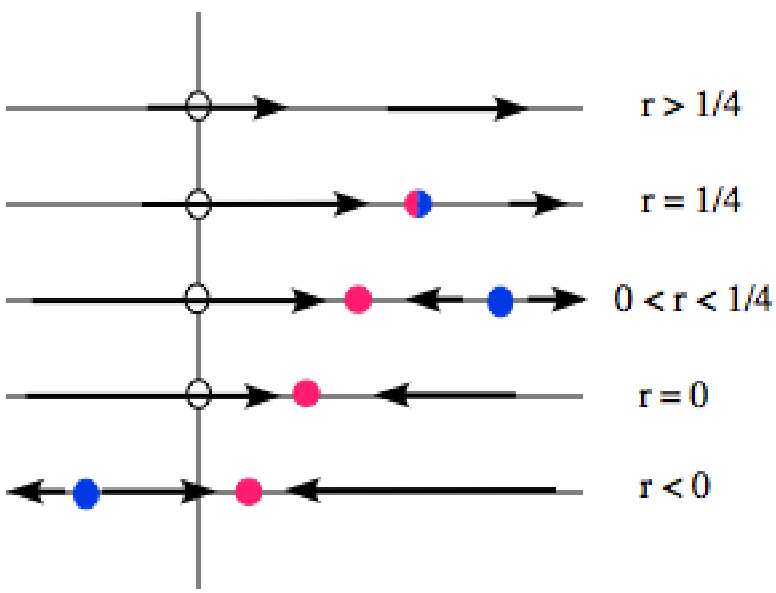
$$\dot{x} = 1 - x + rx^2 = f(x; r)$$

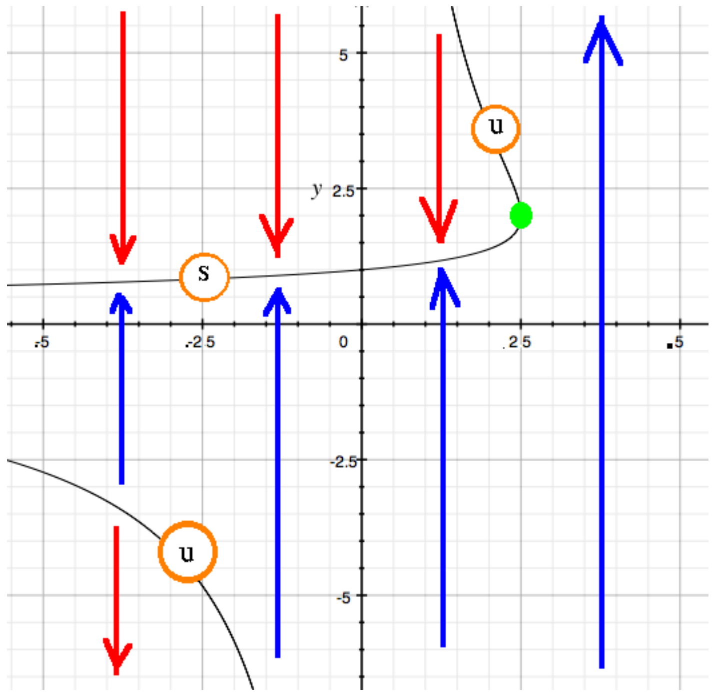
The critical points are zeros of $f(x; r)$, namely $x = 1$ if $r = 0$ and if $r \neq 0$,

$$x = \frac{1 \pm \sqrt{1 - 4r}}{2r} = \frac{1}{2r} \pm \sqrt{\frac{1}{4r^2} - \frac{1}{r}}$$

Thus there are two roots if $r < 1/4$ and none if $r > 1/4$. The bifurcation point is at $r^* = 1/4$ and $x^* = 2$. A stable and an unstable rest point collide as r increases through $r = 1/4$ so it is a saddle-node bifurcation. It is easier to plot $f(x; r) = 0$ by solving for r

$$r = \frac{x - 1}{x^2}$$





3. *Let $\theta(t)$ be the phase in the circle of firefly's flashing rhythm, where $\theta(t) = 0$ corresponds to the instant when the flash is emitted. Assume that the firefly's natural frequency is ω. If it senses a stimulus $\psi(t)$ at frequency Ω, then it tries to adjust according to the system. Show that for Ω close enough to ω, the firefly manages to synchronize with the stimulus, but if Ω is sufficiently different, it fails to synchronize. How close is "close enough"?*

$$\dot{\psi} = \Omega$$
$$\dot{\theta} = \omega + \sin(\psi - \theta)$$

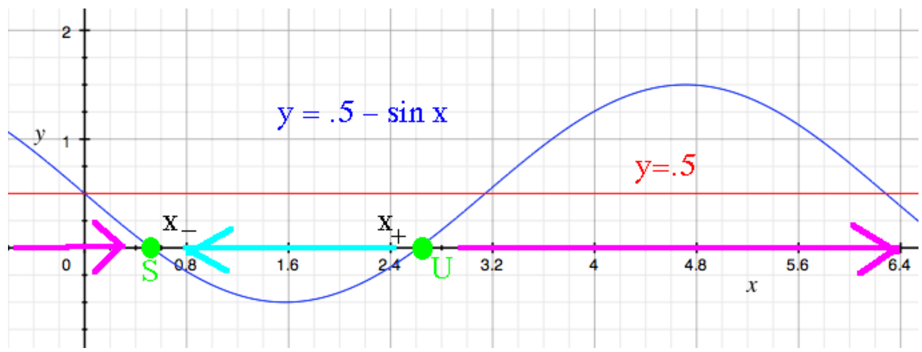
Let $\varphi = \psi - \theta$. Then the phases synchronize if φ tends to a constant, which is the phase delay between flashes of the signal and the firefly. The equation satisfied by φ is

$$\dot{\varphi} = \dot{\psi} - \dot{\theta} = \Omega - \omega - \sin\varphi = f(\varphi; \Omega - \omega)$$

If $|\Omega - \omega| \le 1$ then $f(\varphi; \Omega - \omega) = 0$ has solutions, namely

$$x_{\pm} = \arcsin(\Omega - \omega) = \frac{\pi}{2} \pm \epsilon.$$

If $\Omega - \omega = 1$ then $\varphi = \pi/2$ is the only rest point , which is neutrally stable. If $\Omega - \omega = -1$ then $\varphi = 3\pi/2$ is the only rest point, which is neutrally stable. If $-1 < \Omega - \omega < 1$ then there are two rest points. In this case x_+ is unstable and x_- is stable. All flow on the circle is attracted to this point, so the flashing synchronizes. Figure shows $\Omega - \omega = 1/2$.



If $\Omega - \omega > 1$ then $f(\varphi; \Omega - \omega) > 0$ for all φ, so that $\dot{\varphi} \ge c > 0$ and φ increases in time and does not synchronize. If $\Omega - \omega < -1$ then $f(\varphi; \Omega - \omega) < 0$ for all φ, so that $\dot{\varphi} \le c < 0$ and φ decreases in time and does not synchronize either.

4. *Consider the model of a fishery where t is time, $N(t) \ge 0$ is the fish population and A, H, K and r are positive constants.*

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{HN}{A+N}$$

What are the dimensions of A, H, K and r? Find dimensionless quantities x, τ, a and h so that the equation can be put into the dimensionless form

$$\frac{dx}{d\tau} = x(1-x) - \frac{hx}{a+x} = f(x; a, h)$$

Suppose that $a = 1/2$. Show that the system can have one, two or three rest points, depending on the values of h. Classify the stability of the rest points in each case. Sketch the bifurcation diagram for $a = 1/2$. What is hysteresis? Does the system (a) with $a = 1/2$ exhibit hysteresis?

The left side of the equation has dimension (fish)(time)$^{-1}$ where "fish" is "number of fish." Thus N, A and K have units (fish), r has units (time)$^{-1}$ and H has units (fish)(time)$^{-1}$. Letting $x = \frac{N}{K}$ we can rewrite the equation as

$$K\frac{dx}{dt} = rKx\,(1-x) - \frac{HKx}{A+Kx} = rKx\,(1-x) - \frac{Hx}{\frac{A}{K}+x}$$

Dividing by rK and setting $\tau = rt$ yields

$$\frac{dx}{d\tau} = \frac{1}{r}\frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{r}\frac{dx}{dt} = x\,(1-x) - \frac{\frac{H}{rK}x}{\frac{A}{K}+x}$$

Thus the non-dimensionalization is completed by taking $h = \frac{H}{rK}$ and $a = \frac{A}{K}$. Note that with these definitions, x, τ, a and h are all dimensionless.

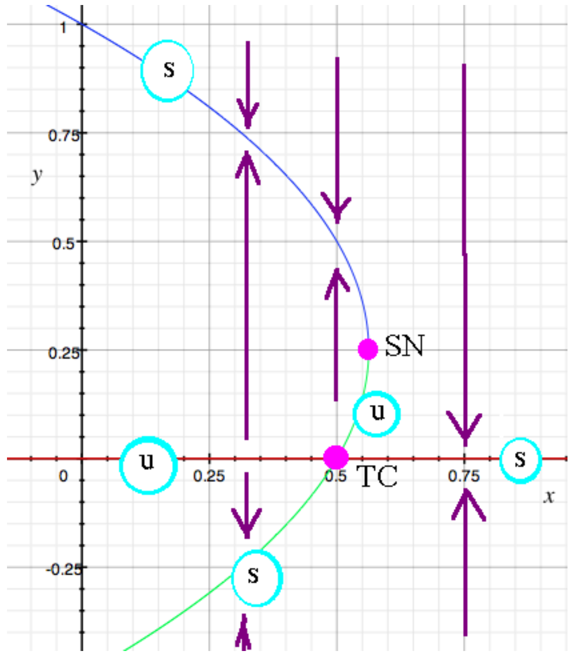
For $a = \frac{1}{2}$, the right side can be factored

$$f(x, \frac{1}{2}, h) = x\left(1 - x - \frac{h}{\frac{1}{2}+x}\right)$$

which has zeros $x = 0$ or when the second factor vanishes, which happens when

$$-\left(x - \frac{1}{4}\right)^2 + \frac{9}{16} = -x^2 + \frac{1}{2}x + \frac{1}{2} = (1-x)\left(\frac{1}{2}+x\right) = h$$

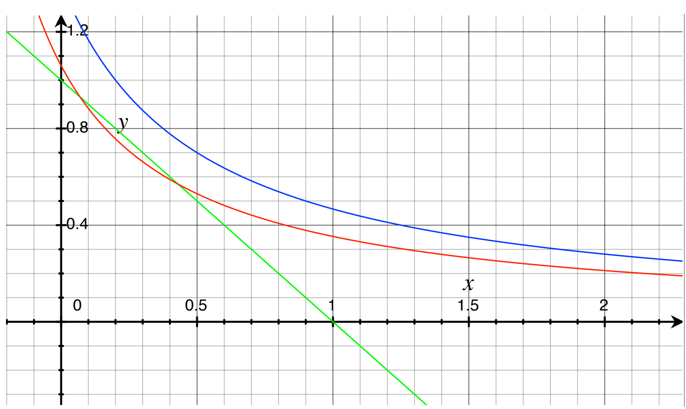
The plots of $x = 0$ and the parabola gives the bifurcation diagram. (h on the horizontal axis and x on the vertical.)



Solving for rest points x we find, for $h \leq \frac{9}{16}$,

$$x_{\pm} = \frac{1}{4} \pm \sqrt{\frac{9}{16} - h}$$

Thus for $h > \frac{9}{16}$ there is only one rest point $x = 0$; for $h = \frac{9}{16}$ there are two $x = 0$ and $x = \frac{1}{4}$; for $h = \frac{1}{2}$ there are two $x = 0$ and $x = \frac{1}{2}$; and for other h there are three, $x = 0, x_+, x_-$. Note that for $h = 1$ we have $\frac{h}{\frac{1}{2}+x} > 1 - x$ so $f < 0$ for $x > 0$. As h decreases, upper curve $y = \frac{h}{\frac{1}{2}+x}$ touches $y = 1 - x$ and then intersects $y = 1 - x$ in two points. Between the points and for $x > 0$ we have $f > 0$ and for $x < 0$ we have $f < 0$. ($h = .7$, blue and $h = .54$, red.)

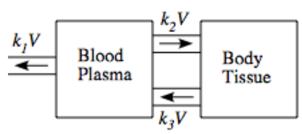


To summarize, $x = 0$ is a stable rest point for $h > \frac{1}{2}$ and unstable for $h < \frac{1}{2}$. The rest point x_+ in the upper half or the parabola is stable for all $h < \frac{9}{16}$. The rest point in the lower half of the parabola, x_- is unstable for $\frac{1}{2} < h < \frac{9}{16}$ and stable for $h < \frac{1}{2}$. There are two bifurcation points. The one at $(h^*, x^*) = \left(\frac{9}{16}, \frac{1}{4}\right)$ is a saddle-node bifurcation because two rest points collide and vanish as h is increased through $h^* = \frac{9}{16}$. The one at

$(h^*, x^*) = \left(\frac{1}{2}, 0\right)$ is a transcritical bifurcation because two rest points cross and swap their stability as h passes through $h^* = \frac{1}{2}$.

Hysteresis is when a rest point jumps as the parameter is varied. It often results in non-reversible phenomena. This model exhibits hysteresis. Say that $h = \frac{1}{4}$ and $x(0) = \frac{3}{4}$. The point flows to the upper part of the parabola. Increasing the harvesting parameter h, the fish population undergoes a catastrophic collapse to the zero rest point as h passes $h^* = \frac{9}{16}$. Now as fishing rate is reduced below h^* to $\frac{1}{2} < h < \frac{9}{16}$, the fish population stays at the stable rest point zero instead of recovering to the level it was before the catastrophe.

5. *Kaplan and Glass's model for intravenous drug administration can be described by a two compartment model, with the first compartment being blood plasma and the second being body tissue, both having about the same volume $V > 0$. Let $x(t)$ be the drug concentration in blood plasma, $y(t)$ the drug concentration is body tissue, and $k_i V > 0$ the flow rates. Find a general solution to the system assuming that $k_2 = 1/2$ and $k_1 = k_3 = 1$. Solve for initial conditions $x(0) = N > 0$ and $y(0) = 0$. What happens as $t \to \infty$? Sketch the phase plane for this system. If the eigenvalues are real, indicate the eigen-directions. Identify the trajectory with $x(0) = N$ and $y(0) = 0$. A schematic and the system is*



$$\begin{cases} \dot{x} = -(k_1 + k_2)x + k_3 y \\ \dot{y} = k_2 x - k_3 y \end{cases}$$

Plugging in the parameters $k_2 = \frac{1}{2}$ and $k_1 = k_3 = 1$ the system becomes

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are found by solving

$$0 = \begin{vmatrix} -\frac{3}{2} - \lambda & 1 \\ \frac{1}{2} & -1 - \lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1$$

which gives

$$\lambda_1, \lambda_2 = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = -\frac{1}{2}, -2$$

We find eigenvectors by inspection.

$$0 = (A - \lambda_1 I)\mathbf{x}_1 = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$0 = (A - \lambda_2 I)\mathbf{x}_2 = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The general solution is thus

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_1 e^{\lambda_2 t}\mathbf{x}_2 = c_1 e^{-\frac{1}{2}t}\begin{pmatrix}1\\1\end{pmatrix} + c_1 e^{-2t}\begin{pmatrix}2\\-1\end{pmatrix}$$

To find the solution of the initial value problem, we solve for c_1 and c_2 when $t = 0$

$$\begin{pmatrix}N\\0\end{pmatrix} = \mathbf{x}(0) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_1 e^{\lambda_2 t}\mathbf{x}_2 = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_1\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

which implies $c_1 = \frac{N}{3}$ and $c_2 = \frac{N}{3}$. The solution of the initial value problem is

$$\mathbf{x}(t) = \frac{N}{3}e^{-\frac{1}{2}t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{N}{3}e^{-2t}\begin{pmatrix}2\\-1\end{pmatrix} = \frac{N}{3}\begin{pmatrix}e^{-\frac{t}{2}} + 2e^{-2t}\\e^{-\frac{t}{2}} - e^{-2t}\end{pmatrix}$$

Thus $x(t)$ decreases from N to 0 monotonically but $y(t)$ increases from and then decreases to zero.

The phase plane shows a stable node at the origin, with fast decay along the $(2,-1)$ direction and slow decay along the $(1,1)$ direction. Hence, almost all trajectories approach zero by coming in tangent to the $(1,1)$ line. On the trajectory starting form $(N,0)$, $y(t)$ moves up and down as the $x(t)$ decreases.

In the Maple© plot, the $N = .9$ trajectory is red and the $\mathbf{x}_1$ and $\mathbf{x}_2$ directions are shown in navy and blue.