Math 545 § 1. **First Midterm Exam** Name:____________________

Treibergs Feb. 1, 1995

This is a closed book test. No books, papers, calculators.
There are [60] total points.

1.______	/12
2.______	/12
3.______	/12
4.______	/12
5.______	/12
Total______	/60

1. [12] Find the solution of the Laplace Equation with

$$\begin{array}{llll} \text{(PDE)} & u_{xx} + u_{yy} = 0 & & \text{for } 0 \le x, y \le 1 \\ \text{(BC)} & u(0,y) = 0, & u(1,y) = 0 & \text{for } 0 \le y \le 1; \\ & u(x,0) = \sin \pi x, & u(x,1) = \sin 2\pi x & \text{for } 0 \le x \le 1; \end{array}$$

2. [12] Find a series solution for

(PDE)	$u_t = u_{xx}$		for $0 \le x \le \pi$, $0 \le t$;
(BC)	$u(0,t) = 0,$	$u(\pi,t) = \frac{\pi}{2} e^{-\frac{t}{2}}$	for $0 \le t$;
(IC)	$u(x,0) = \frac{x}{2}$		for $0 \le x \le \pi$.

Hint: Solution of (PDE) with homogeneous BC's $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \frac{x}{2}$ is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} e^{-k^2 t} \sin kx.$$

3. [12] The Fourier Series for $f(x) = \dfrac{x}{2}$ on $0 < x < \pi$ is

$$\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k}.$$

(a) Define convergence of an infinite series of function in the mean square sense (L^2-sense.) Does this series converge in the L^2-sense? Explain or prove or give a counterexample.

(b) Find:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

4. [12] let the Fourier Series expression for a 2π-periodic function $f(x)$ be given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right).$$

The partial sums are given by

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos kx + b_k \sin kx \right).$$

Write $S_N(x) - f(x)$ as an integral expression involving f. Then show that the Fourier Series converges pointwise at $x = 0$ for the function $f(x) = \left|\sin \frac{x}{2}\right|$.

Hints: You may assume properties of the Dirichlet kernel, $K_N(x) = \dfrac{\sin[N+\frac{1}{2}]\theta}{\sin\frac{\theta}{2}}$. Show $\lim\limits_{N\to\infty} S_N(0) = 0 = f(0)$.

5. [12] Suppose $u(x,y) \in \mathcal{C}^2([0,a]\times[0,b])$ is a solution of the BVP on a rectangle with $0 \le a \le b$.

(PDE)	$u_{xx} + u_{yy} = -1$	on the rectangle $0 \le x \le a$, $0 \le y \le b$;
(BC)	$u(x,0) = u(x,b) = 0,$	for $0 \le x \le a$;
	$u(0,y) = u(a,y) = 0,$	for $0 \le y \le b$.

Show that $u(x,y) \le \dfrac{a^2}{8}$. Hint: Maximum principle using easy solutions of the PDE.