Math 5440 § 1. **Final Exam** Name: Practice

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Problems from the last third of the class. Half of the final exam will be on problems since the Final The other half will be comprehensive.

1. *Solve*

$$\begin{aligned}
&\text{(PDE)} && u_{tt} - c^2 u_{xx} = e^t \sin 5x, && \text{for } 0 < x < \pi,\ 0 < t;\\
&\text{(IC)} && u(x,0) = 0, && \\
& && u_t(x,0) = \sin 3x, && \text{for } 0 < x < \pi;\\
&\text{(BC)} && u(0,t) = 0, && \\
& && u(\pi,t) = 0, && \text{for } 0 < t.
\end{aligned}$$

Let us express the solution as a sine series with coefficients depending on time.

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin nx$$

where

$$u_n(t) = \frac{2}{\pi} \int_0^{\pi} u(t,x) \sin nx \, dx$$

The time derivatives

$$u_t(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin nx$$

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin nx$$

where

$$\begin{aligned}
v_n(t) &= \frac{2}{\pi} \int_0^{\pi} u_t(t,x) \sin nx \, dx \\
&= \frac{2}{\pi} \frac{\partial}{\partial t} \int_0^{\pi} u_(t,x) \sin nx \, dx \\
&= u_n'(t) \\
w_n(t) &= \frac{2}{\pi} \int_0^{\pi} u_{tt}(t,x) \sin nx \, dx \\
&= \frac{2}{\pi} \frac{\partial^2}{\partial t^2} \int_0^{\pi} u_(t,x) \sin nx \, dx \\
&= u_n''(t)
\end{aligned}$$

assuming that the first and second derivatives $u_t(t,x)$ and $u_{tt}(t,x)$ is continuous on $[0,\pi] \times [0,\infty)$. The spatial derivative has the expansion

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin nx$$

where

$$\begin{aligned} y_n(t) &= \frac{2}{\pi}\int_0^\pi u_{xx}(t,x)\sin nx\,dx \\ &= -\frac{2n}{\pi}\int_0^\pi u_x(t,x)\cos nx\,dx + \frac{2}{\pi}\Big[u_x(t,x)\sin nx\Big]_0^\pi \\ &= -\frac{2n^2}{\pi}\int_0^\pi u(t,x)\sin nx\,dx - \frac{2n}{\pi}\Big[u(t,x)\cos nx\Big]_0^\pi \\ &= -n^2 u_n(t) \end{aligned}$$

using the fact that both $\sin nx$ and $u(t,x)$ both vanish at $x=0$ and $x=\pi$. The source term is already a Fourier sine series

$$f(t,x) = e^t \sin 5x = \sum_{n=1}^\infty f_n(t)\sin nx$$

where $f_5(t) = e^t$ and $f_n(t) = 0$ for other n. Expanding the PDE we find for $n = 1, 2, 3, \ldots,$

$$u_n''(t) + c^2 n^2 u_n(t) = f_n(t)$$

The initial conditions say

$$0 = u(x,0) = \sum_{n=1}^\infty u_n(0)\sin nx$$
$$\sin 3x = u_t(x,0) = \sum_{n=1}^\infty u_n'(0)\sin nx$$

It follows that $u_n(0) = 0$ for all n and $u_3'(0) = 1$ and the rest $u_n'(0) = 0$. It follows that $n=3$ and $n=5$ are special cases and the rest are handled the same.

$$u_3'' + 9c^2 u_3 = 0, \qquad u_3(0) = 0, \qquad u_3'(0) = 1.$$

Hence the general solution is

$$u_3(t) = A_3 \cos 3ct + B_3 \sin 3ct.$$

The initial conditions tell us $A_3 = 0$ and $1 = 3cB_3$ so

$$u_3(t) = \frac{1}{3c}\sin 3ct.$$

For $n = 5$,

$$u_5'' + 25c^2 u_5 = e^t, \qquad u_5(0) = 0, \qquad u_5'(0) = 0.$$

Using the method of undetermined coefficients (*i.e.*, guessing) we try a particular solution

$$u_p(t) = Ce^t$$

Plugging gives

$$u_p'' + 25c^2 u_p = C(1 + 25c^2)e^t = e^t,$$

so

$$u_p(t) = \frac{1}{1+25c^2}e^t.$$

The general solution is the particular solution plus the general solution of the homogeneous problem

$$u_5(t) = A_5 \cos 5ct + B_5 \sin 5ct + \frac{1}{1+25c^2} e^t.$$

The zero initial conditions imply $0 = A_5 + (1+25c^2)^{-1}$ and $0 = 5cB_5 + (1+25c^2)^{-1}$ so

$$u_5(t) = \frac{5ce^t - 5c\cos 5ct - \sin 5ct}{5c(1+25c^2)}.$$

The remaining n satisfy

$$u_n'' + n^2c^2u_5 = 0, \qquad u_n(0) = 0, \qquad u_n'(0) = 0.$$

so $u_n(t) = 0$. Summing all nonvanishing terms we find the solution

$$u(t,x) = \frac{1}{3c}\sin 3ct \sin 3x + \frac{5ce^t - 5c\cos 5ct - \sin 5ct}{5c(1+25c^2)} \sin 5x.$$

2. *Find the temperature of a metal rod that is in the shape of a truncated cone whose radius is*

$$R(x) = d(\xi + a), \qquad 0 < \xi < b$$

where a and b are positive constants. Assume that the rod is insulated on its sides, is maintained at zero temperature on its ends and it has an unspecified initial temperature distribution. (Text problem 5.6.10.)

The cross section at $x = a + \xi$ is a circle so the cross sectional area is

$$A(x) = \pi d^2 x^2, \qquad a < x < \ell = a + b$$

We assume that the temperature depends only on x, the coordinate along the rod. Recall the derivation of the heat equation. Let c be the specific heat, κ the heat conductivity and ρ the density. Let H be the total heat in the rod between $x_0 \le x \le x_1$

$$H = \int_{x_0}^{x_1} c\rho A(x) u(t,x)\, dx$$

The rate of change of energy is

$$\frac{dH}{dt} = \int_{x_0}^{x_1} c\rho A(x) u_t(t,x)\, dx = \text{flow in} - \text{flow out} = A(x_1)\kappa u_x(t,x_1) - A(x_0)\kappa u_x(t,x_0)$$

Differentiating with respect to x_1 yields the PDE

$$c\rho A(x) u_t(t,x) = \big[A(x)\kappa u_x(t,x)\big]_x$$

or

$$x^2 u_t(t,x) = k\big[x^2 u_x(t,x)\big]_x, \qquad a < x < \ell$$

where $k = \kappa / c\rho$, the heat constant.

To solve this equation, we make the change of dependent variable

$$u(t,x) = \frac{v(t,x)}{x}.$$

The PDE becomes

$$\begin{aligned} u_x &= \frac{v_x}{x} - \frac{v}{x^2} \\ x^2 u_x &= x v_x - v \\ [x^2 u_x]_x &= x v_{xx} + v_x - v_x = x v_{xx} \end{aligned}$$

so the PDE becomes

$$x v_t = k x v_{xx}$$

which is the standard heat equation for $v(t,x)$. The initial condition is $\varphi(x) = u(x,0) = v(x,0)/x$. Thus the initial-boundary value problem has been transformed to

$$\begin{aligned} &\text{(PDE)} \quad && v_t = k v_{xx}, && \text{for } a < x < \ell,\ 0 < t; \\ &\text{(IC)} \quad && v(x,0) = x\varphi(x-a), && \text{for } a < x < \ell; \\ &\text{(BC)} \quad && v(a,t) = 0, && \\ & && v(\ell,t) = 0, && \text{for } 0 < t. \end{aligned}$$

This is the standard problem except we have translated to $a < x < \ell$ where $\ell = a + b$. Separating variables $v(t,x) = T(t)X(x)$ we get as usual

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

which produces the eigenvalue problem with boundary conditions

$$X'' + \lambda X = 0, \qquad X(a) = 0, \qquad X(\ell) = 0.$$

The solutions are the translated versions of the usual

$$X_n(x) = \sin\left(\frac{n\pi(x-a)}{\ell - a}\right), \qquad \lambda_n = \frac{n^2\pi^2}{(\ell-a)^2}, \qquad n = 1, 2, 3, \ldots.$$

The time equation is the usual

$$T_n' + k\lambda_n T_n = 0$$

whose solution is

$$T_n(t) = \exp\left(-\frac{kn^2\pi^2 t}{(\ell-a)^2}\right).$$

Thus the series solution is

$$v(t,x) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2 t}{(\ell-a)^2}\right) \sin\left(\frac{n\pi(x-a)}{\ell-a}\right)$$

where

$$B_n = \frac{2}{\ell - a} \int_a^\ell x\varphi(x-a) \sin\left(\frac{n\pi(x-a)}{\ell-a}\right) dx$$

Rewriting in terms of $u(t,\xi)$ where $\xi = x - a$ we find for $0 < \xi < b$ and $0 < t$,

$$u(t,\xi) = \frac{1}{a+\xi} \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2 t}{b^2}\right) \sin\left(\frac{n\pi\xi}{b}\right).$$

3. *The damped wave equation has a resistance term with* $r > 0$ *constant. (Text problem 5.6.13.)*

$$
\begin{array}{rll}
\text{(PDE)} & u_{tt} + r u_t = c^2 u_{xx}, & \text{for } 0 < x < \ell,\ 0 < t; \\
\text{(IC)} & u(x,0) = \varphi(x), & \\
 & u_t(x,0) = \psi(x), & \text{for } 0 < x < \ell; \\
\text{(BC)} & u(0,t) = 0, & \\
 & u(\ell,t) = Ae^{i\omega t}, & \text{for } 0 < t.
\end{array}
$$

(a) *Show that the PDE and BC are satisfied by*

$$\mathcal{U}(t,x) = Ae^{i\omega t}\frac{\sin \beta x}{\sin \beta \ell}$$

where $\beta^2 c^2 = \omega^2 - ir\omega$.

(b) *No matter what* $\varphi(x)$ *and* $\psi(x)$ *are, show that* $\mathcal{U}(t,x)$ *is the asymptotic form of* $u(t,x)$ *as* $t \to \infty$.

(c) *Show that you get resonance as* $r \to 0$ *if* $\omega = m\pi c/\ell$ *for some positive integer* m.

(d) *Show that friction can prevent resonance from occurring.*

We check BC's are satisfied for all $t > 0$,

$$\mathcal{U}(t,0) = Ae^{i\omega t}\frac{\sin \beta \cdot 0}{\sin \beta \ell} = 0; \qquad \mathcal{U}(t,\ell) = Ae^{i\omega t}\frac{\sin \beta \cdot \ell}{\sin \beta \ell} = Ae^{i\omega t}.$$

We check the PDE is satisfied

$$\mathcal{U}_{tt} + r\mathcal{U}_t - c^2\mathcal{U}_{xx} = Ae^{i\omega t}\frac{\sin \beta x}{\sin \beta \ell}\left[-\omega^2 + ri\omega + c^2\beta^2\right] = 0.$$

We solve the equation using the method of shifting the data. Putting

$$u(t,x) = \mathcal{U}(t,x) + v(t,x),$$

the function $v(t,x)$ satisfies the initial boundary value problem with Dirichlet boundary conditions

$$
\begin{array}{rll}
\text{(PDE)} & v_{tt} + r v_t = c^2 v_{xx}, & \text{for } 0 < x < \ell,\ 0 < t; \\
\text{(IC)} & v(x,0) = \varphi(x) - \mathcal{U}(x,0), & \\
 & u_t(x,0) = \psi(x) - \mathcal{U}_t(x,0), & \text{for } 0 < x < \ell; \\
\text{(BC)} & v(0,t) = 0, & \\
 & v(\ell,t) = 0, & \text{for } 0 < t.
\end{array}
$$

Separating variables, inserting $v(t,x) = T(t)X(x)$ into the PDE yields

$$\frac{T''(t) + rT'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

resulting in the eigenvalue problem $X(x)$

$$X'' + \lambda X = 0; \qquad X(0) = X(\ell) = 0.$$

The solutions are well known by now

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \qquad \lambda_n = \frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, 3, \ldots.$$

The corresponding time equation is

$$T_n'' + rT_n' + c^2\lambda_n T_n = 0$$

The roots of the characteristic equation

$$\mu_n^2 + r\mu_n + c^2\lambda_n = 0$$

are

$$\mu_n^\pm = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda_n}}{2}$$

There are three cases yielding different general solutions. If $r^2 > 4c^2\lambda_n$ then both $\mu_n^\pm$ are negative and

$$T_n(t) = A_n e^{\mu_n^+ t} + B_n e^{\mu_n^- t}$$

which decay as $t \to \infty$.

If $r^2 = 4c^2\lambda_n$ then both $\mu_n = -r/2$ and the general solution is

$$T_n(t) = A_n e^{-rt/2} + B_n t e^{-rt/2}$$

which also decays to zero.

If $r^2 < 4c^2\lambda_n$ then $\mu_n^\pm = -r/2 \pm \gamma_n i$ where $\gamma_n = (c^2\lambda_n - r^2/4)^{1/2} > 0$.

$$T_n(t) = e^{-rt/2}\left\{A_n \cos\gamma_n t + B_n \sin\gamma_n t\right\}$$

which decays and oscillates about zero.

No matter what the initial conditions are, they determine A_n and B_n by the formulae

$$T_n(0) = \frac{2}{\ell}\int_0^\ell [\varphi(x) - \mathcal{U}(x,0)]\sin\left(\frac{n\pi x}{\ell}\right)\,dx, \quad T_n'(0) = \frac{2}{\ell}\int_0^\ell [\psi(x) - \mathcal{U}_t(x,0)]\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

As we have shown, all $T_n(t) \to 0$ as $t \to \infty$ so

$$u(t,x) = \mathcal{U}(t,x) + v(t,x) \to \mathcal{U}(t,x) + 0 \qquad \text{as } t \to \infty.$$

Resonance is the phenomenon that a periodic driving input at the right frequency will cause larger and larger vibrations. Suppose the boundary condition is driven at the suggested frequency $\omega = m\pi c/\ell$ for some positive integer m. Then the m-th mode goes crazy. Indeed, this is the case here. Let us suppose that A is large so that φ and ψ are relatively smaller, and let us track the m-th mode due to the $\mathcal{U}$ part of the initial data. $r^2 < 4c^2\lambda_m$ since it is close to zero. Then $\gamma_m = (c^2\lambda_m - r^2/4)^{1/2} > 0$. The initial times have the formulae which decays and oscillates about zero.

$$T_m(0) = A_m = \frac{2A}{\ell\sin\beta\ell}\int_0^\ell \sin\beta x \sin\left(\frac{n\pi x}{\ell}\right)\,dx$$

$$T'(0) = -\frac{r}{2}A_m + \gamma_m B_m = \frac{2A\omega i}{\ell\sin\beta\ell}\int_0^\ell \sin\beta x \sin\left(\frac{n\pi x}{\ell}\right)\,dx$$

Computing

$$\int_0^\ell \sin\beta x \sin\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_0^\ell \cos\left(\beta - \frac{n\pi x}{\ell}\right) + \cos\left(\beta + \frac{n\pi x}{\ell}\right)\,dx$$
$$= \frac{1}{2}\left[\frac{\sin\left(\beta - \frac{n\pi x}{\ell}\right)}{\beta - \frac{n\pi x}{\ell}} - \frac{\sin\left(\beta + \frac{n\pi x}{\ell}\right)}{\beta + \frac{n\pi x}{\ell}}\right]_0^\ell$$
$$= \frac{1}{2}\left[\frac{\sin(\beta\ell - n\pi x)}{\beta - \frac{n\pi x}{\ell}} - \frac{\sin(\beta\ell + n\pi x)}{\beta + \frac{n\pi x}{\ell}}\right]$$
$$= \frac{(-1)^m \beta \sin\beta\ell}{\beta^2 - \frac{n^2\pi^2}{\ell^2}}$$
$$= \frac{(-1)^m \beta c\ell i \sin\beta\ell}{\pi m r}$$

where we use $\omega = \pi mc/\ell$ and $\beta^2 c^2 = \omega^2 - ir\omega$. Thus

$$A_m = \frac{2A\beta c(-1)^m i}{\pi m r}, \qquad B_m = \frac{A\beta c(-1)^m(-\omega + ri)}{\gamma_m \pi m r}$$

Since $\beta c \to \omega$ as $r \to 0$, we see that for fixed t, the magnitude of the m-th mode

$$|T_m(t)X_m(x)| \to \infty \qquad \text{as } r \to 0$$

which is resonance.

On the other hand, if a modicum of drag is present $r \ge r_0 > 0$, all v modes are bounded by some C/r_0 where C is a constant depending on φ, ψ c, ℓ A and ω and solutions decay at an exponential rate to the steady periodic solution.

4. *Solve the boundary value problem for Poisson's Equation in a spherical shell in three space. Take the limit of your answer as $a \to 0$ and interpret the result. (Text problem 6.1.8.)*

$$\begin{aligned}
&\text{(PDE)} \qquad && \Delta u = 1, \qquad && \text{for } a < r < b;\\
&\text{(BC)} && u = 0, && \text{for } r = a;\\
& && u_r = 0, && \text{for } r = b.
\end{aligned}$$

The Laplacian in spherical coordinates (r, ϕ, θ) (ϕ is from x-axis in (x, y)-plane and θ is angle from z axis)

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + (\sec^2\theta)u_{\phi\phi}\right]$$

Assuming rotational symmetry, we seek functions $u(r)$ which satisfy

$$u'' + \frac{2}{r}u' = 1, \qquad u(a) = 0, \qquad u_r(b) = 0.$$

Multiplying by r^2,

$$(r^2u')' = r^2u'' + 2ru' = r^2$$

whose general solution is

$$r^2u'(r) = c_1 + \frac{1}{3}r^3, \qquad u(r) = c_2 - \frac{c_1}{r} + \frac{1}{6}r^2.$$

The initial conditions tell us

$$0 = b^2u_r(b) = c_1 + \frac{b^3}{3} \qquad \Longrightarrow \qquad c_1 = -\frac{b^3}{3},$$

and

$$0 = u(a) = c_2 + \frac{b^3}{3a} + \frac{a^2}{6} \qquad \implies c_2 = -\frac{b^3}{3a} - \frac{a^2}{6}$$

so

$$u(r) = -\frac{b^3}{3}\left(\frac{1}{a} - \frac{1}{r}\right) + \frac{r^2 - a^2}{6}$$

As $a \to 0$, the minimum of u which occurs at $r = b$ tends to $-\infty$. If the bottom didn't drop

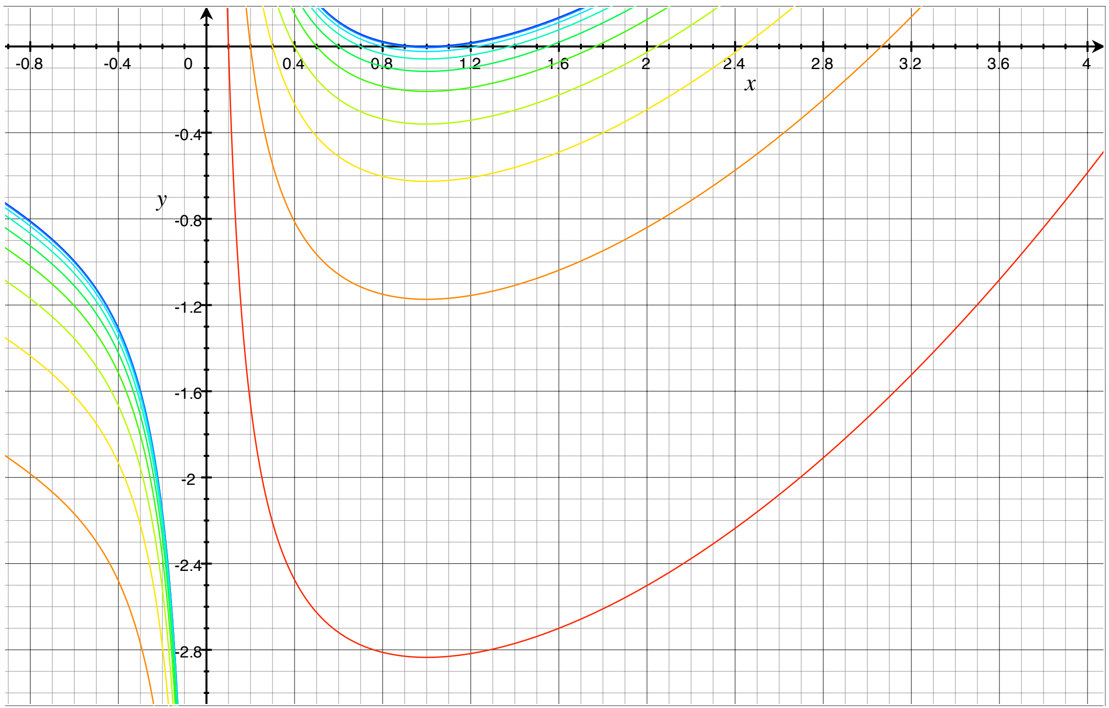

Figure 1: Plots of $u(r)$ for $a = .1, .2, \ldots, 1.0$ and $b = 1$.

out of the limit, then there would be a limiting surface with a finite point singularity at the origin in the shape of a thumb tack. However, bounded solutions of Poisson's equation don't admit point singularities.

5. *Solve the boundary value problem for Laplace's equation.*

$$\begin{aligned} &\text{(PDE)} & u_{xx} + u_{yy} &= 0, && \text{for } 0 < x < \pi,\ 0 < y < 1; \\ &\text{(BC)} & u(x,0) = u(x,1) &= \sin^3 x, && \text{for } 0 < x < \pi; \\ && u(0,y) &= \sin \pi y, && \\ && u(\pi,y) &= 0, && \text{for } 0 < y < 1. \end{aligned}$$

Let us solve for three simpler BVP's, and add their solutions. The problem with data only on the $y = 0$ side is

$$\begin{aligned}
&\text{(PDE)} \qquad & v_{xx} + v_{yy} &= 0, \qquad && \text{for } 0 < x < \pi,\ 0 < y < 1;\\
&\text{(BC)} & v(x,0) &= \sin^3 x, && \\
& & v(x,1) &= 0, && \text{for } 0 < x < \pi;\\
& & u(0,y) = u(\pi,y) &= 0, && \text{for } 0 < y < 1.
\end{aligned}$$

Separate variables $v(x,y) = X(x)Y(y)$ and plug into the PDE

$$X''Y + XY'' = 0$$

so

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = -\lambda.$$

The x equation yields the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X(\pi) = 0.$$

The solutions are as usual,

$$X_n(x) = \sin nx, \qquad \lambda_n = n^2, \qquad n = 1, 2, 3, \ldots.$$

The y equation is

$$Y_n'' - \lambda_n Y_n = 0, \qquad Y(0) = 1, \qquad Y(1) = 0.$$

Its solution is

$$Y_n(y) = \frac{\sinh n(1-y)}{\sinh n}$$

The general solution is

$$v(x,y) = \sum_{n=1}^{\infty} A_n \frac{\sinh n(1-y)}{\sinh n} \sin nx$$

Note

$$\sin^3 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{-8i} = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

At $y = 0$,

$$\frac{3}{4}\sin x - \frac{1}{4}\sin 3x = \sin^3 x = v(x,0) = \sum_{n=1}^{\infty} A_n \sin nx$$

which means that all but two Fourier coefficients vanish and the solution is

$$v(x,y) = \frac{3\sinh(1-y)}{4\sinh 1}\sin x - \frac{\sinh 3(1-y)}{4\sinh 3}\sin 3x.$$

The transformation $x' = x$ and $y' = 1 - y$ swaps top and bottom of the rectangle. Since harmonic functions are invariant under rigid motions, the solution with data on $y = 1$ is

$$w(x,y) = \frac{3\sinh y}{4\sinh 1}\sin x - \frac{\sinh 3y}{4\sinh 3}\sin 3x.$$

Finally we consider the problem with the only nonzero data on the left.

$$
\begin{aligned}
&\text{(PDE)} \qquad & z_{xx} + z_{yy} &= 0, \qquad & &\text{for } 0 < x < \pi,\ 0 < y < 1;\\
&\text{(BC)} \qquad & z(x,0) = z(x,1) &= 0, \qquad & &\text{for } 0 < x < \pi;\\
& & z(0,y) &= \sin \pi y, & &\\
& & z(\pi,y) &= 0, & &\text{for } 0 < y < 1.
\end{aligned}
$$

This time we take the y equation for the eigenvalue problem

$$Y'' - \lambda Y = 0, \qquad Y(0) = Y(1) = 0.$$

The solutions are as usual,

$$Y_n(y) = \sin n\pi y, \qquad \lambda_n = -n^2\pi^2, \qquad n = 1, 2, 3, \ldots.$$

The X equation is

$$X_n'' + \lambda_n X_n = 0, \qquad X(0) = 1, \qquad X(\pi) = 0.$$

Its solution is

$$X_n(y) = \frac{\sinh n(\pi - x)}{\sinh n\pi}$$

The general solution is

$$z(x,y) = \sum_{n=1}^{\infty} B_n \frac{\sinh n(\pi - x)}{\sinh n\pi} \sin n\pi y$$

At $x = 0$,

$$\sin \pi y = z(0,y) = \sum_{n=1}^{\infty} B_n \sin nx$$

which means that all but one Fourier coefficients vanish and the solution is

$$z(x,y) = \frac{\sinh(\pi - x)}{\sinh \pi} \sin \pi y.$$

By superposition, we reach the solution of the original problem

$$
\begin{aligned}
u(x,y) &= v(x,y) + w(x,y) + z(z,y)\\
&= \frac{3\sin x}{4\sinh 1}(\sinh(1-y) + \sinh y) - \frac{\sin 3x}{4\sinh 3}[\sinh 3(1-y) + \sinh 3y] + \frac{\sinh(\pi - x)}{\sinh \pi} \sin \pi y.
\end{aligned}
$$

6. *Suppose that u is a harmonic function defined in the exterior of a piecewise smooth bounded domain $\mathcal{D} \subset \mathbf{R}^d$ which decays uniformly to zero at infinity. Show that*

$$|u(x)| \le \sup_{\partial\mathcal{D}} |u(x)| = M$$

for all $x \in \mathbf{R}^d \backslash \mathcal{D}$. Show that if u is harmonic on $\mathbf{R}^d$ and decays uniformly to zero at infinity, then $u(x) = 0$ for all $x \in \mathbf{R}^d$.

This is a simple application of the maximum principle. First we argue that $|u(x)| \le M$ for any choice of $x \in \mathbf{R}^d \backslash \mathcal{D}$. For any $\epsilon > 0$, there is a large number R such that $R > |x|$, the R-ball contains the domain $B_R(0) \supset \mathcal{D}$ and by the convergence to zero at infinity,

$$|u(x)| < \epsilon \qquad \text{for all } x \in \mathbf{R}^d \text{ such that } |x| = R.$$

By the maximum principle applied to the annular region $\mathcal{D}_R = B_R(0)\backslash\mathcal{D}$ whose boundary is $\partial\mathcal{D} \cup \partial B_R(0)$, we have

$$-M - \epsilon \le \inf_{x\in\partial\mathcal{D}_R} |u(x)| \le u(x) \le \sup_{x\in\partial\mathcal{D}_R} |u(x)| < M + \epsilon.$$

But since $\epsilon > 0$ was arbitrary, we conclude $|u(x)| \le M$. The argument given allows the possibility that $M = 0$.

Now suppose that u is harmonic on all of $\mathbf{R}^d$ and decays uniformly at infinity. Choose $x \in \mathbf{R}^d$. We claim that $u(x) = 0$. If this is not the case, then there is $R > 0$ such that $R > |x|$ and

$$M = \sup_{y\in\partial B_R(0)} |u(y)| < |u(x)|. \tag{1}$$

Applying the maximum principle to $B_R(0)$ yields a contradiction because $x \in B_R(0)$

$$-M \le \inf_{y\in\partial B_R(0)} |u(y)| \le u(x) \le \sup_{y\in\partial B_R(0)} |u(y)| \le M$$

so $|u(x)| \le M$ contrary to (1).

7. *Prove uniqueness of the Robin problem with $a > 0$ constant in a smooth, bounded domain $\mathcal{D} \subset \mathbf{R}^d$.*

$$\begin{aligned} &\text{(PDE)} && \Delta u = 0, && \text{for } x \in \mathcal{D};\\ &\text{(BC)} && \frac{\partial u}{\partial n} + au = 0, && \text{for } x \in \partial\mathcal{D}. \end{aligned}$$

Suppose u and v are two $\mathcal{C}^2$ solutions and let $w = u - v$. By linearity of the PDE and BC, w also satisfies the Robin problem

$$\begin{aligned} &\text{(PDE)} && \Delta w = 0, && \text{for } x \in \mathcal{D};\\ &\text{(BC)} && \frac{\partial w}{\partial n} + aw = 0, && \text{for } x \in \partial\mathcal{D}. \end{aligned}$$

Consider the energy. Integrating by parts, and using the Robin condition,

$$\begin{aligned} \int_{\mathcal{D}} |\nabla w|^2 \, dV &= -\int_{\mathcal{D}} w \, \Delta \, w \, dV + \int_{\partial\mathcal{D}} w \frac{\partial w}{\partial n} \, dS \\ &= 0 - \int_{\partial\mathcal{D}} aw^2 \, dS \\ &\le 0. \end{aligned}$$

It follows that the energy is zero, and because $\nabla w \in \mathcal{C}^1$, the integrand is identically zero. In other words $\nabla w = 0$ for all points in the domain, which means since domains are connected, $w =$ const. Hence at the boundary, $\frac{\partial w}{\partial n} + aw = 0 + aw = 0$, which implies that the constant is zero $w = 0$ on $\mathcal{D}$: the two solutions agree so solutions are unique.

8. *Consider Laplace's equation on the unit disk.*

$$\begin{aligned} &\text{(PDE)} && u_{xx} + u_{yy} = 0, && \text{for } x^2 + y^2 < 1;\\ &\text{(BC)} && u = x^2, && \text{for } x^2 + y^2 = 1. \end{aligned}$$

Find the Rayleigh-Ritz (least energy) approximation to the solution of the form

$$x^2 + c_1(1 - x^2 - y^2) + c_2 x^2 (1 - x^2 - y^2).$$

The harmonic function is the energy minimizer among all functions satisfying the boundary conditions. If we restrict to the finite family of functions

$$v = w_0 + c_1 w_1 + \cdots c_n w_n$$

where $w_0 = x^2$ on the boundary and $w_j = 0$ on the boundary for $j \geq 1$. The energy is in terms of $\mathcal{L}^2$-inner product $(u, v) = \int_{\mathcal{D}} uv \, dA$

$$E = \frac{1}{2}(\nabla v, \nabla v) = \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} c_j c_k (\nabla w_j, \nabla w_k)$$

where $c_0 = 1$ for simplicity. At the minimum, for each $i = 1, 2, \ldots, d$

$$\begin{aligned} 0 = \frac{\partial E}{\partial c_i} &= c_i(\nabla w_i, \nabla w_i) + \frac{1}{2} \sum_{k \neq i} c_k (\nabla w_i, \nabla w_k) + \frac{1}{2} \sum_{j \neq i} c_j (\nabla w_j, \nabla w_i) \\ &= \sum_{j=1}^{n} c_j (\nabla w_j, \nabla w_i) + (\nabla w_j, \nabla w_0) \end{aligned}$$

which is a linear system for c_j. We need to find the inner products using

$$w_0 = x^2, \qquad w_1 = 1 - x^2 - y^2, \qquad w_2 = x^2(1 - x^2 - y^2),$$

The gradients are

$$\nabla w_0 = (2x, 0), \qquad \nabla w_1 = (-2x, -2y), \qquad \nabla w_2 = (2x - 4x^3 - 2xy^2, -2x^2 y)$$

We shall need the following tricks.

$$\begin{aligned} 2\int_{\mathcal{D}} x^2 \, dA = \int_{\mathcal{D}} x^2 + y^2 \, dA &= \int_0^1 \int_0^{2\pi} r^3 \, d\theta \, dr = \frac{\pi}{2}, \\ \int_{\mathcal{D}} x^4 \, dA &= \int_0^1 \int_0^{2\pi} r^5 \cos^4\theta \, d\theta \, dr = \int_0^1 r^5 \, dr \int_0^{2\pi} \cos^2(1 - \sin^2\theta)\theta \, d\theta \\ &= \frac{1}{6} \int_0^{2\pi} \cos^2\theta - \frac{1}{4}\sin^2 2\theta \, d\theta = \frac{\pi}{8} \\ \int_{\mathcal{D}} x^2 y^2 \, dA &= \int_0^1 \int_0^{2\pi} r^5 \cos^2\theta \, \sin^2\theta \, d\theta \, dr = \frac{1}{4} \int_0^1 r^5 \, dr \int_0^{2\pi} \sin^2 2\theta \, d\theta = \frac{\pi}{24} \\ \int_{\mathcal{D}} x^4(x^2 + y^2) \, dA &= \int_0^1 \int_0^{2\pi} r^7 \cos^4\theta \, d\theta \, dr = \int_0^1 r^7 \, dr \int_0^{2\pi} \cos^2(1 - \sin^2\theta)\theta \, d\theta = \frac{3\pi}{32} \\ \int_{\mathcal{D}} x^2 y^2 (x^2 + y^2) \, dA &= \int_0^1 \int_0^{2\pi} r^7 \cos^2\theta \, \sin^2\theta \, d\theta \, dr = \frac{1}{4} \int_0^1 r^7 \, dr \int_0^{2\pi} \sin^2 2\theta \, d\theta = \frac{\pi}{32} \end{aligned}$$

so

$$\begin{aligned} (\nabla w_0, \nabla w_0) &= \int_{\mathcal{D}} 4x^2 \, dA = \pi, \\ (\nabla w_0, \nabla w_1) &= \int_{\mathcal{D}} -4x^2 \, dA = -\pi, \\ (\nabla w_0, \nabla w_2) &= \int_{\mathcal{D}} 4x^2 - 8x^4 - 4x^2 y^2 \, dA = -\frac{\pi}{6}, \\ (\nabla w_1, \nabla w_1) &= \int_{\mathcal{D}} 4x^2 + 4y^2 \, dA = 2\pi, \\ (\nabla w_1, \nabla w_2) &= \int_{\mathcal{D}} -4x^2 + 8x^4 + 8x^2 y^2 \, dA = \frac{\pi}{3}, \\ (\nabla w_2, \nabla w_2) &= \int_{\mathcal{D}} 4x^2 - 16x^4 + 16x^4(x^2 + y^2) + 4x^2 y^2 (x^2 + y^2) - 8x^2 y^2 \, dA = \frac{3\pi}{8}. \end{aligned}$$

The coefficients satisfy

$$\begin{pmatrix} (\nabla w_1, \nabla w_1) & (\nabla w_1, \nabla w_2) \\ (\nabla w_2, \nabla w_1) & (\nabla w_2, \nabla w_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} (\nabla w_0, \nabla w_1) \\ (\nabla w_0, \nabla w_2) \end{pmatrix}$$

or

$$\begin{pmatrix} 2\pi & \frac{\pi}{3} \\ \frac{\pi}{3} & \frac{3\pi}{8} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} -\pi \\ -\frac{\pi}{6} \end{pmatrix}$$

whose solution is $c_1 = \frac{1}{2}$ and $c_2 = 0$.

It happens that this is a harmonic function that satisfies the BC. One sees that

$$x^2 + \frac{1}{2}(1 - x^2 - y^2) = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}$$

which is a sum of harmonics.

9. *Solve the same problem for Laplace's equation on the unit disk. This time, use Fourier series.*

$$\begin{array}{lrl} \text{(PDE)} & u_{xx} + u_{yy} = 0, & \text{for } x^2 + y^2 < 1; \\ \text{(BC)} & u = x^2, & \text{for } x^2 + y^2 = 1. \end{array}$$

In polar coordinates, $u = R(r)\Theta(\theta)$ satisfies Laplace's Equation

$$0 = \Delta\, u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta''.$$

Multiply by $r^2/R\Theta$ and separate.

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.$$

The θ equation gives the eigenvalue problem

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta \text{ is } 2pi\text{-periodic.}$$

This has the eigenfunctions

$$\cos n\theta, \quad \sin n\theta, \qquad \lambda_n = n^2, \qquad 0, 1, 2, 3, \ldots.$$

The corresponding r equation is

$$r^2 R_n'' + rR_n' - \lambda_n R_n = 0, \qquad R_n \text{ is bounded in } x^2 + y^2 < 1.$$

These are Euler equations. For $n \ge 0$ we guess $R = r^\alpha$. Inserting

$$0 = r^2\alpha(\alpha - 1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = (\alpha(\alpha - 1) - \lambda_n)r^\alpha.$$

The characteristic equation is

$$\alpha^2 - n^2 = 0$$

whose roots are $\pm n$ and general solution

$$R_n(r) = C_n r^n + D_n r^{-n}$$

so D_n must vanish to keep the solution bounded at zero. For $n = 0$,

$$0 = r^2 R_0'' + rR' = r(rR_0')'$$

whose general solution is

$$R_0(r) = C_0 + D_0 \log r.$$

Again, D_n must vanish to keep the solution bounded at zero. The general solution is thus

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left\{A_n \cos n\theta + B_n \sin n\theta\right\}.$$

The initial condition tells us on $r = 1$

$$x^2 = \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n \left\{A_n \cos n\theta + B_n \sin n\theta\right\}$$

so

$$1 = A_0, \qquad \frac{1}{2} = A_2, \qquad \text{The rest of the } A_n,\ B_n\text{'s vanish.}$$

Hence the Fourier Series solution is

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2} r^2 \cos(2\theta)$$

which is the same as

$$u = \frac{1}{2} + \frac{1}{2} r^2 \left[\cos^2\theta - \sin^2\theta\right] = \frac{1}{2} - \frac{1}{2}[x^2 - y^2]$$

which agrees with the solution noticed at the end of the previous problem!

10. *Suppose $f(\theta)$ is a continuoius 2π-periodic function. Show that the Poisson integral*

$$P[f](r,\theta) \to f(\theta_0) \qquad \text{as } (r,\theta) \to (a,\theta_0).$$

This problem is more difficult than you would see on an exam, but you should be able to do it as, say, a homework problem. The ideas are similar to the showing the pointwise convergence of Fourier Series. The Poisson Integral is given by,

$$P[f](r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)\,d\phi}{a^2 - 2ar\cos(\phi - \theta) + r^2}$$

Choose $\epsilon > 0$. By continuity, there is a $\delta > 0$ so that

$$|f(\phi - \theta_0) - f(\theta_0)| < \frac{\epsilon}{2} \qquad \text{whenever } |\phi - \theta_0| < \delta.$$

Now take θ such that $|\theta - \theta_0| < \delta/2$.

Observe that since the harmonic function with constant boundary values $f(\theta) = 1$ is $u(r,\theta) = 1$ for all $0 \le r < a$ and θ. We could also observe that we can integrate term by term the uniformly convergent series when $r < a$. In either case, we get

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \left\{1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\phi - \theta)\right\} d\phi = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{d\phi}{1 - 2r\cos(\phi - \theta) + r^2} \qquad (2)$$

It follows that

$$P[f](r,\theta) - f(\theta_0) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi) - f(\theta_0)}{a^2 - 2ar\cos(\phi - \theta) + r^2} d\phi$$

Substitute $\psi = \phi - \theta_0$, and use the fact that the integrand has period 2π

$$P[f](r,\theta) - f(\theta_0) = \frac{a^2-r^2}{2\pi}\int_{-\theta_0}^{2\pi-\theta_0} \frac{f(\psi+\theta_0)-f(\theta_0)}{a^2-2ar\cos(\psi+\theta_0-\theta)+r^2}d\psi$$
$$= \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\psi+\theta_0)-f(\theta_0)}{a^2-2a\cos(\psi+\theta_0-\theta)+r^2}d\psi$$

Split into three integrals

$$P[f](r,\theta) - f(\theta_0) = I + II + III$$
$$= \frac{a^2-r^2}{2\pi}\left[\int_{-\pi}^{-\delta} \frac{f(\psi+\theta_0)-f(\theta_0)}{a^2-2a\cos(\psi+\theta_0-\theta)+r^2}d\psi + \int_{-\delta}^{\delta} \frac{f(\psi+\theta_0)-f(\theta_0)}{1-2r\cos(\psi+\theta_0-\theta)+r^2}d\psi + \int_{\delta}^{\pi} \frac{f(\psi+\theta_0)-f(\theta_0)}{1-2r\cos(\psi+\theta_0-\theta)+r^2}d\psi\right]$$

In the the middle integral, using the continuity and the positivity of the integrand in (2),

$$\left|\frac{a^2-r^2}{2\pi}\int_{-\delta}^{\delta} \frac{f(\psi+\theta_0)-f(\theta_0)}{a^2-2ar\cos(\psi+\theta_0-\theta)+r^2}d\psi\right| \le$$
$$\le \frac{1-r^2}{2\pi}\int_{-\delta}^{\delta} \frac{|f(\psi+\theta_0)-f(\theta_0)|}{a^2-2ar\cos(\psi+\theta_0-\theta)+r^2}d\psi$$
$$\le \frac{\epsilon}{2}\cdot\frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{d\psi}{a^2-2ar\cos(\psi+\theta_0-\theta)+r^2}d\psi \le \frac{\epsilon}{2}$$

On the other hand, $|\theta-\theta_0| < \delta/2$ and $|\psi| \ge \delta$ we have

$$|\psi+\theta_0-\theta| \ge |\psi| - |\theta-\theta_0| \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

so

$$a^2 - 2ar\cos(\psi+\theta_0-\theta) + r^2 + r^2 \ge a^2 - 2ar\cos\left(\frac{\delta}{2}\right) + r^2.$$

Thus the other two integrals are bounded by

$$|I + III| \le \frac{a^2-r^2}{2\pi\left(a^2-2ar\cos\left(\frac{\delta}{2}\right)+r^2\right)}\left[\int_0^{2\pi}|f(\phi)|d\phi + 2\pi|f(\theta_0)|\right]$$

which tends to zero as $r \to a-$. Hence there is $\eta > 0$ so that

$$|I + III| \le \frac{\epsilon}{2} \qquad \text{whenever } a-\eta < r < a.$$

It follows that

$$|P[f](r,\theta) - f(\theta_0)| < \epsilon \qquad \text{whenever } |\theta-\theta_0| < \frac{\delta}{2} \text{ and } a-\eta < r < a.$$

which is to say

$$\lim_{(t,\theta)\to(a,\theta_0)} P[f](\theta) = f(\theta_0).$$

Note that the convergence in Poisson's formula is better than for Fourier Series. The pointwise proof required that f' be piecwise continuous too. Indeed, there are continuous functions for which the Fourier series does not even converge, whereas they do in Poisson's formula.

11. *Use Poisson's Formula to prove Weierstrass's Approximation Theorem: if f is a 2π-periodic, continuous function, then for every $\epsilon > 0$ there is a trigonometric polynomial (finite Fourier series) $S_N(\theta)$ such that*

$$|f(\theta) - S_N(\theta)| < \epsilon \qquad \text{for all } \theta.$$

This is an example of how PDE techniques can be used to prove results in real analysis. The Weierstrass Approximation Theorem is sometimes covered in first year analysis courses. Another version says continuous functions on a compact interval may be uniformly approximated by ordinary polynomials.

The idea is to take r close to a and approximate f by the Poisson integral. Then to approximate the infinite sum by a finite one.

We showed in the previous problem that the Poisson integral approximates continuous functions pointwise. Because the unit circle $0 \le \theta_0 \le 2\pi$ is compact, it follows from a result in first quarter analysis that the continuous function $h(\phi)$ is uniformly continuous. Hence neither the the δ nor the η depend on θ so the approximation is uniform. Thus it says, for every $\epsilon > 0$ there is an $\eta > 0$ so that

$$|P[f](r,\theta_0) - f(\theta_0)| < \frac{\epsilon}{2} \qquad \text{for any } \theta_0 \text{ and whenever } a - r < r < a. \tag{3}$$

Now recall that the Poisson Formula was derived by summing the Fourier Series

$$P[f](r,\theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left\{C_n \cos n\theta + D_n \sin n\theta\right\}. \tag{4}$$

Since we have moved a^n back into the formula, for $n = 0, 1, 2, \ldots$, this leaves

$$C_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi, \qquad D_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi$$

The coefficients are bounded

$$|C_n|, \quad |D_n| \quad \le \frac{1}{\pi}\int_0^{2\pi} |f(\phi)|\, d\phi = M$$

which means that the series (4) is uniformly convergent because the summands decay geometrically since $r/a < 1$. Recall that the proof of Poisson's Formula follows by replacing C_n and D_n by their integrals, and then exchanging summation and integration by uniform convergence of the series and then summing the series inside the integrals.

The geometric decay allows us to estimate how well the partial sums of (4) approximate the total, thereby completing our approximation. Fix $a - \eta < r < a$, say $r_0 = a - \eta/2$. Then the error approximating the partial sum is bounded by

$$\begin{aligned} |P[f](r_0,\theta) - S_N(\theta)| &= \left| \sum_{n=N+1}^{\infty} \left(\frac{r_0}{a}\right)^n \left\{C_n \cos n\theta + D_n \sin n\theta\right\} \right| \\ &\le \sum_{n=N+1}^{\infty} \left(\frac{r_0}{a}\right)^n \left\{|C_n| + |D_n|\right\} \\ &\le 2M \sum_{n=N+1}^{\infty} \left(\frac{r_0}{a}\right)^n \\ &= \frac{2M\left(\frac{r_0}{a}\right)^{N+1}}{1 - \left(\frac{r_0}{a}\right)} \end{aligned} \tag{5}$$

which tends to zero as $N \to \infty$. Thus for $N_0 = N(a, M, r_0)$ sufficiently large, the sum is less than $\epsilon/2$. Combining (3) and (5), it follows that

$$|f(\theta_0) - S_{N_0}(\theta_0)| < \epsilon \qquad \text{for all } \theta_0.$$

where the trigonometric polynomial is given by

$$S_{N_0}(\theta) = \frac{C_0}{2} + \sum_{n=1}^{N_0} \left(\frac{r_0}{a}\right)^n \{C_n \cos n\theta + D_n \sin n\theta\}.$$

We remark again that the series (4) may not converge at all if $r = a$ because the Fourier series may not converge if f is only continuous. The approximating trigonometric polynomial has the additional $(a/r)^n$ convergence factors, which is known as an Abel Sum in the theory of trigonometric series.

12. *Solve the BVP for Laplace's Equation in the quarter circle*

$$\begin{aligned}
&\text{(PDE)} \qquad && \Delta u = 0, \qquad && \text{for } x^2 + y^2 < 1 \text{ and } 0 < \theta < \frac{\pi}{2};\\
&\text{(BC)} && u(1, \theta) = \theta, && \text{for } 0 < \theta < \frac{\pi}{2},\\
& && u(r, 0) = 0, && \\
& && u_\theta(r, \frac{\pi}{2}) = 0, && \text{for } 0 < r < 1.
\end{aligned}$$

In polar coordinates, $u = R(r)\Theta(\theta)$ satisfies Laplace's Equation

$$0 = \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Multiply by $r^2/R\Theta$ and separate.

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.$$

The θ equation gives the eigenvalue problem

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = 0, \qquad \Theta'(\tfrac{\pi}{2}) = 0.$$

These are symmetric boundary conditions so the eigenvalues are positive $\lambda = \beta^2$ where $\beta > 0$. The general solution is

$$\Theta(\theta) = A\cos\beta\theta + B\sin\beta\theta$$

The boundary condition $\Theta(0) = 0$ says $A = 0$. The boundary condition $0 = \Theta'(\frac{\pi}{2}) = B\beta\cos(\beta\frac{\pi}{2})$ says $\beta = 1, 3, 5, \ldots$. Thus the eigenvalus and eigenfunctions are

$$\lambda_n = (2n+1)^2, \qquad \Theta_n(\theta) = \sin(2n+1)\theta, \qquad n = 0, 1, 2, 3, \ldots$$

The corresponding r equation is

$$r^2 R_n'' + rR_n' - \lambda_n R_n = 0, \qquad R_n \text{ is bounded in } r < 1.$$

As in Problem 9, the solution is

$$R_n(r) = r^{2n+1}.$$

The general solution is thus

$$u(r,\theta) = \sum_{n=0}^{\infty} B_n r^{2n+1} \sin(2n+1)\theta.$$

The initial condition tells us on $r = 1$

$$\theta = u(1,\theta) = \sum_{n=0}^{\infty} B_n \sin(2n+1)\theta$$

Picking off the coefficients using the inner product, and integrating by parts

$$\begin{aligned} B_n &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \theta \sin(2n+1)\theta \, d\theta \\ &= \frac{4}{\pi} \left\{ -\left[\frac{\theta \cos(2n+1)\theta}{2n+1} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)\theta \, d\theta}{2n+1} \right\} \\ &= \frac{4}{\pi} \left[\frac{\sin(2n+1)\theta}{(2n+1)^2} \right]_0^{\frac{\pi}{2}} = \frac{4 \sin(n+\frac{1}{2})\pi}{\pi(2n+1)^2} = \frac{4(-1)^n}{\pi(2n+1)^2} \end{aligned}$$

Hence the Fourier Series solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} \frac{4(-1)^n r^{2n+1}}{\pi(2n+1)^2} \sin(2n+1)\theta.$$

13. *Let $\mathcal{D} \subset \mathbf{R}^2$ be an open domain. Suppose u is continuous on the closure $\overline{\mathcal{D}}$ and harmonic in $\mathcal{D}$. Suppose that $B_R(x) \subset \mathcal{D}$, the open disk of radius R and center x is completely contained in the domain. Show the mean value property also holds on the whole disk*

$$u(x) = \frac{1}{\pi R^2} \int_{B_R(x)} u(x) \, dA(x).$$

Let $0 \le r \le R$ and consider the circle of radius r about x. For $y = x + \rho(\cos\theta, \sin\theta)$ where $0 \le \rho < r$ we have Poisson's Formula

$$u(y) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u\big(x + \rho(\cos\phi, \sin\phi)\big) \, d\phi}{r^2 - 2\rho r \cos(\theta - \phi) + \rho^2}$$

We have the mean value property for circles simply by substituting $\rho = 0$ into Poisson's formula.

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u\big(x + r(\cos\phi, \sin\phi)\big) \, d\phi$$

Substituting this for the inside integral over the whole disk gives the result

$$\frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u\big(x + r(\cos\phi, \sin\phi)\big) \, d\phi \, r \, dr = \frac{1}{\pi R^2} \int_0^R 2\pi u(x) \, r \, dr = \frac{2u(x)}{R^2} \cdot \frac{R^2}{2} = u(x).$$

14. *Let A and B be two disjoint smooth bounded domains in $\mathbf{R}^3$ and $Q > 0$ constant. Denote by $\mathcal{D} = \mathbf{R}^3 - (A \cup B)$ be the exterior domain. Thus $\partial\mathcal{D} = \partial A \cup \partial B$. Let u be a harmonic function on $\mathcal{D}$ which tends to zero at infinity and which is constant on ∂A and constant on ∂B, and satisfies*

$$\int_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0 \qquad \text{and} \qquad \int_{\partial A} \frac{\partial u}{\partial n} dS = 0 \tag{6}$$

where n denotes the outer normal from $\mathcal{D}$. [Interpretation: u is the electrostatic potential outside two conductors, A carries charge Q and B is uncharged.] (Text problem 7.1.6.)

(a) Show the solution is unique. [Hint: Use Hopf's Maximum Principle.]

(b) Show that $u \geq 0$ on $\mathcal{D}$. [Hint: If not, u has a negative minimum. Use Hopf's Maximum Principle again.]

(c) Show $u > 0$ in $\mathcal{D}$.

Let u and v be two solutions. Their difference $w = u - v$ is harmonic, vanishes at infinity and is constant on ∂A and ∂B (although we don't know that w is zero on either ∂A or ∂B) and

$$\int_{\partial A} \frac{\partial w}{\partial n} dS = \int_{\partial B} \frac{\partial w}{\partial n} dS = 0. \tag{7}$$

Let α and β be the values of w on the respective boundaries and suppose that the larger of the two magnitudes occurs on ∂A, so $\alpha \neq 0$ and $|\alpha| \geq |\beta|$. If the maximum occurs on B, we just swap the roles of A and B in the argument. By replacing w by $-w$ we may assume $\alpha > 0$. If one takes a large disk B_R about zero that contains A and B, then for R large enough, $\alpha > |u(x)|$ for all $x \in \partial B_R$. It follows by the maximum principle that the maximum of u on $B_R - (A \cup B)$ occurs on the boundary, in fact on ∂A. By Hopf's Maximum Principle, the derivative is strictly increasing to a maximum from inside $\mathcal{D}$

$$\frac{\partial w}{\partial n}(a) > 0 \qquad \text{for all } a \in \partial A.$$

But this contradicts (7). It follows that $\alpha = \beta = 0$ and the maximum occurs on ∂B_R. By taking $R \to \infty$, both the maximum and minimum of w on ∂B_R tend to zero, which means that at a point $x \in \mathcal{D}$ inside, $|w(x)|$ also tends to zero. Thus $w \equiv 0$ in $\mathcal{D}$ and the solution is unique.

We argue that $u \geq 0$ on $\mathcal{D}$. Let α and β be the constants on ∂A and ∂B. First we argue that neither is negative. Suppose one of them is the more negative, say $\alpha < 0$ and $\beta \geq \alpha$. As before, for large R, $u > \alpha$ on ∂B_R for R large enough. Thus the weak maximum principle says the minimum of u is on the boundary ∂A. By Hopf's Maximum Principle, the derivative is strictly decreasing to a minimum from inside $\mathcal{D}$

$$\frac{\partial u}{\partial n}(a) < 0 \qquad \text{for all } a \in \partial A.$$

Hence

$$\int_{\partial A} \frac{\partial w}{\partial n} dS < 0$$

which contradicts (6). If the minimum were on ∂B instead, then we would likewise get

$$\int_{\partial B} \frac{\partial w}{\partial n} dS < 0$$

which contradicts (6) also.

As before, if $u(x) < 0$, for large R, $u(y) > u(x)$ on $y \in \partial B_R$. Then there is a negative minimum point $z \in \mathcal{D}$ such that $u(z) \leq u(x)$. But this contradicts the usual maximum principle on $B_R - (A \cup B)$. Thus we have shown $u \geq 0$ on $\mathcal{D}$.

Finally, we claim $u > 0$ on $\mathcal{D}$. First we must have $\alpha > 0$. If not $\alpha = 0$ and because $u \geq 0$ then u is a minimum on ∂A. But this means that

$$\frac{\partial u}{\partial n}(a) < 0 \qquad \text{for all } a \in \partial A.$$

contrary to (6). By the same argument $\beta > 0$. Let us rule out $u(x) = 0$ for any interior point $x \in \mathcal{D}$. Because $u \geq 0$ in $\mathcal{D}$ ir follows that $\nabla u(x) = 0$ at the minimum. Put a tiny sphere $\Omega \subset \mathcal{D}$ such that $x \in \partial\Omega$. Now $u \geq 0$ in Ω says that u is minimum of u in $\overline{\Omega}$ at x. Let ν be the outward normal of Ω. By Hopf's maximum Principle,

$$\frac{\partial u}{\partial \nu}(x) < 0,$$

contrary to $\nabla u(x) = 0$.

15. *Suppose $u(x,y) \in \mathcal{C}^2([0,a]\times[0,b])$ is a solution of the boundary value problem for Poisson's equation on the rectangle. Show that $u(x,y) \leq \frac{a^2}{8}$. (Hint: maximum principle using easy solutions of the PDE.)*

$$\begin{array}{lll} \text{(PDE)} & u_{xx} + u_{yy} = -1, & \text{for } 0 < x < \pi,\ 0 < y < 1; \\ \text{(BC)} & u(x,0) = u(x,b) = 0, & \text{for } 0 < x < a; \\ & u(0,y) = u(a,y) = 0,, & \text{for } 0 < y < b. \end{array}$$

Let's consider a quadratic polynomial, whose second derivative is -1 and that is zero at endpoints.

$$v(x,y) = \frac{1}{2}x(a-x).$$

Hence $\Delta\, v = -1$. Put $w(x,y) = u(x,y) - v(x,y)$ to show $w \leq 0$. w satisfies

$$\begin{array}{lll} \text{(PDE)} & w_{xx} + w_{yy} = -1 - (-1) = 0, & \text{for } 0 < x < \pi,\ 0 < y < 1; \\ \text{(BC)} & w(x,0) = w(x,b) = 0 - \frac{1}{2}x(a-x), & \text{for } 0 < x < a; \\ & w(0,y) = w(a,y) = 0 - 0 = 0, & \text{for } 0 < y < b. \end{array}$$

Because $w(x,y) \leq 0$ on all four sides, we have by the maximum principle

$$w(x,y) \leq 0 \qquad \text{for all } 0 < x < a \text{ and } 0 < y < b.$$

Hence we obtain the desired estimate by maximizing the quadratic function

$$u(x,y) \leq \frac{1}{2}x(a-x) \leq \frac{a^2}{8} \qquad \text{for all } 0 < x < a \text{ and } 0 < y < b.$$

16. *Let $\varphi(x)$ be any $\mathcal{C}^2$ function on three dimensional space that vanishes outside some sphere. Show the equality. (Text problem 7.2.2.)*

$$\varphi(0) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\Delta\,\varphi(x)}{|x|}\, dV(x)$$

Let us integrate by parts on the annulus $D(\epsilon, R)$ where $R > 0$ is so big that it contains the support of φ and $0 < \epsilon < R$. The formula will be the result of taking the limit $\epsilon \to 0$. Applying the divergence theorem to $u\nabla v - v\nabla u$ we get,

$$\int_{D(\epsilon,R)} u\,\Delta\, v - v\,\Delta\, u\, dV = \int_{\partial D(\epsilon,R)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dA.$$

Insert the choice $u = |x|^{-1} = r^{-1}$ and $v = \varphi$, we have $\Delta\, u = 0$ in $D(\epsilon, R)$ which is away from the singularity. Note that $\varphi(x) = 0$ near $|x| = R$ so that both $\phi(x) = \partial_r\varphi(x) = 0$

when $|x| = R$ so that the integrals on the outer boundary of $D(r, R)$ vanish. On the inner boundary $\partial/\partial n = -\partial/\partial r$ because the normal points toward the origin. It follows that

$$\begin{aligned}\int_{D(\epsilon,R)} \frac{\Delta\,\varphi(x)}{|x|}\,dV(x) &= -\int_{\partial B_\epsilon(0)} \left(\frac{1}{r}\frac{\partial\varphi}{\partial r} - \varphi(x)\frac{\partial}{\partial r}\frac{1}{r}\right)\,dA \\ &= -\int_{\partial B_\epsilon(0)} \left(\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\varphi(x)}{\epsilon^2}\right)\,dA.\end{aligned} \tag{8}$$

Since φ is $\mathcal{C}^2$, its gradient is bounded $|\nabla\varphi(x)| \le M$ for $x \in \overline{B_R(0)}$. It follows that

$$\left|\int_{\partial B_\epsilon(0)} \left(\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)\,dA\right| \le \frac{M}{\epsilon}\cdot 4\pi\epsilon^2 \to 0 \qquad \text{as } \epsilon \to 0.$$

On the other hand $\phi(x) = \phi(0) + \mathcal{R}(x)$ where $|\mathcal{R}(x)| \le M|x - 0|$ since M bounds the Lipschitz constant for φ. Hence

$$\begin{aligned}\left|\frac{1}{\epsilon^2}\int_{\partial B_\epsilon(0)} \phi(x)\,dA - 4\pi\varphi(0)\right| &= \left|\frac{1}{\epsilon^2}\int_{\partial B_\epsilon(0)} \varphi(x) - \phi(0)\,dA\right| \\ &= \frac{1}{\epsilon^2}\int_{\partial B_\epsilon(0)} |\mathcal{R}(x)|\,dA \\ &\le \frac{M|\epsilon|\cdot 4\pi\epsilon^2\varphi(0)}{\epsilon^2} \to 0 \qquad \text{as } \epsilon \to 0.\end{aligned}$$

Thus, taking $\epsilon \to 0$ and $R \to \infty$ in (8) yields the desired equation

$$\int_{\mathbf{R}^3} \frac{\Delta\,\varphi(x)}{|x|}\,dV(x) = -4\pi\varphi(0).$$

17. *Show that distributions may be differentiated. If f is any distribution on $\mathbf{R}$, then f' is a distribution defined by the formula. (Text problem 12.1.2.)*

$$(f', \varphi) = -(f, \varphi') \qquad \text{for all test functions } \varphi.$$

To be a distribution, it must be linear and continuous on test functions. To check linearity, let φ and ψ be test functions ($\mathcal{C}^\infty$ functions with compact support) and let a and b be constants.

$$\begin{aligned}(f', a\varphi + b\psi) &= -(f, [a\varphi + b\psi]') && \text{definition of derivative distribution.} \\ &= -(f, a\varphi' + b\psi') && \text{linearity of derivative of a smooth function.} \\ &= -a(f, \varphi') - b(f, \psi') && \text{linearity of the pairing } f \text{ with test function.} \\ &= a(f', \varphi) + b(f', \psi) && \text{definition of derivative distribution.}\end{aligned}$$

To check continuity, let $\varphi_n \to \varphi$ be a sequence of test functions that all vanish outside a common finite interval which for which each derivative converges uniformly to the derivative of φ. Then, as $n \to \infty$,

$$\begin{aligned}(f', \varphi_n) &= -(f, \varphi_n') && \text{definition of derivative distribution.} \\ &\to -(f, \varphi') && \text{continuity of } f \text{ for the converging sequence } \{\varphi_n'\}. \\ &= (f', \varphi) && \text{definition of derivative distribution.}\end{aligned}$$

18. *Construct a piecewise constant approximate identity* χ_a *as follows. Show that* $\chi_a \to \delta_0$ *weakly as* $a \to 0$. *(Text problem 12.1.10.)*

$$\chi_a = \begin{cases} \dfrac{1}{2a}, & \text{if } -a < x < a; \\ \\ 0, & \text{otherwise.} \end{cases}$$

Weak convergence for distributions means

$$(\chi_a, \varphi) \to 0 \qquad \text{as } a \to 0- \text{ for every test function } \varphi.$$

Since χ_a is a function, as a distribution its evaluates on test functions φ as

$$(\chi_a, \varphi) = \int_{-\infty}^{\infty} \chi_a(s)\phi(s)\, ds = \frac{1}{2a}\int_{-a}^{a} \varphi(s)\, ds$$

Since φ is continuous at zero, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(s) - \varphi(0)| < \epsilon \qquad \text{whenever } |x| < \delta.$$

It follows that if $0 < a < \delta$,

$$\begin{aligned} |(\chi_a, \varphi) - \varphi(0)| &\le \left| \frac{1}{2a}\int_{-a}^{a} \varphi(s) - \varphi(0)\, ds \right| \\ &\le \frac{1}{2a}\int_{-a}^{a} |\varphi(s) - \varphi(0)|\, ds \\ &< \frac{1}{2a}\int_{-a}^{a} \epsilon\, ds = \epsilon \end{aligned}$$

Hence

$$(\chi_a, \varphi) \to \varphi(0) = (\delta_0, \varphi) \qquad \text{as } a \to 0.$$

Since φ was arbitrary we have shown $\chi_a \to \delta_0$ weakly.

19. *Verify that the Fourier Transform of the square pulse equals the sine. (Text problem 12.3.1b.)*

$$\mathcal{F}[H(a - |x|)](k) = \frac{2}{k}\sin ak.$$

The square pulse equals one if $|x| < a$ and zero if $|x| > a$. Thus its Fourier transform equals

$$\begin{aligned} \mathcal{F}[H(a - |x|)](k) &= \int_{-\infty}^{\infty} H(a - |x|)e^{-ikx}\, dx \\ &= \int_{-a}^{a} e^{-ikx}\, dx \\ &= \left[\frac{e^{-ikx}}{-ik}\right]_{-a}^{a} \\ &= \frac{1}{-ik}\left(e^{-ika} - e^{ika}\right) \\ &= \frac{2}{k}\sin ka. \end{aligned}$$

20. *Verify the Fourier Transform of the derivative formula. (Text problem 12.3.2a.)*

$$\mathcal{F}\left[\frac{\partial f}{\partial x}\right](k) = ik\mathcal{F}\left[f\right](k).$$

In case f is a $\mathcal{C}^1$ function with compact support, we can integrate by parts. Suppose that the support of f is contained in the interval $(-R, R)$. Thus

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial f}{\partial x}\right](k) &= \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x)e^{-ikx}\,dx \\
&= \int_{-R}^{R} \frac{\partial f}{\partial x}(x)e^{-ikx}\,dx \\
&= \Big[f(x)e^{-ikx}\Big]_{-R}^{R} + ik\int_{-R}^{R} f(x)e^{-ikx}\,dx \\
&= 0 + ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\
&= ik\mathcal{F}[f](k).
\end{aligned}$$

In case f is a distribution, for any test function φ, the definition of the Fourier transform is

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]).$$

We get for the derivative and for any test function (compactly supported)

$$\begin{aligned}
(\mathcal{F}\left[f'\right], \phi) &= (f', \mathcal{F}\left[\phi\right]) \\
&= -\big(f, (\mathcal{F}\left[\phi\right])'\big) \\
&= -\left(f, \frac{\partial}{\partial k}\int_{-\infty}^{\infty} e^{-ikx}\phi(x)\,dx\right) \\
&= -\left(f, \int_{-\infty}^{\infty} \frac{\partial}{\partial k}e^{-ikx}\phi(x)\,dx\right) \\
&= -\left(f, \int_{-\infty}^{\infty} -ixe^{-ikx}\phi(x)\,dx\right) \\
&= \big(f, \mathcal{F}\left[ix\phi(x)\right]\big) \\
&= \big(\mathcal{F}\left[f\right](k), ik\phi(k)]\big) \\
&= \big(ik\mathcal{F}[f](k), \phi(k)\big).
\end{aligned}$$

21. *Assuming $\varphi(x)$ is bounded and continuous, show that there is at most one solution of the initial value problem for the heat equation on the line. In particular, if φ and u are bounded, prove the maximum principle $|u| \le \sup|\varphi|$.*

(PDE)	$u_t = u_{xx},$	for $-\infty < x < \infty$, $0 < t < t_0$;
(IC)	$u(0, x) = \varphi(x),$	for $-\infty < x < \infty$;
(BC)	$u(t, x)e^{x^2/4t_0} \to 0$	uniformly in t as $|x| \to \infty$.

Hint: make a change of dependent variables [Problem from Weinberger, First Course in PDE, Xerox Pub., 1965, p. 320.]

$$u(t, x) = \frac{v(t, x)}{\sqrt{t_0 - t}}\exp\left(\frac{x^2}{4(t_0 - t)}\right)$$

Take the kernel

$$H(t,x) = \frac{1}{\sqrt{t_0 - t}} \exp\left(\frac{x^2}{4(t_0 - t)}\right).$$

It satisfies the heat equation

$$H > 0, \qquad H_x = \frac{xH}{2(t_0 - 1)}, \qquad H_t = H_{xx}$$

so

$$0 = u_t - u_{xx} = (v_t - v_{xx})H - 2v_x H_x + v(H_t - H_{xx}) = \left(v_t - v_{xx} - \frac{x v_x}{t_0 - t}\right) H.$$

Thus v satisfies the maximum principle in any region $[-R, R] \times [0, t_0)$. Now choose any $-\infty < \xi < \infty$ and $0 < \tau < t_0$.

Since we assume $u(x,t)e^{-x^2/4t_0} \to 0$ uniformly in t, for every $\epsilon > 0$ there is an $R > 0$ so that $R > |\xi|$ and $|u(x,t)| \le \epsilon e^{R^2/4t_0}$ whenever $0 \le t < t_0$ and $|x| \ge R$. Then

$$|v(t,x)| \le \epsilon \sqrt{t_0 - t} \exp\left(\frac{x^2}{4}\left\{\frac{1}{t_0} - \frac{1}{(t_0 - t)}\right\}\right) \le \epsilon\sqrt{t_0}.$$

for all $0 < t \le t_0$ and $|x| \ge R$. Since the initial condition

$$v(0,x) = \varphi(x)\sqrt{t_0} \exp\left(\frac{-x^2}{4t_0}\right),$$

by the maximum principle on $[-R, R] \times [0, \tau]$, we have

$$|v(\tau, \xi)| \le \max\left[\epsilon\sqrt{t_0}, \sqrt{t_0} \sup\left\{|\varphi(x)| \exp\left(\frac{-x^2}{4t_0}\right)\right\}\right]$$

Since $\epsilon > 0$ was arbitrary,

$$|v(\tau, \xi)| \le \sqrt{t_0} \sup\left\{|\varphi(x)| \exp\left(\frac{-x^2}{4t_0}\right)\right\}.$$

It follows that for every $(\xi, \tau) \in (-\infty, \infty) \times [0, t_0]$,

$$|u(\tau, \xi)| \le \frac{\sqrt{t_0}}{\sqrt{t_0 - \tau}} \exp\left(\frac{\xi^2}{4(t_0 - \tau)}\right) \sup\left\{|\varphi(x)| \exp\left(\frac{-x^2}{4t_0}\right)\right\}.$$

Now, if w is the difference of two solutions of the IVP, this estimate shows that $w \equiv 0$ so the solution is unique. One can continue the solutions to $t_0 \le t \le 2t_0$ and then to $2t_0 \le t \le 3t_0$ and so on to prove $w \equiv 0$ for all t.

If u and φ are bounded, then the condition holds for t_0 arbitrarily large. Fix $(\xi, \tau) \in (-\infty, \infty) \times [0, \infty)$, and take $t_0 > \tau$. Letting $t_0 \to \infty$ in the bound gives

$$|u(\tau, \xi)| \le \frac{\sqrt{t_0}}{\sqrt{t_0 - \tau}} \exp\left(\frac{\xi^2}{4(t_0 - \tau)}\right) \sup\left\{|\varphi(x)| \exp\left(\frac{-x^2}{4t_0}\right)\right\} \to \sup|\varphi(x)|.$$

22. *Using Fourier transforms, solve the initial value problem for the wave equation on the real line.*

$$\begin{array}{lll} \text{(PDE)} & u_{tt} = c^2 u_{xx}, & \text{for } -\infty < x < \infty,\ 0 < t; \\ \text{(IC)} & u(x,0) = 0, & \\ & u_t(x,0) = \delta(x), & \text{for } -\infty < x < \infty; \end{array}$$

Take Fourier transform with respect to x, using $\widehat{u_t} = \hat{u}_t$, $\widehat{u_x} = i\omega\hat{u}$ and $\hat{\delta} = 1$,

$$\hat{u}_{tt} = -c^2\omega^2\hat{u}, \qquad \hat{u}(0,\omega) = 0, \qquad \hat{u}_t(0,\omega) = 1.$$

This ODE in t has the solution

$$\hat{u}(t,\omega) = \frac{1}{c\omega}\sin c\omega t = \frac{1}{2c}\cdot\frac{2}{\omega}\sin c\omega t$$

From Problem 19,

$$\mathcal{F}[H(a-|x|)](\omega) = \frac{2}{\omega}\sin a\omega.$$

Taking $a = ct$,

$$u(t,x) = \frac{1}{2c}H(ct-|x|).$$

This is the fundamental solution. An initial unit impulse of velocity at the origin results in a widening square wave.

23. *Using the source function from Problem 22, solve the wave equation on the line, where φ and ψ are bounded, piecewise continuously differentiable functions.*

$$\begin{aligned} &\text{(PDE)} & u_{tt} &= c^2 u_{xx}, && \text{for } -\infty < x < \infty,\ 0 < t;\\ &\text{(IC)} & u(x,0) &= \varphi(x), &&\\ && u_t(x,0) &= \psi(x), && \text{for } -\infty < x < \infty;\end{aligned}$$

Decompose the solution into $u = v + w$ where v solves

$$\begin{aligned} &\text{(PDE)} & v_{tt} &= c^2 v_{xx}, && \text{for } -\infty < x < \infty,\ 0 < t;\\ &\text{(IC)} & v(x,0) &= \varphi(x), &&\\ && v_t(x,0) &= 0, && \text{for } -\infty < x < \infty.\end{aligned}$$

and w solves

$$\begin{aligned} &\text{(PDE)} & w_{tt} &= c^2 w_{xx}, && \text{for } -\infty < x < \infty,\ 0 < t;\\ &\text{(IC)} & w(x,0) &= 0, &&\\ && w_t(x,0) &= \psi(x), && \text{for } -\infty < x < \infty.\end{aligned}$$

Since the source function kernel solves the wave equation (at least in the distributional sense) then

$$w(t,x) = \frac{1}{2c}\int_{-\infty}^{\infty} H(ct-|x-y|)\psi(y)\,dy = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy.$$

solves the PDE and IC for w. The integral makes sense by the piecewise continuity of ψ. Of course this recovers the second part of d'Alembert's solution. We also use the fact that the derivative z_t solves the wave equation if z does. Consider the following IVP for z Decompose the solution into $u = v + w$ where v solves

$$\begin{aligned} &\text{(PDE)} & z_{tt} &= c^2 z_{xx}, && \text{for } -\infty < x < \infty,\ 0 < t;\\ &\text{(IC)} & z(x,0) &= 0, &&\\ && z_t(x,0) &= \varphi(x), && \text{for } -\infty < x < \infty.\end{aligned}$$

As for w, the solution is

$$z(t,x) = \frac{1}{2c}\int_{-\infty}^{\infty} H(ct-|x-y|)\varphi(y)\,dy.$$

Now use the fact that as distributions, $H' = \delta$. To see this, for any test function ζ, for R larger than the support of ζ,

$$(H', \zeta) = -(H, \zeta') = -\int_{-\infty}^{\infty} H(s)\zeta'(s)\,ds = -\int_0^R \zeta'(s)\,ds = -\zeta(R) + \zeta(0) = \zeta(0) = (\delta, \zeta).$$

Differentiating we find

$$\begin{aligned} z_t(t,x) &= \frac{1}{2c}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} H(ct - |x-y|)\varphi(y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} H'(ct - |x-y|)\varphi(y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \delta(ct - |x-y|)\varphi(y)\,dy \\ &= \frac{1}{2}\big[\varphi(ct-x) + \varphi(ct+x)\big]. \end{aligned}$$

One checks that $z_{tt}(0,x) = 0$ so it solves the IC's for v, thus $v = z_t$. Of course, this is the first half of d'Alembert's formula so we are not surprised.

24. *Find by means of Plancherel's Formula. [Problem from Weinberger, First Course in PDE, Xerox Pub., 1965.]*

 (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$

 (b) $\displaystyle\int_{-\infty}^{\infty} \left|\frac{1 - e^{-iax}}{ix}\right|^2\,dx$

Plancherel's formula for $f \in \mathcal{L}^2$ is

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$$

By problem 19,

$$\mathcal{F}\left[\frac{H(1-|x|)}{2}\right](\omega) = \frac{1}{\omega}\sin\omega.$$

Thus from the Plancherel formula

$$\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}\,d\omega = \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty}|f(x)|^2\,dx = 2\pi\int_{-\infty}^{\infty}\left|\frac{H(1-|x|)}{2}\right|^2\,dx = \frac{\pi}{2}\int_{-1}^{1}dx = \pi.$$

Also, observing that

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \int_0^a e^{i\omega x}\,dx = \frac{1 - e^{-ia\omega}}{ix}$$

where f is a pulse from 0 to a,

$$f(x) = H(x) - H(x-a).$$

Thus from the Plancherel formula

$$\begin{aligned} \int_{-\infty}^{\infty}\left|\frac{1-e^{-ia\omega}}{i\omega}\right|^2\,d\omega &= \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty}|f(x)|^2\,dx \\ &= 2\pi\int_{-\infty}^{\infty}|H(x) - H(x-a)|^2\,dx = 2\pi\int_0^a dx = 2\pi a. \end{aligned}$$