Final practise exam (5440)

Name and Unid: ________________________________

Carefully Read The Instructions:

Instructions: This exam will last two hours and consists of 8 exercises plus one bonus exercise. Provide solutions to the exercises in the space provided or if you need extra space to work, there are two pages at the end of the test, but please idicate the exercise. All solutions MUST be sufficiently justified to receive all the credit. Illegible answers will receive deductions. Calculators, books and notes are not allowed.

Advice: If you get stuck on a exercise or a question don't panic! Move on and come back to it later.

We recall:

- the formula of a cosine series of a function ϕ defined $[0, l]$

$$A_0/2 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right),$$

  where

$$A_0 = \frac{2}{l}\int_0^l \phi(x)\mathrm{d}x$$

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)\mathrm{d}x,\ \forall n \in \mathbb{N}^*.$$

- the formula of a full Fourier series of a function ϕ defined $[-l, l]$

$$A_0/2 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$A_0 = \frac{1}{l}\int_{-l}^{l} \phi(x)\mathrm{d}x,$$

and

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{and} \quad B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \ \forall n \in \mathbb{N}^*.$$

- the formula of a the partial sum of geometric series:

$$\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}, \quad \text{for q} \neq 1.$$

Exercise 1(Classification of partial differential equations).

1) Prove that the following partial differential equation is linear:

$$x^2\,\frac{\partial^2 u}{\partial t^2}(x,t) + \sin(x)\,u(x,t) = 0.$$

2) For each of the following equations, state the order and wether if it is nonlinear, linear inhomogeneous or linear homogenous (we don't ask any justification only for this question):

$$a) \qquad \frac{\partial^5 u}{\partial x^5}(x,t) + \exp(u(x,t)) = 0,$$

$$b) \qquad \frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) + t^4 u(x,t) + \sinh(t) = 0, ,$$

$$c) \qquad \frac{\partial^3 u}{\partial x^3}(x,t) + 2x^2\,\frac{\partial u}{\partial t}(x,t) + \sin(x)\frac{\partial u}{\partial x}(x,t) + 2x^3\,u(x,t) = 0$$

3) We consider the following linear homogeneous second order partial differential equation:

$$x\,\frac{\partial^2 u}{\partial x^2}(x,y) + 2\,\frac{\partial^2 u}{\partial x \partial y}(x,y) + \, y\frac{\partial^2 u}{\partial y^2}(x,y) + 3y^3\frac{\partial u}{\partial y}(x,y) = 0\,. \qquad (1)$$

Find the regions in the xy plane where the equation (1) is elliptic, parabolic and hyperbolic. Sketch them.

Exercise 2 (Transport equation)

We consider the following transport equation

$$\frac{\partial u}{\partial x}(x,y) + \sin(x)\frac{\partial u}{\partial y}(x,y) = 0 \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{2}$$

1) Find the solutions of the transport equation (2).

2) Find the solution associated to the initial condition: $u(0,y) = y^2$.

<u>Exercise 3</u> (Fourier transform)

We recall that the Fourier transform $\hat{f}$ of a function u is given by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-i\xi x} \mathrm{d}x, \ \forall \xi \in \mathbb{R}.$$

1) Compute the Fourier tramsform of the function

$$f(x) = \mathbb{1}_{[-\pi,\pi]}(x)\, e^{-ix}, \ \forall x \in \mathbb{R}$$

where $\mathbb{1}_{[-\pi,\pi]}$ is the indicatrix function of the interval $[-\pi, \pi]$ defined by $\mathbb{1}_{[-\pi,\pi]}(x) = 1$ if $x \in [-\pi, \pi]$ and 0 if $x \in \mathbb{R} \setminus [-\pi, \pi]$.

Exercice 4 (Separation of variables of the wave equation with Neumann boundary conditions)

We consider the following wave equation:

$$\left|\begin{array}{l}\dfrac{\partial u^2}{\partial t^2}(x,t) = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial x^2}(x,t), \qquad \text{for } 0 < x < l \text{ and } t > 0, \\[2ex] \dfrac{\partial u}{\partial x}(0,t) = 0 \;\text{ and }\; \dfrac{\partial u}{\partial x}(l,t) = 0,\end{array}\right. \tag{3}$$

1) Apply the separations of variables method by looking for non zero solutions

$$u(x,t) = X(x)\,T(t)$$

of this problem. Find one boundary value problem for X (ordinary differential equation+boundary conditions at $x = 0$ and $x = l$) and one ODE for T.

2) Prove that the boundary value problem satisfied by X admits only non negative eigenvalues.

3) Find the eigenvalues λ and the eigenfunctions associated to the boundary value problem satisfied by X.

4) Solve the time ODE for λ an eigenvalue.

5) By "using the superposition principle" (no justification is asked here) express the general form of the solution.

6) We suppose that the waves equations (3) admits as initial condition

$$u(x,0) = 1 \;\text{ and }\; \frac{\partial u}{\partial t}(x,0) = 0 \qquad \text{for } 0 \le x \le L.$$

By computing the unknown coefficients which appears in the general form of the solution of the question 5, precise the expression of the solution with this particular initial condition.

<u>Exercice 5</u> (convergence of a series of functions)

Let f_n be the function

$$f_n(x) = x^{2n}, \ \forall x \in [-1,1], \ \forall n \in \mathbb{N}.$$

1) Compute $\sum_{n=0}^{N} f_n(x)$ on $(-1,1)$.

2) Does the series $\sum_{n=0}^{\infty} f_n$ converge pointwise on $(-1,1)$?

3) Does the series $\sum_{n=0}^{\infty} f_n$ converge unformly on $[-1,1]$, on $[-1/2,1/2]$?

4)Does the series $\sum_{n=0}^{\infty} f_n$ converge in the L^2 sense on $(-1/2,1/2)$?

<u>Exercice 6</u> (Fourier Series)

Let f be the function defined on $[-l, l]$ by

$$f(x) = |x| \text{ on } [-l, l].$$

1) Sketch the fucntion $f(x)$?

2) Give the pointwise limit of the Fourier series of f on $[-l, l]$? Does the Fourier series converge uniformly to f on $[-l, l]$? Does the Fourier series converge to f in the L^2 sense on $[-l, l]$?

3) Compute the Fourier coefficients of this Fourier series.

<u>Exercice 7</u> (Orthogonality of eigenfunctions)

1) We consider two real functions: X_1 and X_2 twice differentiable on $[-l, l]$ which satisfy both the periodic boundary conditions:

$$X(-l) = X(l) \quad \text{and} \quad \frac{\mathrm{d}X}{\mathrm{d}x}(-l) = \frac{\mathrm{d}X}{\mathrm{d}x}(l). \tag{4}$$

Prove that X_1 and X_2 satisfy the identity:

$$\int_{-l}^{l} -\frac{\mathrm{d}^2 X_1}{\mathrm{d}x^2}(x)\, X_2(x) + X_1(x)\frac{\mathrm{d}^2 X_2}{\mathrm{d}x^2}(x)\, \mathrm{d}x = 0\,. \tag{5}$$

(referred as the second Green's identity).

2) Use the identity (5) to prove that if X_1 and X_2 are two real eigenfucntions associated respectively to different eigenvalues λ_1 and λ_2 (assumed to be real) of the operators $-\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}$ with periodic boundary conditions (4) then:

$$(X_1, X_2) = \int_{-l}^{l} X_1(x)\, X_2(x) \mathrm{d}x = 0$$

in other words that the eigenfunctions X_1 and X_2 are orthogonal.

Exercice 8 (Energy of the heat equation) We consider the following heat equation:

$$\frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) \text{ for } t > 0 \text{ and } 0 < x < l, \tag{6}$$

with initial condtions:

$$u(x,0) = \phi(x)$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = g(t) \text{ and } u(l,t) = h(t)$$

where f, ϕ, g and h are four given functions.

1) If u_1 and u_2 are two solutions of this problem what heat equation, initial condition and boundary conditions satisfy the function $u = u_2 - u_1$?

2) By using an energy method prove that the function $u = 0$ and that the problem (6) (with its initial and boundary conditions) admits a unique solution.

<u>Exercice 9</u> (bonus exercise: inner product)

1) Let $\mathbb{R}_2[X]$ be the vector space of polynomials with real coefficients of degree at most two.

Prove that:

$$(P, Q) = P(0)Q(0) + P(1)Q(1) + P(2)Q(2), \, \forall P, Q \in \mathbb{R}_2[X]$$

define a real inner product on $\mathbb{R}_2[X]$.

2) Let H be a vector space equipped with a real inner product $(\, , \,)$ which define a L^2 norm $\|\cdot\|$ by:

$$\|X\| = (X, X)^{\frac{1}{2}}, \; , \forall X \in H \tag{7}$$

Prove the following identity:

$$\|X + Y\|^2 + \|X - Y\|^2 = 2(\|X\|^2 + \|Y\|^2), \; \forall X, Y \in H \tag{8}$$

referred as the parallelogram identity.

3) Interpret geometrically the identity (8) in the case where $H = \mathbb{R}^2$ and

$$(X, Y) = x_1 y_1 + x_2 y_2, \; \forall X = (x_1, x_2), \, Y = (y_1, y_2) \in \mathbb{R}^2.$$

(*Hint: think to a parallelogram.)*