Math 5410 § 1. **First Midterm Exam** Name: Solutions
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1. *Consider a population maodel with exponential growth and periodic harvesitng with rate constants $a > 0$ and $h > 0$ of the form*

$$x' = ax - he^{\sin t}.$$

Find the general solution. Find the Poincaré map for 2π periodic solutions. Is there a 2π-periodic solution? Is it unique? Why? What happens to a periodic solution when the harvesting constant h is increased?

This is a linear equation. Multiplying by the integrating factor e^{-at} yields

$$\left(e^{-at}x\right)' = e^{-at}(x' - ax) = -he^{-at+\sin t}$$

Integrating from 0 to t using $x(0) = x_0$,

$$e^{-ta}x(t) - x_0 = -h\int_0^t e^{-as+\sin s}\,ds$$

so the general solution is

$$x(t) = e^{at}\left(x_0 - h\int_0^t e^{-as+\sin s}\,ds\right).$$

The Poincaré Map, the time 2π map is

$$\wp(x_0) = x(2\pi) = e^{2a\pi}x_0 - he^{2a\pi}\int_0^{2\pi} e^{-as+\sin s}\,ds.$$

The system has a 2π periodic solution if there is an x_0 such that $x_0 = \wp(x_0)$, *i.e.,* the solution returns to the same value after time 2π. Thus we have to solve

$$\left(1 - e^{2a\pi}\right)x_0 = -he^{2a\pi}\int_0^{2\pi} e^{-as+\sin s}\,ds.$$

The coefficient for x_0 is nonzero since $e^{2a\pi} > 1$ so this linear equation has the unique solution

$$x_0 = \frac{he^{2a\pi}}{e^{2a\pi} - 1}\int_0^{2\pi} e^{-as+\sin s}\,ds.$$

There is a unique periodic solution. Since the integral is positive, one sees that x_0 increases as h increases. To balance a higher harvesting rate, the periodic population has to be greater.

2. *Solve the initial value system*

$$X' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} X \qquad X(0) = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

Finding the eigenvalues, the characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 8 = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4$$

Thus the eigenvalues satisfy $(\lambda+1)^2 = -4$ which implies $\lambda = -1 \pm 2i$. We only need one complex solution since its real and imaginary parts will be independent real solutions. For the eigenvalue $\lambda = -1+2i$ the eigenvector satisfies

$$0 = (A - \lambda I)V = \begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \end{pmatrix}.$$

Checking the second equation, $2 \cdot 2 + (-2-2i)(1-i) = 4 - 2 + 2i - 2i - 2 = 0$. Thus a complex solution is given by

$$\begin{aligned} Z(t) = e^{(-1+2i)t} \begin{pmatrix} 2 \\ 1-i \end{pmatrix} &= e^{-t}(\cos 2t + i \sin 2t) \begin{pmatrix} 2 \\ 1-i \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + ie^{-t} \begin{pmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \end{aligned}$$

The real and imaginary parts are independent real solutions. Thus, the general solution is given by

$$X(t) = c_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

To solve the initial value problem, we see that at time $t = 0$,

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} = X(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{7}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Thus the solution of the initial; value problem is

$$X(t) = \frac{5}{2} e^{-t} \begin{pmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} - \frac{7}{2} e^{-t} \begin{pmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} = e^{-t} \begin{pmatrix} 5\cos 2t - 7\sin 2t \\ 6\cos 2t - \sin 2t \end{pmatrix}$$

3. *For the system, determine the canonical form for this equation. Find the matrix T so that $X = TY$ puts it into canonical form. Show that by changing variables using your matrix, Y satisfies the canonical form equation.*

$$X' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} X.$$

Finding the eigenvalues, the characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

so we have a double eigenvalue $\lambda = 2, 2$. The matrix

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

has rank one, so that there is only one independent eigenvector. It follows that the canonical form is

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solving for an eigenvector we see

$$0 = (A - \lambda I)V = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using the recipe in the text, choose any independent vector, say

$$W = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = AW = \mu V + \nu W = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

As predicted $2 = \mu \neq 0$ and $2 = \nu = \lambda$. One sets

$$T = \left(V, \ \frac{1}{\mu} W \right) = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Then this matrix is the desired change of variables. To see it we show $M = T^{-1}AT$, or what is the same,

$$AT = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = TM$$

Another way to find T is to look for cyclic vectors, which is explained later in the text.

4. *Find the flows ϕ_t^X and ϕ_t^Y. Find an explicit topological conjugacy between the flows. Check that your conjugacy works.*

$$X' = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} X, \qquad Y' = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} Y.$$

Both systems decouple making each component an exponential. The flows are given by

$$\varphi_t^A\begin{pmatrix}x_1\\ \\ x_2\end{pmatrix}=\begin{pmatrix}x_1\,e^{-2t}\\ \\ x_2\,e^{t}\end{pmatrix}\qquad\qquad \varphi_t^B\begin{pmatrix}y_1\\ \\ y_2\end{pmatrix}=\begin{pmatrix}y_1\,e^{-3t}\\ \\ y_2\,e^{4t}\end{pmatrix}$$

The topological conjugacy has to stretch each x component into the corresponding y component. The conjugating homeomorphism is given by

$$h\begin{pmatrix}x_1\\ \\ x_2\end{pmatrix}=\begin{pmatrix}\operatorname{sgn}(x_1)\,|x_1|^{3/2}\\ \\ \operatorname{sgn}(x_2)\,|x_1|^{4}\end{pmatrix}$$

It remains to check that the topological conjugacy equation holds. For every t and x we have using $e^{-2y}>0$ and $e^t>0$,

$$h\circ\varphi_t^A\begin{pmatrix}x_1\\ \\ x_2\end{pmatrix}=h\begin{pmatrix}x_1\,e^{-2t}\\ \\ x_2\,e^{t}\end{pmatrix}=\begin{pmatrix}\operatorname{sgn}(x_1\,e^{-2t})\,|x_1\,e^{-2t}|^{3/2}\\ \\ \operatorname{sgn}(x_2\,e^{t})\,|x_2\,e^{t}|^{4}\end{pmatrix}=\begin{pmatrix}\operatorname{sgn}(x_1)|x_1|^{3/2}\,e^{-3t}\\ \\ \operatorname{sgn}(x_2)|x_2|^{4}\,e^{4t}\end{pmatrix},$$

$$\varphi_t^B\circ h\begin{pmatrix}x_1\\ \\ x_2\end{pmatrix}=\varphi_t^B\begin{pmatrix}\operatorname{sgn}(x_1)\,|x_1|^{3/2}\\ \\ \operatorname{sgn}(x_2)\,|x_1|^{4}\end{pmatrix}=\begin{pmatrix}\operatorname{sgn}(x_1)\,|x_1|^{3/2}\,e^{-3t}\\ \\ \operatorname{sgn}(x_2)\,|x_1|^{4}\,e^{4}t\end{pmatrix}$$

which are the same, proving the conjugation equation.

5. *Consider the family of differential equations depending on the parameter a.*

$$x'=ax+x^2+x^3$$

Find the bifurcation points. Sketch the bifurcation diagram for this family of equations. Identify the rest points on the bifurcation diagram as sources, sinks or neither. Sketch the phase lines for values of a above and below the bifurcation values.

The bifurcation diagram plots the locus of rest points in the $a-x$ plane. For the ODE

$$x'=ax+x^2+x^3=f(x;a)=x(a+x+x^2)$$

It is the locus of zeros $f(x;a)=0$, which are the two lines

$$x=0\qquad\text{or}\qquad a=-x-x^2.$$

Thus the rest points are on the x-axis and on the left-opening parabola $a=-x-x^2$ which passes through the points $(a,x)=(0,0)$ and $(0,-1)$. The maximum of a on the parabola occurs at the point $P=(\frac{1}{4},-\frac{1}{2})$.

The bifurcation points are the origin O, where the two lines cross, and at P which is extreme for the parabola. The origin is a trans critical bifurcation: two rest points come together, bounce and separate as a increases through zero. P is a fold bifurcation: two rest points come together and annihilate as a increases through $\frac{1}{4}$.

For any a the function $f(x;a)>0$ for very positive x or for very negative x. If $a>\frac{1}{4}$ then there is a zero at $x=0$ and no other zeros, so o is an unstable point. When $0<a<\frac{1}{4}$ then there are three zeros of $f(x;a)$, say $z_1<z_2<0$. $f>0$ for $z_1<x<z_2$ and for $0<x$ and

$f < 0$ for $x < z_1$ or for $z_2 < x < 0$, making z_1 and 0 unstable and z_2 stable. When $a < 0$ then there are three zeros of $f(x; a)$, say $z_1 < 0 < z_2$. $f > 0$ for $z_1 < x < 0$ and for $z_2 < x$ and $f < 0$ for $x < z_1$ or for $0 < x < z_2$, making z_1 and 0 unstable and z_2 stable. The three phase lines are shown in the figure.

The stability of the rest point is indicated in the phase diagram. The x-axis is stable for $a < 0$ and upper branch of the parabola, z_2, is stable for $0 < a < \frac{1}{4}$. The origin and P are bifurcation points. The rest of the zeros are unstable.