Math 5410 § 1. **Final Exam** Name:________________________

Treibergs December 12, 2022

1.______	/13
2.______	/13
3.______	/13
4.______	/13
5.______	/13
6.______	/13
7.______	/12
8.______	/13
Total_____	/90

This is a closed book exam except that you are allowed a "cheat sheet," a single 8.5" × 11" page with notes on both sides. Other notes, books, calculators, tablets, laptops, phones and text messaging devices are prohibited. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [90] total points. **DO ONLY SEVEN PROBLEMS** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1. [13] Find e^{tA}. [Hint: $\lambda = 2, 2, 2$]

$$A = \begin{pmatrix} 4 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

2. [13] Prove that there is a nontrivial periodic solution : $\ddot{x} + (x^2 - 1)\dot{x} + 2\dot{x}^3 + x = 0$

3. For any real $n \times n$ matrix A and $V \in \mathbf{R}^n$, consider the initial value problem in $\mathbf{R}^n$

$$X' = AX + |X|^2 V; \qquad X(0) = X_0. \tag{IVP}$$

(a) [3] Find the integral equation (IE) satisfied by solutions of the (IVP). Give the recursion formula for the Picard Iterates $U_k(t)$ for the integral equation.

(b) [6] Briefly explain why there is a $T > 0$ such that iterates $\{U_k(t)\}$ converge in the space of continuous functions $\mathcal{C}([0, T], \mathbf{R}^n)$.

(c) [4] Assuming that the $\{U_k(t)\}$ converge in $\mathcal{C}([0, T], \mathbf{R}^n)$, explain why the limit satisfies the (IE).

4. Consider the Lorenz System with $r > 0$, $\sigma = 10$ and $b = \frac{8}{3}$. Answer the following questions and justify your answers with computations and by applying theorems. Explain why the required hypotheses hold. State any theorems that you quote completely in your answers, don't just give its page or section number.

$$(L) \qquad \begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

(a) [5] Why does the solution $\varphi(t, c)$ with initial value c exists for all $t \geq 0$?

(b) [5] The system undergoes bifurcations as the parameter r is increased. Describe the bifurcations of (L) that occur at $r = 1$ and $r = r^* = 24.74$.

(c) [3] Define *attractor.* Explain why the Lorenz system at $r = 28$ has an attractor.

5. [13] Consider the system given in polar coordinates of the plane (r, θ), where $r > 0$, by

$$\begin{aligned} \dot{r} &= 1 - r \\ \dot{\theta} &= 1 \end{aligned}$$

Find the Poincaré Map for the limit cycle $C = \{r = 1\}$. Determine whether C is stable and explain what sense of stability you mean.

6. Consider a central force field system for the motion of a particle $X(t) \in \mathbf{R}^3$ given by

$$\ddot{X} = -\frac{f(r)}{r} X \tag{1}$$

where $f : (0, \infty) \to \mathbf{R}$ is continuous function that gives the dependence of force on the distance $r = \sqrt{x^2 + y^2}$.

(a) [3] Show that the angular momentum $Y = X \times \dot{X}$ is conserved. Hence the particle stays in a plane through the origin.

(b) [5] Find an energy for this system and show that it is conserved.

(c) [5] Assuming that the motion is in the $x_3 = 0$ plane, write (1) in polar coordinates $(r(t), \theta(t))$ where $X(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t), 0)$. Find Kepler's second law for this system. [Find a first integral of your $\ddot{\theta}$ equation.]

7. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [4] STATEMENT: Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a continuously differentiable such that $|F(X)| \le |X| + 1$ for all X. If $X(t)$ is a solution of $\dot{X} = F(X)$ defined for $0 \le t < T$ then $X(t)$ is bounded on $0 \le t < T$. TRUE: ◯ FALSE: ◯

   (b) [4] STATEMENT: Suppose $X(t) \in \mathbf{R}^3$ satisfies Newton's Equation $\ddot{X} = -\dfrac{X}{|X|^3}$ for all $t \in [0, \infty)$. Then the omega limit set is empty $\omega(X) = \emptyset$. TRUE: ◯ FALSE: ◯

   (c) [4] STATEMENT: Real hyperbolic 3×3 matrices are generic. TRUE: ◯ FALSE: ◯

8. Consider the predator prey system for populations $x, y \geq 0$. The equilibrium points are $(0,0)$, $(6,0)$, $(0,6)$ and $(2,2)$.

$$\begin{cases} x' = x(6 - x - 2y) \\ y' = y(2x + y - 6) \end{cases}$$

(a) [8] Find the null-clines. For each equilibrium point, determine the local behavior.

(b) [4] Sketch the global flow pattern. Be sure to indicate directions of flow in each region, stable and unstable directions at the saddles and any limit cycles or seperatrices.

(c) [1] Determine whether the species always die out, or if they can coexist.