Math 5410 § 1. **Third Midterm Exam** Name: Solutions

Treibergs Nov. 16, 2022

1. *Find the values of the parameter a where bifurcations occur. Describe the nature of the bifurcations. Sketch the phase plane for various a's.*

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 - y - a \end{cases}$$

The rest points are at $0 = \dot{x} = y$ and $0 = \dot{y} = x^2 - y - a = x^2 - a$ so that if $a < 0$ there are no rest points and if $a \ge 0$ the rest points are $(\pm\sqrt{a}, 0)$. The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 2x & -1 \end{pmatrix}$$

so $D = det(J(\pm\sqrt{a}, 0) = \mp 2\sqrt{a}$ and trace $T = -1$. It follows that $(\sqrt{a}, 0)$ is a saddle and $(-\sqrt{a}, 0)$ is a stable node. Thus the system undergoes a saddle node bifurcation as a increases through the bifurcation point $a = 0$.

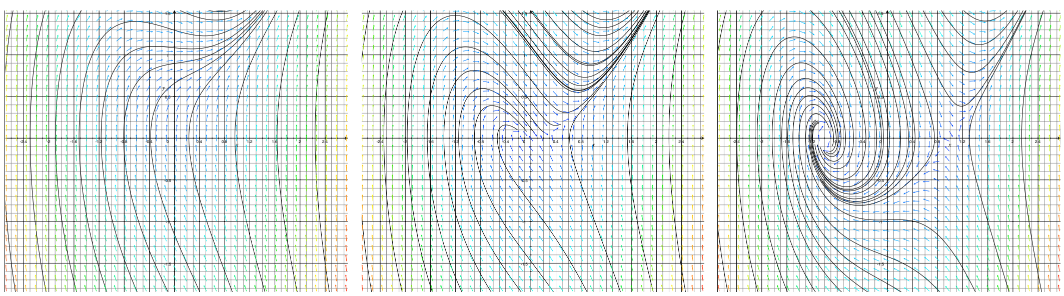

Figure 1: Phase portraits for $a = -1$, $a = 0$ and $a = 1$.

2. *Suppose $b = 0$. Find the equilibrium points. Find an energy for the system. Sketch the phase portrait showing at least four equilibrium points. Suppose the constant $b > 0$. Determine the stability of the equilibrium points. Sketch the phase portrait that includes at least four equilibrium points. For a stable equilibrium point, find a set in its basin of attraction.*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\cos(x) - by \end{cases}$$

When $b = 0$ the corresponding second order equation is

$$\ddot{x} + \cos(x) = 0.$$

Multiplying by $\dot{x}$ and integrating gives the first integral (energy)

$$E = \frac{1}{2}\dot{x}^2 + \sin(x) + 1 = \frac{1}{2}y^2 + \sin(x) + 1$$

Energy is preserved because

$$\dot{E} = y\dot{y} + \cos(x)\dot{x} = y(-\cos(x)) + \cos(x)y = 0.$$

The energy function has relative minima at the minima of sine, namely at $(-\frac{\pi}{2}+2\pi k, 0)$ and saddles at the maxima of sine, at $(\frac{\pi}{2}+2\pi k, 0)$ where k is an integer. The trajectories are level curves of $E(x,y)$ which are centers at $(-\frac{\pi}{2}+2\pi k, 0)$ and saddles at $(\frac{\pi}{2}+2\pi k, 0)$ for all integers k. The high energy orbits loop around with nonvanishing velocity.

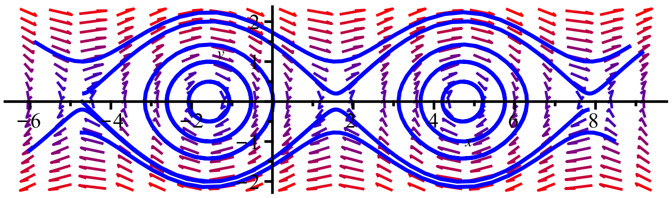

Figure 2: $b = 0$ trajectories.

When $b > 0$ then

$$\dot{E} = y\dot{y} + \cos(x)\dot{x} = y\left(-\cos(x) - \frac{b}{2}\right) + \cos(x)y = -by^2 \le 0.$$

Thus the rest points at $(-\frac{\pi}{2}+2\pi k, 0)$ are Liapunof Stable. In fact, the set $Z = \{(x,y) : \dot{E}(x,y) = 0\} = \{y = 0\}$ has no invariant subsets other than $\{(-\frac{\pi}{2}+2\pi k, 0)\}$ in the trapping region $P_{k,\delta} = \{(x,y) : E(x,y) \le \delta, |-\frac{\pi}{2}+2\pi k||\pi\}$ for fixed integer k and $\delta < 2$. Thus by the Lasalle Invariance Principle, $(-\frac{\pi}{2}+2\pi k, 0)$ is asymptotically stable and $P_{k,\delta}$ is in its basin of attraction. The points $(\frac{\pi}{2}+2\pi k, 0)$ are saddles since the determinant of the Jacobian

$$\det\left(J\left(\tfrac{\pi}{2}+2\pi k, 0\right)\right) = \begin{vmatrix} 0 & 1 \\ \sin(\frac{\pi}{2}+2\pi k) & -b \end{vmatrix} = -1.$$

All orbits eventually get trapped and spiral into one of the stable rest points.

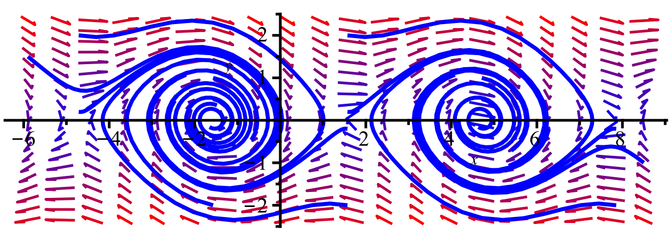

Figure 3: $b = .2$ trajectories.

3. *Determine whether the given equilibrium point for the given system is Asymptotically Stable, is Liapunov Stable but Not Asymptotically Stable, or is Not Stable. Give a brief explanation.*

   (a) $r = 1$, $\theta = \pi$ for $\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 + \cos\theta \end{cases}$

   Not Stable. Although $r = 1$ is stable for $\dot{r} = r(1-r^2)$, the other equation has $\dot{\theta} = 1 + \cos\theta$ which is positive except at $\theta = \pi$. Thus for every small neighborhood about $(r,\theta) = (1,\pi)$, starting from a point $(1, \pi+\epsilon)$ in the neighborhood for $\epsilon > 0$ small enough, the trajectory moves in the positive θ direction until it exits the neighborhood, and in fact loops around until it reenters it. Recall, that this is an example of a rest point which is attractive, but not Liapunov stable.

(b) $r = 1,\ \theta = \pi$ for $\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = \sin\theta \end{cases}$

Asymptotically Stable. The Jacobian

$$J(r, \theta) = \begin{pmatrix} 1 - 3r^2 & 0 \\ 0 & \cos\theta \end{pmatrix}, \qquad J(1, \pi) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

is a stability matrix at $(1, \pi)$ since both eigenvalues are negative. By the linearization stability theorem, $(1, \pi)$ is asymptotically stable.

(c) $X = (0, 0)$ for $\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x - y^3 \end{cases}$

Asymptotically Stable. Consider the Liapunov function

$$V(x, y) = \frac{1}{2}(x^2 + y^2).$$

We have

$$\dot{V} = x\dot{x} + y\dot{y} = x(-y - x^3) + y(x - y^3) = -(x^4 + y^4)$$

which is strictly negative for $(x, y) \neq (0, 0)$. Thus by Liapunov's stability theorem, $(0, 0)$ is asymptotically stable.

(d) [4] $X = (0, 0, 0, 0)$ for $\dot{X} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix} X.$

Stable but Not Asymptotically Stable The system decouples into two 2×2 systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{d}{dt}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

The first has eigenvalues $-1, -1$ thus zero is asymptotically stable. The second has eigenvalues $\pm\sqrt{2}\, i$, thus zero is a center. Thus for the composite system, trajectories converge to elliptical orbits and so $(0, 0, 0, 0)$ is stable but not asymptotically stable.

4. *Consider the solution $x(t)$ of the IVP for $t \geq 0$ Find estimates for $x(t)$ and $\dot{x}(t)$ in terms of t and (u_0, u_1). Does the solution exist for all $t \in [0, \infty)$? Why?*

$$\ddot{x} + (2 + \sin t)\dot{x} + x = 0, \qquad x(0) = u_0, \quad \dot{x}(0) = u_1.$$

Convert to a system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 - \sin t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A(t)\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix function satisfies

$$|A(t)X|^2 = y^2 + (-2 - \sin t)^2 x^2 \le 9x^2 + 9y^2 = 9|X|^2.$$

The integral equation for the system is

$$X(t) = U_0 + \int_0^t A(s)X(s)\,ds$$

where $U_0 = (u_0, u_1)$. Estimating this,

$$|X(t)| \le |U_0| + \int_0^t |A(s)X(s)|\,ds \le |U_0| + 3\int_0^t |X(s)|\,ds.$$

Thus, by Gronwall's Inequality, for all $t \ge 0$,

$$|X(t)| \le |U_0| e^{3t}. \tag{1}$$

That a solution for the ODE exists for all $t \ge 0$ follows from the short and long time existence theorems. Since the right side $F(t, X) = A(t)X$ is continuously differentiable for all (t, X), the ODE has a short time solution starting at any initial point U_0 and time t_0. If the maximal interval of existence for a solution $X(t)$ starting from U_0 is $0 \le t < T$, then by the long time existence theorem, $X(t)$ would leave any compact set as $t \to T-$. However, by (1), the solution stays bounded for the whole time $|X(t)| \le |U_0| e^{3T}$ for all $0 \le t < T$. Thus the solution does not cease existing at any $T > 0$, thus exists for all time.

5. *You may assume that this competing species system is defined only for* $x, y \ge 0$. *The equilibrium points are* $(0,0)$, $(0,1.5)$, $(1.5,0)$ *and* $(1,1)$. *Draw the* x *and* y *nullclines and use this information to sketch the global phase portrait of this nonlinear system. At the interior equilibrium point* $(1,1)$, *give a detailed description of the behavior of the linearized system.*

$$\begin{aligned} \dot{x} &= x(3 - 2x - y) \\ \dot{y} &= y(3 - x - 2y) \end{aligned}$$

The nullclines for $\dot{x} = 0$ are $x = 0$ and $2x + y = 3$ where the flow is vertical. For (x, y) outside the line, $\dot{x} < 0$. For $\dot{y} = 0$ the nullclines are $y = 0$ and $x + 2y = 3$ where the flow is horizontal. For points outside this line, $\dot{y} < 0$. Thus going clockwise around $(1,1)$ starting outside both slant lines the general flow direction is SW, SE, NE, NW, resp. suggesting that $(0,0)$ might be a stable node.

The Jacobian is

$$J(x,y) = \begin{pmatrix} 3 - 4x - y & -x \\ -y & 3 - x - 4y \end{pmatrix}$$

At the interior rest point

$$J(1,1) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

whose eigengalues are $\lambda_1 = -1$ and $\lambda_2 = -3$. Thus $(1,1)$ is a stable node. The slow incoming flow is in the λ_1 eigenvector $V_1 = \pm(1,-1)$ directions. The fast incoming flow is in the λ_2 eigenvector $V_2 = \pm(1,1)$ directions. Thus the incoming trajectories are tangent

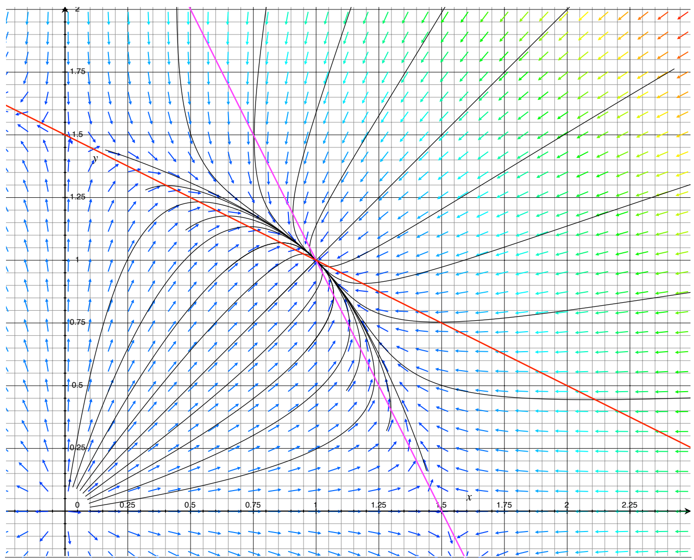

Figure 4: Competing Species System.

to V_1 at $(0,0)$. The general flow pattern may be concluded from this information. For al starting values (x_0, y_0) with both coordinates posotive tend to the equilibrium point $(1,1)$. The species can coexist.

We complete the local analyses of the remaining rest points which was not required in your answer. At the origin rest point

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

whose eigengalues are $\lambda_1 = \lambda_2 = 3$. Thus $(0,0)$ is a source. The eigenvalues are the same so the unstable space is the whole plane: every vector through the origin is tangent to an outgoing trajectory.

At the x-axis rest point

$$J(1.5,0) = \begin{pmatrix} -3 & -1.5 \\ 0 & 1.5 \end{pmatrix}$$

whose eigengalues are $\lambda_1 = -3$ and $\lambda_2 = 1.5$. Thus $(1.5,0)$ is a saddle. The incoming stable curve is in the λ_1 eigenvector $V_1 = \pm(1,0)$ direction. The outgoing unstable curve is in the λ_2 eigenvector $V_2 = (-1,3)$ direction.

At the y-axis rest point

$$J(0,1.5) = \begin{pmatrix} 1.5 & 0 \\ -1.5 & -3 \end{pmatrix}$$

whose eigengalues are $\lambda_1 = 1.5$ and $\lambda_2 = -3$. Thus $(0,1.5)$ is a saddle. The outgoing unstable curve is in the λ_1 eigenvector $V_1 = (3,-1)$ direction. The incoming stable curve is in the λ_2 eigenvector $V_2 = \pm(0,1)$ direction.