Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find $e^{tA}$ where*

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

$A = 2I + N$ where

$$N = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

The identity matrix commutes with all matrices so $(2I)N = N(2I)$. Thus we may decompose $e^{tA} = e^{t(2I+N)} = e^{2tI}e^{tN}$. Note that $e^{2tI} = e^{2t}I$,

$$N^2 = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $N^3 = 0$. Thus we may sum the exponential series

$$\begin{aligned} e^{tN} &= I + tN + \frac{t^2}{2}N^2 + \frac{t^3}{6}N^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3t & 4t + \frac{15}{2}t^2 \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Finally,

$$e^{tA} = e^{2tI}e^{tN} = e^{2t}\begin{pmatrix} 1 & 3t & 4t + \frac{15}{2}t^2 \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{pmatrix}.$$

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *For infinitely many $\omega \in \mathbf{R}$ with $\omega > 0$, the solution $(x(t), y(t))$ of the harmonic oscillator equations $\ddot{x} + x = 0$, $\ddot{y} + \omega^2 y = 0$ is not periodic.*
   True. The system of harmonic oscillators has periodic trajectories if and only if the ratio of angular velocities $\frac{\omega}{1}$ is rational. Thus the trajectories are not periodic for infinitely many irrational $\omega$.

(b) STATEMENT: *If $f : \mathbf{R} \to \mathbf{R}$ is continuous, then any short time solution of the IVP $\dot{x} = f(x)$ and $x(0) = 0$ is unique.*

FALSE. The function $f(x) = \sqrt{|x|}$ is continuous on $\mathbf{R}$, but the IVP has many solutions, $x(t) = 0$ for all $t$ is one and for each $k \ge 0$ there is another

$$x(t) = \begin{cases} \frac{1}{4}(t-k)^2, & \text{if } t > k; \\ 0, & \text{if } t \le k. \end{cases}$$

(c) STATEMENT: *The set $S$ of real $2 \times 2$ matrices that have distinct eigenvalue is open and dense in the set of real $2 \times 2$ matrices.*

TRUE. For the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the eigenvalues satisfy

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

so that by the quadratic formula

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

The eigenvalues are repeated if and only if

$$f(A) = (a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc = 0.$$

Thus $S = f^{-1}(\mathbf{R}\backslash\{0\})$ is an open set because it is the pullback of an open set under a continuous mapping. $S$ is dense because every $A \notin S$ may be approximated by matrices in $S$. Choose $A \notin S$, which means $(a-d)^2 + 4bc = 0$. Consider the approximating matrices

$$A_\epsilon = \begin{pmatrix} a+\epsilon & b \\ c & d-\epsilon \end{pmatrix}.$$

$A_\epsilon \to A$ as $\epsilon \to 0$. For these

$$f(A_\epsilon) = (a-d+2\epsilon)^2 + 4bc = (a-d)^2 + 4bc + 4(a-d)\epsilon + \epsilon^2 = 4(a-d)\epsilon + \epsilon^2$$

is nonzero for all but at most two $\epsilon$'s. Thus a sequence can be chosen $\epsilon_i \to 0$ as $i \to \infty$ such that $f(A_{\epsilon_i}) \ne 0$, thus $A_{\epsilon_i} \in S$, and $A_{\epsilon_i} \to A$ as $i \to \infty$.

3. *Solve the initial value problem using the Variation of Parameters Formula*

$$X' = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} X + \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \qquad \textit{Hint:} \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

The solution is to apply the variation of parameters formula which requires the evaluation of $e^{tA}$. By the hint, $A = PDP^{-1}$, thus

$$\begin{aligned} e^{tA} = Pe^{tD}P^{-1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} & -2e^{3t} \\ 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{3t} & 2e^{5t} - 2e^{3t} \\ 0 & e^{5t} \end{pmatrix}. \end{aligned}$$

The solution of $X' = AX + b(t)$, $X(0) = x_0$ is given by the variation of parameters formula

$$\begin{aligned} X(t) &= e^{tA}\left(x_0 + \int_0^t e^{-sA} b(s) ds\right) \\ &= \begin{pmatrix} e^{3t} & 2e^{5t} - 2e^{3t} \\ 0 & e^{5t} \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-3s} & 2e^{-5s} - 2e^{-3s} \\ 0 & e^{-5s} \end{pmatrix} \begin{pmatrix} e^{3s} \\ 0 \end{pmatrix} ds \right\} \\ &= \begin{pmatrix} e^{3t} & 2e^{5t} - 2e^{3t} \\ 0 & e^{5t} \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds \right\} \\ &= \begin{pmatrix} e^{3t} & 2e^{5t} - 2e^{3t} \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1+t \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} (1+t)e^{3t} + 4e^{5t} - 4e^{3t} \\ 2e^{5t} \end{pmatrix}. \end{aligned}$$

4. *Find the first few Picard iterates of the system. Show that they converge to a solution of the IVP.*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Start at the initial condition

$$X_0(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then iterating

$$X_{n+1}(t) = X_0 + \int_0^t F(X_n(s))\, ds$$

we get

$$\begin{aligned} X_1(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} 1 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} ds = \begin{pmatrix} 1+t \\ 1+2t \end{pmatrix} \\ X_2(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} 1+t \\ 1+2t \end{pmatrix} ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+2t \\ 2 \end{pmatrix} ds = \begin{pmatrix} 1+t+t^2 \\ 1+2t \end{pmatrix} \\ X_3(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} 1+t+t^2 \\ 1+2t \end{pmatrix} ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+2t \\ 2 \end{pmatrix} ds = \begin{pmatrix} 1+t+t^2 \\ 1+2t \end{pmatrix} \end{aligned}$$

The sequence stabilizes: $X_{n+1} = X_n$ for all $n \ge 2$. Thus the limit of the Picard iteration is the function

$$X(t) = \begin{pmatrix} 1+t+t^2 \\ 1+2t \end{pmatrix}.$$

This solves the IVP since

$$\frac{d}{dt}X(t) = \begin{pmatrix} 1+2t \\ 2 \end{pmatrix} = F\begin{pmatrix} 1+t+t^2 \\ 1+2t \end{pmatrix} = F(X(t)), \qquad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

5. *Delay Differential Equations are a type of ODE we haven't discussed, but local existence may be derived by the techniques from the class. Let $0 < \alpha < 1$. Consider the integral equation (IE). For a continuous function $y : \mathbf{R} \to \mathbf{R}$, let $J[y](t) = 1 + \int_0^t y(\alpha s)\,ds$. Let $y_0(t) = 1$ and $y_{n+1}(t) = J[y_n](t)$. The first four iterations are given. Explain briefly why $\{y_n(t)\}$ converges to a continuous function $x(t)$ satisfying (IE) for $t \in I$ where $I = [0, \frac{1}{2}]$. Why is $x(t)$ continuously differentiable? State the initial value problem satisfied by $x(t)$. What do you expect the solution of (IE) to be?*

$$\text{(IE)} \qquad x(t) = 1 + \int_0^t x(\alpha s)\,ds. \qquad \begin{cases} y_0(t) = 1, \\ y_1(t) = 1 + t, \\ y_2(t) = 1 + t + \frac{1}{2}\alpha t^2, \\ y_3(t) = 1 + t + \frac{1}{2}\alpha t^2 + \frac{1}{6}\alpha^3 t^3, \\ y_4(t) = 1 + t + \frac{1}{2}\alpha t^2 + \frac{1}{6}\alpha^3 t^3 + \frac{1}{24}\alpha^6 t^4. \end{cases}$$

The solution is found in the space of continuous real functions on $I$ with sup norm $\|x\| = \sup_{t \in I} |x(t)|$. The sequence $\{y_n\}$ is shown to be a Uniformly Cauchy Sequence, hence uniformly convergent to a continuous function $x$. This follows by showing $y_{n+1} - y_n$ decay geometrically. Uniform convergence $y_n \to x$ implies $y_{n+1} = J[y_n]$ may be taken to the limit to show $x$ satisfies (IE). Since $J$ applies to all continuous functions, there is no need to show that the $y_n$'s stay in a fixed ball. The $y_n$'s are a Uniformly Cauchy Sequence, hence bounded.

First we show that the $y_n$'s are continuous. This is done by induction. $y_0(t) = 1$ which is continuous. For the induction step, assume that $y_n$ is continuous on $I$ for some $n$. Then

$$y_{n+1}(t) = 1 + \int_0^t y_n(\alpha s)\,ds$$

is the integral of the continuous function $y_n(\alpha s)$, hence is continuously differentiable.

Second, we show that consecutive terms $\{y_n\}$ are close to each other, namely we show for every $n$ that

$$\|y_{n+1} - y_n\| \le \frac{1}{2^{n+1}}. \qquad (1)$$

By induction, the base case is from the first few iterates,

$$|y_1(t) - y_0(t)| = |1 + t - 1| = |t|.$$

Taking sup over $t \in I$ yields

$$\|y_1 - y_0\| \le \frac{1}{2}.$$

Assuming (1) for some $n \ge 0$ we have

$$\begin{aligned} |y_{n+2}(t) - y_{n+1}(t)| &= |J[y_{n+1}](t) - J[y_n](t)| \\ &= \left| 1 + \int_0^t y_{n+1}(\alpha s)\,ds - 1 - \int_0^t y_n(\alpha s)\,ds \right| \\ &\le \int_0^t |y_{n+1}(\alpha s) - y_n(\alpha s)|\,ds \\ &\le \int_0^t \|y_{n+1} - y_n\|\,ds \\ &= t\|y_{n+1} - y_n\| \\ &\le \frac{1}{2} \cdot \frac{1}{2^{n+1}} \end{aligned}$$

Taking sup over $t \in I$ yields

$$\|y_{n+2} - y_{n+1}\| \le \frac{1}{2^{n+2}}$$

and the induction step is proved.

Third we show that $\{y_n\}$ is a Uniformly Cauchy Sequence. Choose $\epsilon > 0$. Let $N \in \mathbf{R}$ satisfy $\frac{1}{2^N} = \epsilon$ . Then for every $p, q \in \mathbb{N}$ such that $p > N$ and $q > N$, we may suppose that $p > q$. If $p = q$ then $\|y_p - y_q\| = 0 < \epsilon$. If $p < q$ then we swap the roles of $p$ and $q$. Then by the telescoping sum trick,

$$\begin{aligned}
\|y_p - y_q\| &= \|(y_p - y_{p-1}) + (y_{p-1} - y_{p-2}) + \cdots + (y_{q+1} - y_q)\| \\
&\le \|y_p - y_{p-1}\| + \|y_{p-1} - y_{p-2}\| + \cdots + \|y_{q+1} - y_q\| \\
&\le \frac{1}{2^p} + \frac{1}{2^{p-1}} + \cdots + \frac{1}{2^{q+1}} \\
&= \frac{1}{2^{q+1}} \sum_{\ell=0}^{p-q-1} \frac{1}{2^\ell} \\
&= \frac{1}{2^{q+1}} \frac{1 - \frac{1}{2^{p-q}}}{1 - \frac{1}{2}} \\
&< \frac{1}{2^q} < \frac{1}{2^N} = \epsilon.
\end{aligned}$$

It follows that $y_n \to x$ uniformly to some function $x : I \to \mathbf{R}$. Since the $y_n$ are continuous, and the convergence is uniform, $x$ must also be continuous.

Fourth we show that (IE) holds for $x$. Note that $y_n(\alpha t) \to x(\alpha t)$ as $n \to \infty$ converges uniformly on $I$. But since the convergence is uniform, we may exchange limit and integral

$$\begin{aligned}
x(t) = \lim_{n\to\infty} y_{n+1}(t) &= \lim_{n\to\infty} \left(1 + \int_0^t y_n(\alpha s)\, ds\right) \\
&= 1 + \int_0^t \left(\lim_{n\to\infty} y_n(\alpha s)\right) ds = 1 + \int_0^t x(\alpha s)\, ds.
\end{aligned}$$

Fifth, $x(t)$ is continuously differentiable for $t \in I$ because in (IE),

$$x(t) = 1 + \int_0^t x(\alpha s)\, ds$$

it is the integral of a continuous function.

Sixth, it satisfies an IVP. Using the Fundamental Theorem of Calculus on (IE) and evaluating at $t = 0$,

$$\frac{dx}{dt}(t) = x(\alpha t); \qquad x(0) = 1.$$

Seventh, we see that the Picard Iterates are the partial sums of a power series that actually converges faster than the exponential series for all $t \in \mathbf{R}$. The only missing detail is what is the correct power of $\alpha$? Let us write the integer valued function $m(k)$, where according to the first few iterates takes the values $m(0) = 0$, $m(1) = 0$, $m(2) = 1$, $m(3) = 3$ and $m(4) = 6$. We claim

$$y_n(t) = \sum_{k=0}^{n} \frac{\alpha^{m(k)}}{k!} t^k.$$

Arguing by induction, the base case is true because it agrees with the first few listed $y_n$'s. For the induction case, assume this is true for some $n \in \mathbb{N}$. Substituting into

$$y_{n+1}(t) = 1 + \int_0^t y_n(\alpha s)\, ds$$

we find that

$$y_{n+1}(t) = 1 + \int_0^t \left( \sum_{\ell=0}^{n} \frac{\alpha^{m(\ell)}}{\ell!} (\alpha s)^\ell \right) ds = 1 + \sum_{\ell=0}^{n} \frac{\alpha^{m(\ell)}\alpha^\ell}{(\ell+1)\ell!} t^{\ell+1} = \sum_{k=0}^{n+1} \frac{\alpha^{m(k-1)}\alpha^{k-1}}{k!} t^k$$

proving the claim. The $k = n+1$ term says

$$m(n+1) = m(n) + n.$$

Each new term is gotten by adding $n$ to the old term. Thus we see that for $n \geq 1$,

$$m(n) = \sum_{\ell=0}^{n-1} \ell = \frac{1}{2} n(n-1),$$

which agrees with the first few $m(n)$'s. It follows that the solution of the IVP is

$$x(t) = \sum_{k=0}^{\infty} \frac{\alpha^{\frac{1}{2}k(k-1)}}{k!} t^k.$$

Note that this power series is majorized by the exponential series, thus converges for all $t$. Or, we see it by applying the ratio test

$$\lim_{n\to\infty} \frac{\frac{\alpha^{\frac{1}{2}(n+1)n}}{(n+1)!} t^{n+1}}{\frac{\alpha^{\frac{1}{2}n(n-1)}}{n!} t^n} = \lim_{n\to\infty} \frac{\alpha^n t}{n+1} = 0.$$