Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find the general solution of*

$$X' = \begin{pmatrix} 3 & 2 & 1 & 0 \\ -2 & 3 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix} X.$$

The matrix A is the canonical form for repeated eigenvalues $\lambda = 3 \pm 2i, 3 \pm 2i$. Write $A = B + N$ where the matrices are two by two blocks

$$B = \begin{pmatrix} B_2 & 0 \\ 0 & B_2 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the matrices commute $BN = NB$, we have $e^{tA} = e^{tB} e^{tN} =$

$$= e^{3t} \begin{pmatrix} \cos 2t & \sin 2t & 0 & 0 \\ -\sin 2t & \cos 2t & 0 & 0 \\ 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= e^{3t} \begin{pmatrix} \cos 2t & \sin 2t & t\cos 2t & t\sin 2t \\ -\sin 2t & \cos 2t & -t\sin 2t & t\cos 2t \\ 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & \cos 2t \end{pmatrix}$$

The general solution is thus $X(t) = e^{tA} c$ or

$$X(t) = c_1 \begin{pmatrix} e^{3t}\cos 2t \\ -e^{3t}\sin 2t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^{3t}\sin 2t \\ e^{3t}\cos 2t \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} te^{3t}\cos 2t \\ -te^{3t}\sin 2t \\ e^{3t}\cos 2t \\ -e^{3t}\sin 2t \end{pmatrix} + c_4 \begin{pmatrix} te^{3t}\sin 2t \\ te^{3t}\cos 2t \\ e^{3t}\sin 2t \\ e^{3t}\cos 2t \end{pmatrix}$$

2. *Find a basis for both* $\operatorname{Ker} T$ *and* $\operatorname{Range} T$ *where* T *is the matrix*

$$T = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & -2 \\ 3 & 6 & 2 & 2 \end{pmatrix}$$

Do row operations. The matrix T becomes

$$\to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

There are two pivots and two free variables so each space has dimension two. Setting the free variables gives in $Rx = 0$ gives the basis $\mathcal{B}_K$ of $\operatorname{Ker} T$.

$$\mathcal{B}_K = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

On the other hand, the columns one and three of the original matrix corresponding to the pivot variables gives a basis $\mathcal{B}_R$ for the range.

$$\mathcal{B}_R = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

3. *Find a matrix T such that puts A into its canonical form. Find e^{tA}.*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic equation is

$$0 = (2-\lambda)^2(1-\lambda)$$

so the eigenvalues are $\lambda = 2, 2, 1$. We find a chain of generalized eigenvectors of length two for $\lambda = 2$. Namely, if $\lambda = 2$ then

$$0 = (A - \lambda I)V_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \qquad V_1 = (A - \lambda I)V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 1$ we have an eigenvector.

$$0 = (A - \lambda I)V_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

The matrix T is composed of the generalized eigenvectors. We check $AT = TC$ where C is the canonical form

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then since $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = B + N$ and $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ we get

$$e^{t(B+N)} = e^{tB}e^{tN} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

we have since $A = TCT^{-1}$ that $e^{tA} = e^{tTCT^{-1}} = Te^{tC}T^{-1}$ so

$$e^{tA} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} & e^t - e^{2t} + te^{2t} \\ 0 & e^{2t} & e^{2t} - e^t \\ 0 & 0 & e^t \end{pmatrix}.$$

4. it Determine which of the following properties of real 3×3 matrices are generic. Give a brief reason.

   (a) PROPERTY: *A is not a diagonal matrix.*
   GENERIC. If A is non-diagonal then some entry $a_{ij} \neq 0$ where $i \neq j$. If $\|A - B\|$ is sufficiently small then $B_{ij} \neq 0$ so non diagonal matrices are open. Similarly if A is any matrix, then one can choose arbitrarily small ϵ to make $A_{1,2} + \epsilon \neq 0$. Thus every A is arbitrarilyy close to a non-diagonal matrix, thus non-diagonal matrices are dense.

   (b) PROPERTY: *All solutions of $X' = AX$ tend to zero as $t \to \infty$.*
   NOT GENERIC. Solutions tend to zero if and only if $\Re\mathrm{e}\,\lambda < 0$ for all eigenvalues of A. However, such matrices are not dense. If A is a matrix that has an eigenvalue with $\Re\mathrm{e}\,\lambda = \alpha > 0$, because the eigenvalues depend continuously on the matrices, all matrices sufficiently close to A will have an eigenvalue with real parts close to α so positive.

   (c) PROPERTY: *A is diagonalizable: there is a possibly complex change of coordinates T for which $T^{-1}AT = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ where some λ_i may be complex.*

   NOT-GENERIC. The diagonalizable matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is arbitrarily close to the non-diagonalizable matrices $\begin{pmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $\epsilon \neq 0$ but arbitrarily small. Thus the diagonalizable matrices are not open. However, they are dense because matrices with distinct eigenvalues are dense and are diagonalizable.

5. Let $c \in \mathbf{R}$. Consider the initial value problem. Find the first three Picard Iterates. Guess the limit of your iterates. Check that your guess is correct by solving the IVP.

$$\begin{aligned} x' &= 1 & \qquad x(0) &= 0 \\ y' &= x - y & \qquad y(0) &= 0 \end{aligned}$$

The Picard Iterates are the following sequence of approximations.

$$\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = X_0(t) = x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} x_{k+1}(t) \\ y_{k+1}(t) \end{pmatrix} = X_{k+1}(t) = x_0 + \int_0^t f(X_k(s))\,ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ x_k(s) - y_k(s) \end{pmatrix} ds$$

Computing,

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 0 - 0 \end{pmatrix} ds = \begin{pmatrix} t \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} x_2(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ s - 0 \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix},$$

$$\begin{pmatrix} x_3(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ s - \frac{1}{2}s^2 \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{1}{2}t^2 - \frac{1}{3!}t^3 \end{pmatrix},$$

From the pattern we guess X_k converges to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ e^{-t} + t - 1 \end{pmatrix}$$

To check, the system decouples:

$$x' = 1; \qquad x(0) = 0$$

so $x(t) = t$. Then

$$y' + y = t, \qquad y(0) = 0$$

so multiplying by the integrating factor

$$\left(e^t y\right)' = te^t$$

so

$$e^t y(t) - y(0) = \int_0^t se^s \, ds = te^t - e^t + 1.$$

Hence

$$y(t) = e^{-t} + t - 1.$$