Homework for Math 5210 §1, Spring 2022

A. Treibergs, Instructor

April 14, 2022

Our text is by Robert S. Strichartz, *The Way of of Analysis,* revised edition, Jones and Bartlett Publishers, Sudbury (2000). Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on April 21, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Tuesday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before 4 pm Friday afternoon will be considered to be on time.

Please hand in problems A1 – A2 on Friday, January 14.

A1. Countable/Uncountable. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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13[2, 5]
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Copy of these problems.

2. Is the set of all finite subsets of $\mathbb{N}$ countable or uncountable? Give a proof of your assertion.
5. Let $A_1, A_2, A_3, \ldots$ be countably infinite sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined as the set of all sequences $(a_1, a_2, a_3, \ldots)$ where a_k is an element of A_k, Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, \ldots$ has at least two elements.

A2. Bijection. Exhibit an explicit bijection between $(0, 1)$ and $[0, 1]$. (*Cf.* R. Gariepy & W. Ziemer, *Modern Real Analysis,* PWS Publishing, Boston, 1995, p. 33.)

Please hand in problems B on Friday, January 21.

B. Construction of the Reals. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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48[1, 3, 5, 8ad, 9c]
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Please hand in problems C on Friday, January 27.

C. Completeness. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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54[2, 3, 9]
84[1, 5, 9]
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Please hand in problems D on Friday, February 4.

D. Topology of $\mathbb{R}^1$. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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98[1, 2, 7, 8, 10, 11]
384[2, 3]    Problem 10 is postponed.
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Please hand in problems E1 on Friday, February 11.

E1. Metric Spaces. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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Problems from p. 274 are postponed.
366[2, 10, 15]
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Please hand in problems F on Friday, February 18.

F. Compact Metric Spaces. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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385[8, 11, 14, 16]
409[1, 20]
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Please hand in problems G1–G5 on Friday, February 25.

G1. Connectedness and Contractions. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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409[7, 16, 17, 26]
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G2. Fredholm Integal Equation. Let $I = [a, b]$, $g : I \to \mathbf{R}^n$ and $K : I \times I \to M_{n\times n}(\mathbf{R})$ be continuous functions, where $M_{n\times n}(\mathbf{R})$ are real $n \times n$ matrices. Find $\lambda_0 > 0$ so that if $|\lambda| \le \lambda_0$, then there is a unique continuous function $x : I \to \mathbf{R}^n$ that solves the Fredholm Integral Equation. [Haaser & Sullivan, *Real Analysis,* p. 104.]

$$x(t) = \lambda \int_a^b K(t, s)x(s)\, ds + g(t).$$

G3. Hausdorff Metric. Find $d_A(B)$, $d_B(A)$ and $h(A, B)$, the Hausdorff distance for the given A and B.

a. $A = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;

b. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;

c. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 2\}$;

G4. Enlarged Sets. Let $A, B, C, D \in \mathcal{K}(\mathbf{R}^n)$.

a. For $\epsilon > 0$ show $h(A_\epsilon, B_\epsilon) \le h(A, B)$.

b. Let $A \boxplus B = \{a + b : a \in A,\ b \in B\}$ be the Minkowski sum. Show $h(A \boxplus B, C \boxplus D) \le h(A, C) + h(B, D)$.
[Hadwiger, *Vorlesungen Über Inhalt, Oberfläche und Isoperimetrie,* p. 152.]

G5. Decreasing Sequence. Let $K_n \in \mathcal{K}(\mathbf{R}^n)$ such that $K_n \supset K_{n+1}$ for all n. Show that in $(\mathcal{K}(\mathbf{R}^n), h)$,

$$\lim_{n\to\infty} K_n = K_\infty \qquad \text{where} \qquad K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

[D. Burago, Y. Burago & S. Ivanov, *A Course in Metric Geometry.* p. 253.]

Please hand in problems H on Friday, March 4.

H. Weierstrass Approximation Theorem. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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307[2. 3, 4, 10]
314[4, 6]
384[10]
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Please hand in problems I1–I3 on Friday, March 18.

I1. Stone Weierstrass. If X and Y are compact metric spaces, show that the subspace of $C(X \times Y)$ spanned by functions of the form $f(x, y) = g(x)h(y)$ with $g \in C(X)$ and $h \in C(Y)$ is dense in $C(X \times Y)$.

I2. Borel Sets. If $A \subset \mathbf{R}$ and $\epsilon > 0$ there is an open set U such that $A \subset U$ and $m^*U \leq m^*A + \epsilon$ and a $\mathcal{G}_\delta$ set G such that $A \subset G$ and $m^*A = m^*G$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 56.]

I3. Outer Measure. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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409[12, 13, 18]
641[5, 6, 10, 16]
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Please hand in problems J1 – J4 on Friday, March 25.

J1. G-delta. Show that every closed set $C \subset \mathbf{R}$ is a $\mathcal{G}_\delta$. Does it follow at once that every open set is an $\mathcal{F}_\sigma$? [C. Pugh, *Real Mathematical Analysis*, 2nd. ed., Springer, 2010, p. 450.]

J2. Measure Zero. Let $A \subset [0, 1]$ have outer measure zero $\mathrm{m}^*(A) = 0$. Show that every subset $B \subset A$ is Lebesgue measurable.

J3. Counting Measure Let $\mathcal{P}(\mathbf{R})$ be the set of all subsets of $\mathbf{R}$. Define the *counting measure*, $\sharp : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ where $\sharp(S)$, is the number of points of S. Prove that $\sharp$ is an abstract outer measure and that all $S \subset \mathbf{R}$ are measurable. [C. Pugh, *Real Mathematical Analysis*, 2nd. ed., Springer, 2010, p. 450.]

J4. Density of Full Measure Sets. If $A \subset [0, 1]$ and $\mathrm{m}^*(A) = 0$ then A^c is dense in $[0, 1]$. [N. Carothers, *Real Analysis*, 2nd. ed., Cambridge U. Press., 2000, p. 271.]

J5. Local Saturatation.. If $A \subset \mathbf{R}$ has $\mathrm{m}^*(A) > 0$, then given $\alpha \in (0, 1)$ there is an open interval I so that $\mathrm{m}^*(A \cap I) > \alpha\, \mathrm{m}^*(I)$. [N. Carothers, *Real Analysis*, 2nd. ed., Cambridge U. Press., 2000, p. 273.]

Please hand in problems K1–K5 on Friday, April 1.

K1. Products of Measurable Functions. a. Let f be an extended real-valued function with measurable domain D and let $D_1 = \{x \in D : f(x) = \infty\}$ and $D_2 = \{x \in D : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \backslash (D_1 \cup D_2)$ is measurable.

b. Prove that the product of two measurable extended real-valued functions is measurable.

c. If f and g are measurable extended real-valued functions with measurable domains and α a fixed number, then $f + g$ is measurable provided that we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

d. Let f and g be measurable extended real-valued functions which are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

[H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 69[22].]

K2. Composition. Show that if f is a measurable real valued function and g a continuous function defined on $(-\infty, \infty)$ then $g \circ f$ is measurable. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 70[25].]

K3. Measurable Functions. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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668[2, 7, 10, 14]
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Please hand in problems L1 on Friday, April 8.

L1. Dominated Convergence Theorem. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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669[11, 16]
681[2, 3, 4, 5, 7, 10]
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Please hand in problems M1 on Friday, April 8.

M1. Measurable Functions. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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559[6]
682[8]
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M2. Indefinite Integral. Let f be a nonnegative integrable function. Show that the function $F(x)$ defined by

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

is continuous using the Monotone Convergence Theorem. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[5].]

M3. Fatou's Lemma and MCT. Show that we can have strict inequality in Fatou's Lemma. Hint: consider the sequence $\{f_n\}$ where $f_n(x) = 1$ if $n \le x \le n+1$ and $f_n(x) = 0$ otherwise.

Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions. Hint: consider the sequence $\{g_n\}$ where $g_n(x) = 0$ if $x < n$ with $g_n(x) = 1$ otherwise. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[7].]

M4. Convergence on Subsets. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ *a.e.* and suppose that $\int f_n \to \int f < \infty$, Show that for each measurable set E we have $\int_E f_n \to \int_E f$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[9].]

M5. $\mathcal{L}^1$ Convergence. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ *a.e.* with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 90[14].]

M6. Riemann-Lebesgue Lemma. Let f be an integrable function on $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx\, dx = 0.$$

Hint: first show that for every $\varepsilon > 0$ there is a step function ψ such that

$$\int_{-\infty}^{\infty} |f - \psi| < \varepsilon.$$

[H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 90[16].]

Please hand in problems N1 on Friday, Apr. 22.

N. Fubini Theorem. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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704[7, 8, 9, 10]
720[1, 2, 3, 7]
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The FINAL EXAM is Mon., May 2 at 1:00 - 3:00 PM in the usual classroom, JWB 335.