Math 5210 § 1. **First Midterm Exam** Name: Solutions

Treibergs February 8, 2017

1. *Some real numbers $x \in (0,1]$ do not have unique decimal expansions, for example*

$$.375000\ldots = \frac{3}{8} = .374999\ldots.$$

Determine whether the set of those real numbers in $(0,1]$ whose decimal expansions are not unique is countable and prove your result.

We shall show that the numbers without unique decimal expansions is COUNTABLE. Denote the numbers without unique decimal expansion by

$$\mathcal{S} = \{x \in (0,1] : x \text{ does not have a unique decimal expansion}\}$$

$x \in S$ means that x has a terminating decimal expansion

$$\mathcal{S} = \{x = (.a_1 a_2 \ldots) \in (0,1] : a_i \in \{0,1,2,\ldots,9\} \text{ and } (\exists N \in \mathbb{N})(\forall i \ge N)\ a_i = 0.\}$$

Let the decimal fractions with n digits be denoted

$$\mathcal{S}_n = \{x = (.a_1 a_2 \ldots) \in (0,1] : a_i \in \{0,1,2,\ldots,9\} \text{ and } (\forall i > n)\ a_i = 0.\}.$$

We have $\mathcal{S} \subset \cup_{n=1}^{\infty} \mathcal{S}_n$. Equality doesn't hold because we have double-counted fractions that end in zero. Each of the $\mathcal{S}_n$ is finite with cardinality $m(n) = 10^n$. Now a countable union of finite sets is countable. This is most easily seen by a diagonal enumeration. If $\mathcal{S}_n = \{x_{1,1}, x_{1,2}, \ldots, x_{1,m(n)}\}$ then we may enumerate the union as

$$x_{1,1},\ x_{2,1},\ x_{1,2},\ x_{3,1},\ x_{2,2},\ x_{1,3},\ \ldots$$

omitting the entries beyond the last entry of each row.

2. *Let $(X, \|\bullet\|)$ be a nontrivial real normed linear space and $\mathcal{G} = \{x \in X : \|x\| > 3\}$. Define: $\mathcal{G}$ is open. Show that $\mathcal{G}$ is open. Define: x is a limit point (same as cluster point) of $\mathcal{G}$. Determine the limit points of $\mathcal{G}$ and prove your result.*

A subset $E \subset X$ is open if for all $y \in E$ there is $r > 0$ so that $B_r(y) \subset E$. Here, the open ball is given by

$$B_r(y) = \{x \in X : \|x - y\| < r\}.$$

To see that $\mathcal{G}$ given here is open, choose $g \in \mathcal{G}$. Let $r = \|g\| - 3 > 0$. Then $B_r(g) \subset \mathcal{G}$. To see it, choose $z \in B_r(g)$. Then by the triangle inequality,

$$\|z\| = \|z - 0\| \ge \|g - 0\| - \|g - z\| > \|g\| - r = \|g\| - (\|g\| - 3) = 3.$$

Hence $z \in \mathcal{G}$ and so $\mathcal{G}$ is open.

A limit point x of $E \subset X$ is such that for all $r > 0$ there is $y \in \mathcal{G}$ such that $0 < \|y - g\| < r$.

Claim: the set of limit points $\mathcal{G}' = E$ where $E = \{x \in X : \|x\| \ge 3\}$. To see it, choose $g \in E$ and $r > 0$ to show $E \subset \mathcal{G}'$. Let $\alpha = 1 + r/2\|g\|$ and $y = \alpha g$. Since $g \ne 0$ and $\alpha > 1$ we have $g \ne y$. Also, since $\|g\| \ge 3$ we have $\|\alpha g\| = |\alpha|\|g\| > 1 \cdot 3 = 3$ so $y \in \mathcal{G}$. Finally,

$$\|y - g\| = \|\alpha g - g\| = |\alpha - 1|\|g\| = \frac{r}{2\|g\|} \cdot \|g\| = \frac{r}{2} < r$$

so $y \in B_r(g)$. To see $\mathcal{G} \subset E$, suppose $g \notin E$ or $\|g\| < 3$. Then $B_r(g) \cap \mathcal{G} = \emptyset$ if $r = 3 - \|g\|$. Thus g cannot be a limit point, $g \notin \mathcal{G}'$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample. In all problems, let (X, d) be a metric space.*

   (a) *If $K \subset X$ is closed and bounded, then K is compact.*

   FALSE. The closed ball

   $$A = \{f \in \mathcal{C}(\mathbb{R}) : \|f\| \le 1\}$$

   is closed and bounded in $\mathcal{C}(\mathbb{R})$. However A contains the subsequence $\{f_n\}$ where

   $$f_n(x) = \begin{cases} (x-n)(x-n-1), & \text{if } n \le x \le n+1; \\ 0, & \text{otherwise.} \end{cases}$$

   that has the property that if $i \ne j$ then $\|f_i - f_j\| = \frac{1}{4}$. Thus no subsequence can be Cauchy so will not converge. So A is not sequentially compact.

   (b) *Suppose $K \subset X$ is compact and $\delta > 0$. Then there are finitely many points $\{k_1, \ldots, k_n\} \subset K$ such that every point of K is within δ of one of the k_n's: $(\forall x \in K)(\exists \ell \in \{1, \ldots, n\})(d(x, k_\ell) < \delta)$.*

   TRUE. Consider the open cover of K given by the balls $\{B_\delta(k) : k \in K\}$. By the Heine-Borel Property, there is a finite subcollection $\{k_1, \ldots, k_n\} \subset K$ such that

   $$K \subset \bigcup_{i=1}^{n} B_\delta(k_i)$$

   Thus any $k \in K$ is in one of the balls $k \in B_\delta(k_j)$ for some $1 \le j \le n$. Thus $d(k, k_j) < \delta$.

   (c) *Let $\{x_n\} \subset K$ be a sequence in a compact subset $K \subset X$ and let $f : X \to \mathbb{R}$ be a continuous function. Then the real sequence $\{f(x_n)\}$ has a convergent subsequence.*

   TRUE. Since K is sequentially compact, there is a convergent subsequence $x_{n_j} \to x_\infty \in K$ as $j \to \infty$. But since f is continuous, then $f(x_{n_j}) \to f(x_\infty)$ as $j \to \infty$.

4. *The real numbers were defined to be equivalence classes $\mathcal{R} = \mathcal{C}/\sim$, where $\mathcal{C}$ is the set of Cauchy Sequences of rational numbers, and where two sequences are equivalent, $(a_i) \sim (b_i)$, if for every positive rational number ϵ, there is $N \in \mathbb{N}$ so that*

   $$|a_i - b_i| < \epsilon \qquad \text{whenever } i \ge N.$$

   If $[(a_i)], [(b_i)] \in \mathcal{R}$, define $[(a_i)] < [(b_i)]$. Assuming that the rationals are an Archimedean ordered field, show that for every class $[(a_i)] > 0^*$, there is a natural number n such that $1^* < [(a_i)]n^*$, where $q^* \in \mathcal{R}$ means the rational number q viewed as a real number.

   $[(a_i)] < [(b_i)]$ means there is a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that

   $$a_i + \epsilon < b_i \qquad \text{whenever } i \ge N.$$

   Supposing that $0^* = [(0, 0, 0, \ldots)] < [(a_i)]$, there is a rational $\epsilon > 0$ and an $N \in \mathbb{N}$ so that

   $$\epsilon < a_i \qquad \text{whenever } i \ge N.$$

   By the Archimedean Property of the rationals, there is an $n \in \mathbb{N}$ such that $n > 2/\epsilon$. Thus for $\epsilon' = 1$ we have

   $$na_i > n\epsilon > 2 = \epsilon' + 1 \qquad \text{whenever } i \ge N.$$

   Hence

   $$n^*[(a_i)] = [(na_i)] > [(1, 1, 1, \ldots)] = 1^*.$$

5. *Let $\mathcal{F} \subset \mathcal{C}(X)$ be a family of continuous functions on the compact metric space X and $\| \bullet \|_{\text{sup}}$ be the sup-norm. Define: $\mathcal{F}$ is uniformly equicontinuous. Consider the sequence $\{f_n\} \subset \mathcal{C}([0,1])$. Suppose that the derivative $f_n'(x)$ exists for all n and all $x \in [0,1]$ and that for some constant $M < \infty$ the functions satisfy*

$$\|f_n\|_{\text{sup}} + \|f_n'\|_{\text{sup}} \le M \qquad \text{for all } n. \tag{1}$$

Does there exist a convergent subsequence? (You may use theorems proved in class.)

The family $\mathcal{F}$ is uniformly equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \text{whenever } f \in \mathcal{F} \text{ and } x, y \in X \text{ such that } d(x,y) < \delta.$$

The inequality (1) says $|f_n'(x)| \le M$ for every $n \in \mathbb{N}$ and every $x \in [0,1]$. For any n and $x, y \in [0,1]$, by the mean value theorem there is a c between x and y such that

$$|f_n(x) - f_n(y)| = |f'(c)(x-y)| \le M|x-y|$$

Hence $\{f_n\}$ are uniformly M-Lipschitz. It follows that $\{f_n\}$ is uniformly equicontinuous. Indeed, for any $\varepsilon > 0$ let $\delta = \varepsilon/(1+M)$. Then for any $n \in \mathbb{N}$ and any $x, y \in [0,1]$ such that $|x-y| < \delta$, we have

$$|f_n(x) - f_n(y)| \le M|x-y| < M\delta = M \cdot \frac{\varepsilon}{1+M} < \varepsilon,$$

showing $\{f_n\}$ is uniformly equicontinuous. By the (corollary of the) Arzela-Ascoli theorem, there is a subsequence f_{n_j} which converges uniformly on $[0,1]$ to a continuous function as $j \to \infty$.