Math 5010 § 1. **Final Exam** Name: Sample
Treibergs $\sigma\tau$ May 27, 2009

Questions from Mohammud Foondun's 5010 final exam from Fall 2008.

1. You roll two dice. What is the probability of the events:
   (a) They show the same.
   (b) Their sum is seven.

2. An event E is independent of itself. Show that its probability is either 0 or 1.

3. Show that for any real number x and integer j,

$$\binom{x}{j-1} + \binom{x}{j} = \binom{x+1}{j}.$$

4. Let X be uniformly distributed on $\{1, 2, \ldots, n\}$ and let B be the event that $a \le X \le b$ where $1 \le a < b \le n$. Find the mass function of X given B.

5. Let X be exponentially distributed with parameter λ. Show that

$$\mathbf{P}(X > s + t \mid X > s) = P(X > t).$$

   Recall that the probability density $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$.

Questions from Stewart Ethier's 5010 final exam given Dec. 5, 2007.

1. A pizza parlor lists 12 ingredients that can be ordered on pizzas (sausage, pepperoni, green peppers, etc.). How many types of pizzas can be ordered with ...
   (a) 3 ingredients?
   (b) any number of ingredients (0 ingredients = plain pizza; 12 ingredients = pizza with the works)? Simplify.
   (c) 3 ingredients on one half and 3 entirely different ingredients on the other half?
   (d) 3 ingredients on one half and 3 on the other half, but not the exact same 3 ingredients on each half?

   Note: There are no probabilities in this problem.

2. Find the probability that a 13-card bridge hand will contain exactly 1 card of some denomination (i.e., exactly 1 ace or exactly 1 two or ...)

   [Hint: Apply the inclusion-exclusion law.]

3. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, or weak are 0.7, 0.2, and 0.1, respectively.
   (a) How certain is she that she will receive the new job offer?
   (b) Given that she does receive the job offer, how likely should she feel that she received a strong recommendation; a moderate recommendation; a weak recommendation?

(c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation; a moderate recommendation; a weak recommendation?

4. Suppose two teams, A and B, play a series of games that ends when one of them has won 3 games. Suppose that each game played is, independently, won by Team A with probability p.

   (a) Find the probability mass function of the number X of games that are played. (And check that it is a pmf.)

   (b) Find $\mathbf{E}(X)$.

   (c) Show that $\mathbf{E}(X)$ is maximized when $p = 1/2$.

5. Let X be a uniform random variable on $(0, 1)$. Let $Y = \dfrac{X}{1-X}$.

   (a) Find $\mathbf{P}(Y < 1/2)$.

   (b) Find the probability density function of Y .

6. Let X and Y have joint probability density function $f(x, y) = 8xy$ for $0 < x < y < 1$ and zero otherwise. Note that the region where $f > 0$ is triangular.

   (a) Are X and Y independent? (Half credit for correct answer, half credit for justification of answer.)

   (b) Find $\mathbf{P}(X/Y > 1/2$. (Before integrating, draw a picture.)

7. A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of two people of the opposite sex. [Hint: Use indicator random variables.]

8. When a dart is thrown at a circular target of radius 1 foot, assume that the distance X of the dart to the center of the target (in feet) is a random variable with probability density function $f(x) = 2x$, $0 \leq x \leq 1$. Now suppose 100 darts are thrown independently. Use the central limit theorem to approximate the probability that the average distance to the center is at most 7.5 inches (recall 12 inches = 1 foot). Express your answer in terms of $\Phi(z)$, the standard normal cdf.

Some Extra Sample Problems from the Last Third of the Class.

1. Let X_n be a sequence of independent exponential variables with parameter λ for all n. Show that the sample means converge in probability as $n \to \infty$,

$$\frac{1}{n}(X_1 + \cdots + X_n) \xrightarrow{P} \frac{1}{\lambda}.$$

2. Let P and Q be two points chosen independently and uniformly on $(0, 1)$. Let D denote the distance between the two points. Find the probability density function $f_D(d)$.

3. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in months) by

$$f_X(x) = \begin{cases} Cxe^{-x/2}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

   (a) Find C to make $f_X(x)$ a pdf.

   (b) What is the probability that the system functions for at least five months?

   (c) Find the moment generating function $M_X(t)$. For what t is it defined?

   (d) Find $\mathbf{E}(X)$ and $\mathbf{Var}(X)$.

4. Suppose $X_1, X_2, X_3, \ldots$ is a sequence of independent random variables all exponentially distributed with parameter λ. Let

$$Y_n = \frac{S_n - \mathbf{E}(S_n)}{\sqrt{\mathbf{Var}(S_n)}}$$

   where $S_n = X_1 + X_2 + \cdots + X_n$. Show that

$$Y_n \xrightarrow{D} Z \qquad \text{as} \qquad n \to \infty$$

   converges in distribution, where $Z \sim N(0,1)$ is the standard normal variable. (Don't quote CLT.)

5. Suppose X and Y are independent standard normal variables. Find the pdf for $Z = X^2 + Y^2$.

6. Suppose X and Y are independent standard normal variables. Find $\mathbf{E}(\max(|X|, |Y|))$.

7. Let C be a circle of radius R. Find the average length of a random chord. This problem has many possible answers, and that's why it's called **Bertrand's Paradox**. Of course what "random chord" means has not been specified and different notions yield different results.

   (a) Suppose that we pick two endpoints of a chord at random on the circumference of C independently and uniformly. Then what is the expected chord length?

   (b) Let's suppose we pick a random line according to the kinematic density, the one that is invariant under Euclidean motions of the plane. That means, if h is the perpendicular distance of the line from the origin and θ is its direction, then (h, θ) is uniform in $[0, R] \times [0, 2\pi)$. What is the expected chord length now?

8. Suppose that a lighthouse is on an island a distance of A miles from a straight shoreline, that O is the closest point on the shoreline to the lighthouse and that its beacon is rotating at a constant velocity. Let Q be the point on shore where the light is pointing and X the signed distance from O. Given that the beacon is pointing toward the shoreline at a random instant, what is the probability density function of the distance X?

9. Let X_n have negative binomial distribution with parameters n and $p = 1 - q$. Show using generating functions that if $n \to \infty$ in such a way that $\lambda = nq$ remains constant, then

$$\lim_{n \to \infty} \mathbf{P}(X_n = k) = \frac{e^{-\lambda}\,\lambda^k}{k!}.$$

10. Let $X_1, X_2, \ldots, X_n$ be independent random variables uniformly distributed on $[0, 1]$. Let $Y = \max\{X_1, X_2, \ldots, X_n\}$. Find the cumulative distribution function, density, expectation and variance of Y. What is the probability that $Y > 1 - \frac{1}{n}$? What is the limiting probability as $n \to \infty$?

11. Two types of coins are produced in a factory, a fair coin, and a biased one that comes up heads 55% of the time. You have one of the coins, but do not know if it is the fair one or the biased one. In order to determine which coin you have, you perform a statistical test: toss the coin 1000 times. If the coin lands on heads 525 or more times, then you conclude that it is the biased one. If it lands heads less than 525 times, then conclude that it is the fair coin. If the coin is actually fair, what is the probability that we reach a false conclusion? What would it be if the coin were biased?

12. Let U be a uniformly distributed on $(0, 1)$ and $r > 1$. Show how to use U to simulate the distribution

$$F_X(x) = \begin{cases} 0, & \text{if } x \le 1; \\ 1 - x^{-r}, & \text{if } 1 < x. \end{cases}$$

13. Let U and V be uniformly distributed on $(0, 1)$. Find the cumulative distribution function and probability density function of $Z = UV$.

14. Let $0 < \sigma, \tau$ and $|\rho| < 1$ be constants. Suppose X and Y be jointly distributed satisfying the bivariate normal density for $(x, y) \in \mathbf{R}^2$,

$$f(x, y) = \frac{1}{2\pi\sigma\tau\sqrt{1 - \rho^2}} e^{-\frac{1}{2}Q(x,y)}$$

where the quadratic form is given by

$$Q(x, y) = \frac{1}{1 - \rho^2} \left\{ \frac{x^2}{\sigma^2} - \frac{2\rho xy}{\sigma\tau} + \frac{y^2}{\tau^2} \right\}.$$

Find the marginal density $f_Y(y)$ and the correlation coefficient of X and Y.