5010 solutions, Assignment 6. Chapter 4: 1, 2, 5, 11, 16

1. (a) Ω is the set of all subsets of size 3 of a set of 16 distinct grapefruit.
(b) $f_X(x) = \binom{12}{x}\binom{4}{3-x}/\binom{16}{3}$ for $x = 0, 1, 2, 3$.

$$E[X] = 0\frac{\binom{12}{0}\binom{4}{3}}{\binom{16}{3}} + 1\frac{\binom{12}{1}\binom{4}{2}}{\binom{16}{3}} + 2\frac{\binom{12}{2}\binom{4}{1}}{\binom{16}{3}} + 3\frac{\binom{12}{3}\binom{4}{0}}{\binom{16}{3}} = \frac{1260}{560} = \frac{9}{4}.$$

2. This requires some interpretation. With one pair, there is 1. With two pair there are 2. With three of a kind or better, there is 0. (Admittedly, this interpretation is arguable. Some would say a full house contains one pair, but we don't count that.) The expectation is therefore

$$1\frac{1{,}098{,}240}{2{,}598{,}960} + 2\frac{123{,}552}{2{,}598{,}960} = \frac{1{,}345{,}344}{2{,}598{,}960} \approx 0.517647.$$

The answer in the book is improperly rounded.

5. (i) (a) This is a Poisson(2) shifted by 1, and $\sum_1^\infty c2^x/x! = c(e^2-1)$, so $c = (e^2-1)^{-1}$. (b) $E[X] = \sum_1^\infty c2^x/(x-1)! = 2ce^2 = 2e^2(e^2-1)^{-1}$.

(ii) (a) This is geometric(q), where $q = 1-p$. The mass function is $p^{x-1}q$ for $x \geq 1$, so $c = q/p = 1/p - 1$. (b) The mean is $1/q = 1/(1-p)$.

(iii) (a) Let $h(p) = \sum_1^\infty p^x x^{-1}$, so $h'(p) = \sum_1^\infty p^{x-1} = 1/(1-p)$, hence $h(p) = \int_0^p h'(r)\,dr = -\ln(1-p)$, and we conclude that $c = 1/h(p) = -1/\ln(1-p)$. (b) $E[X] = c\sum_1^\infty p^x = -p/[(1-p)\ln(1-p)]$.

(iv) (a) From page 23, $c = (\pi^2/6)^{-1}$. (b) $E[X] = \infty$ since the harmonic series diverges.

(v) (a) $f(x) = c[1/x - 1/(x+1)]$, so $c = 1$ (telescoping series). (b) $E[X] = \infty$ for the same reason as in (iv).

11. This follows from $E[(X-a)^2] = E[(X-\mu+\mu-a)^2] = E[(X-\mu)^2] + (\mu-a)^2 = \mathrm{Var}(X) + (\mu-a)^2$.

16. $E[\cos(\pi X)] = \sum_{n=-\infty}^\infty \cos(\pi n) f(n) = \sum_{n=-\infty}^\infty f(2n) - \sum_{n=-\infty}^\infty f(2n+1)$.
$E[\sin(\pi X)] = \sum_{n=-\infty}^\infty \sin(\pi n) f(n) = 0$, since $\sin(\pi n) = 0$ for all n.