5010 solutions, Assignment 13. Chapter 7: 3, 13, 15, 24, 25, 33, 35.

3. First result is obvious. $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = B(a+2,b)/B(a,b) - [B(a+1,b)/B(a,b)]^2$. It is possible to express $B(a,b)$ in terms of the gamma function and further simplify these formulas, but the problem does not require that.

13. $E[e^{tX}] = \int_0^\infty e^{tX}\lambda e^{-\lambda x}\,dx = \int_0^\infty \lambda e^{-(\lambda-t)x}\,dx = \lambda/(\lambda-t)$, and this is valid for $t < \lambda$.

15. Let $\mu = E[X]$. Then $E[(X-c)^2] = E[(X-\mu+\mu-c)^2] = E[(X-\mu)^2] + (\mu-c)^2$, showing that the function is minimized at $c=\mu$.

24. This is important in statistics. The density of $Y = X^2$ is the derivative of $P(X^2 \le y) = 2\Phi(\sqrt{y}) - 1$, which is $f_Y(y) = (2\pi y)^{-1/2}e^{-y/2}$ for $y > 0$. (See last week's quiz problem.) The mgf is that of the gamma density with parameters $\lambda = 1/2$ and $\alpha = 1/2$, which we derived in class. The result is $(\lambda/(\lambda-t))^\alpha = (1-2t)^{-1/2}$.

25. This is the density $f_X(x) = (\lambda/2)e^{-\lambda|x|}$, x real. Let Y be exponential(λ). Then the mgf of X is

$$\begin{aligned}
E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx}(\lambda/2)e^{-\lambda|x|}\,dx \\
&= \frac{1}{2}\int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx + \frac{1}{2}\int_{-\infty}^0 e^{tx}\lambda e^{\lambda x}\,dx \\
&= \frac{1}{2}\int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx + \frac{1}{2}\int_0^\infty e^{-tx}\lambda e^{-\lambda x}\,dx \\
&= (1/2)[M_Y(t) + M_Y(-t)] \\
&= \frac{1}{2}\left[\frac{\lambda}{\lambda-t} + \frac{\lambda}{\lambda+t}\right] \\
&= \frac{\lambda^2}{\lambda^2 - t^2}.
\end{aligned}$$

The argument requires $|t| < \lambda$.

33. By Chebyshev, $P(|X-\mu| > a\sigma) \le \sigma^2/(a^2\sigma^2) = 1/a^2$. Take complements.

Y has mean $\alpha+\gamma$ and variance $\beta^2\mathrm{Var}(Z) + \gamma^2\mathrm{Var}(Z^2) + 2\alpha\beta\mathrm{Cov}(Z,Z^2)$. The variance of Z^2 is 2 and the covariance term is 0, so the variance is $\beta^2 + 2\gamma^2$. By Chebyshev, $P(|X-\alpha-\gamma| > \alpha/2) \le 4\alpha^{-2}(\beta^2+2\gamma^2)$. Again, take complements.

35. (a) $E[X] = E[\int_0^X 1\,dx] = E[\int_0^\infty 1_{\{X>x\}}\,dx] = \int_0^\infty E[1_{\{X>x\}}]\,dx = \int_0^\infty P(X>x)\,dx$.

(b) $E[X^r] = E[\int_0^X rx^{r-1}\,dx] = E[\int_0^\infty rx^{r-1}1_{\{X>x\}}\,dx] = \int_0^\infty rx^{r-1}E[1_{\{X>x\}}]\,dx = \int_0^\infty rx^{r-1}P(X>x)\,dx$.

(c) $E[e^{\theta X}] = 1 + E[\int_0^X \theta e^{\theta x}\, dx] = 1 + E[\int_0^\infty \theta e^{\theta x} 1_{\{X>x\}}\, dx] = 1 + \int_0^\infty \theta e^{\theta x} E[1_{\{X>x\}}]\, dx = 1 + \theta \int_0^\infty e^{\theta x} P(X > x)\, dx$.

(d) $\sum_{k=0}^\infty s^k P(X > k) = E[\sum_0^\infty s^k 1_{\{X>k\}}] = E[\sum_0^{X-1} s^k] = E[(1 - s^X)/(1 - s) = (1 - G_X(s))/(1 - s)$.