5010 solutions, Assignment 9. Chapter 5: 31, 32, 33, 35, 36, 39.

31. Let $X_1, X_2, \dots, X_k$ be the numbers on the removed balls.

(a) $E[X_1 + \cdots + X_k] = kE[X_1] = k(1 + 2 + \cdots + n)/n = k(n+1)/2$, and $\mathrm{Var}(X_1 + \cdots + X_k) = k\mathrm{Var}(X_1) + k(k-1)\mathrm{Cov}(X_1, X_2)$. Now $\mathrm{Var}(X_1) = (1^2 + 2^2 + \cdots + n^2)/n - [(1+2+\cdots+n)/n]^2 = (n+1)(2n+1)/6 - (n+1)^2/4 = (n^2-1)/12$ and $\mathrm{Cov}(X_1, X_2) = \sum_{i \neq j} ij/[n(n-1)] - (n+1)^2/4 = (\sum_{i,j} ij - \sum_i i^2)/[n(n-1)] - (n+1)^2/4$, etc., etc.

(b) Mean is the same as in (a), variance is $k(n^2-1)/12$ from (a) since the covariance terms are 0.

(c) Let M be the largest number removed. In (a),

$$P(M = m) = \frac{\binom{m}{k}\binom{n-m}{0}}{\binom{n}{k}} - \frac{\binom{m-1}{k}\binom{n-m+1}{0}}{\binom{n}{k}} = \frac{\binom{m-1}{k-1}}{\binom{n}{k}}.$$

and in (b),

$$P(M = m) = (m/n)^k - ((m-1)/n)^k,$$

since in both case you have to choose all balls from $\{1, 2, \dots, m\}$ but not from $\{1, 2, \dots, m-1\}$.

32. $X - Y$ and $X + Y$ assume the values $\pm a$ with probability 1/2 each, and we can check that, and the joint distribution assumes all four possibilities $(a, a), (a, -a), (-a, a), (-a, -a)$ with probability 1/4 each. Thus, the product of the marginal distributions is the joint distribution, which is independence.

33. The joint distribution is $P(U = 1, V = 1) = P(U = 1)P(V = 1 \mid U = 1) = (1/2)(1/3) = 1/6$, $P(U = 1, V = -1) = P(U = 1)P(V = -1 \mid U = 1) = (1/2)(2/3) = 1/3$, $P(U = -1, V = 1) = P(U = -1)P(V = 1 \mid U = -1) = (1/2)(2/3) = 1/3$, and $P(U = -1, V = -1) = P(U = -1)P(V = -1 \mid U = -1) = (1/2)(1/3) = 1/6$.

(a) The condition for real roots is that the discriminant is nonnegative, i.e., $U^2 - 4V \geq 0$. Since U and V are ± 1, it is necessary and sufficient that $V = -1$, and so the probability is 1/2.

(b) $P(U = 1 \mid V = -1) = (1/3)/(1/2) = 2/3$, hence $P(U = -1 \mid V = -1) = 1/3$. If $V = -1$, the largest root of the quadratic is $(-U + \sqrt{U^2 + 4})/2 = (-U + \sqrt{5})/2$. Its expectation is $(2/3)(-1 + \sqrt{5})/2 + (1/3)(1 + \sqrt{5})/2 = -1/6 + \sqrt{5}/2$.

(c) $U + V$ equals 2 with probability 1/6, 0 with probability 2/3, and -2 with probability 1/6. The discriminant is $(U+V)^2 - 4(U+V) = (U+V-4)(U+V)$. This will be nonnegative if $U + V$ is 0 or -2, hence with probability 5/6.

35. (a) Let Y be the number requiring surgery. Then

$$\begin{aligned} P(Y = m) &= \sum_{n \geq n} P(X = n)P(Y = m \mid X = n) \\ &= \sum_{n \geq m} e^{-8} 8^n (n!)^{-1} \binom{n}{m} (1/4)^m (3/4)^{n-m} \end{aligned}$$

$$
\begin{aligned}
&= e^{-8}3^{-m}(m!)^{-1}\sum_{n\geq m} 8^n(3/4)^n/(n-m)! \\
&= e^{-8}3^{-m}(m!)^{-1}6^m e^6 \\
&= e^{-2}2^{-m}(m!)^{-1},
\end{aligned}
$$

which is Poisson(2), whose mean is 2.

(b) Replace 8 by 4 to get Poisson(1) for weekends. For the week, we have 5 days with Poisson(2) and 2 days of Poisson(1). Total is Poisson(12), which has mean and variance 12.

36. This is the hypergeometric distribution. The mean is $E[X_1+\cdots+X_n] = nE[X_1] = nm/M$, and the variance is $n\mathrm{Var}(X_1) + n(n-1)\mathrm{Cov}(X_1, X_2) = n(m/M)(1-m/M) + n(n-1)\{m(m-1)/[M(M-1)] - (m/M)^2\}$, which can be simplified slightly.

39. (a)

$$
\begin{aligned}
P(X_1 < X_2 < X_3) &= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{k=j+1}^{\infty} P(X_1 = i, X_2 = j, X_3 = k) \\
&= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{k=j+1}^{\infty} p_1^{i-1}(1-p_1)p_2^{j-1}(1-p_2)p_3^{k-1}(1-p_3) \\
&= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} p_1^{i-1}(1-p_1)p_2^{j-1}(1-p_2)p_3^{j} \\
&= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} p_1^{i-1}(1-p_1)(p_2p_3)^{j-1}(1-p_2)p_3 \\
&= \sum_{i=1}^{\infty} p_1^{i-1}(1-p_1)(p_2p_3)^{i}(1-p_2)p_3/(1-p_2p_3) \\
&= \sum_{i=1}^{\infty} (p_1p_2p_3)^{i-1}(1-p_1)(1-p_2)p_2p_3^2/(1-p_2p_3) \\
&= (1-p_1)(1-p_2)p_2p_3^2/[(1-p_2p_3)(1-p_1p_2p_3)],
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(X_1 \leq X_2 \leq X_3) &= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\sum_{k=j}^{\infty} P(X_1 = i, X_2 = j, X_3 = k) \\
&= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\sum_{k=j}^{\infty} p_1^{i-1}(1-p_1)p_2^{j-1}(1-p_2)p_3^{k-1}(1-p_3) \\
&= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} p_1^{i-1}(1-p_1)p_2^{j-1}(1-p_2)p_3^{j-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} p_1^{i-1}(1-p_1)(p_2p_3)^{j-1}(1-p_2) \\
&= \sum_{i=1}^{\infty} p_1^{i-1}(1-p_1)(p_2p_3)^{i-1}(1-p_2)/(1-p_2p_3) \\
&= \sum_{i=1}^{\infty} (p_1p_2p_3)^{i-1}(1-p_1)(1-p_2)/(1-p_2p_3) \\
&= (1-p_1)(1-p_2)/[(1-p_2p_3)(1-p_1p_2p_3)],
\end{aligned}
$$

(c) The probability A throws 6 first, B second, and C third is just the answer to part (b) with $p_1 = p_2 = p_3 = 5/6$. We get

$$(1/6)(1/6)/[(1-25/36)(1-125/216)] = 216/1001.$$