Half of the final will be devoted to material since the 3rd midterm. The sample problems on this page reflect this part of the exam. The other half will be comprehensive.

Questions from Dec. 12, 2000 final

1.) *Let $F : \mathbf{R}^5 \to \mathbf{R}^2$ be given by $F = (f_1, f_2)$ where*

$$f_1(v, w, x, y, z) = v + w^2 + x + y,$$
$$f_2(v, w, x, y, z) = vy + wz.$$

Show that there is a neighborhood $U \subseteq \mathbf{R}^3$ containing the point $(3, 4, 5)$ and $\mathcal{C}^1$ functions $G : U \to \mathbf{R}^2$ where $G = (g_1, g_2)$ so that $g_1(3, 4, 5) = 1$, $g_2(3, 4, 5) = 2$ and for all $(x, y, z) \in U$,

$$f_1(g_1(x, y, z), g_2(x, y, z), x, y, z) = 12,$$
$$f_2(g_1(x, y, z), g_2(x, y, z), x, y, z) = 14.$$

What is the differential $DG(3, 4, 5)(h, j, k)$?

2.) *Let $f(x, y) : [a, b] \times [c, d] \to \mathbf{R}$ be a continuously differentiable ($\mathcal{C}^1$) function. Let $a < b$ be finite numbers. Assume that*

$$F(y) = \int_a^b f(x, y)\, dx$$

exists and is a continuous function for all $y \in [c, d]$. [We showed this in class.] Prove that F is differentiable and that for all $y \in [c, d]$, [Hint: Mean Value Theorem]

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx.$$

3.) *Let $f(x, y) : \mathbf{R}^2 \to \mathbf{R}$. Prove if true; give a counterexample if false:*

a.) **Statement.** *Suppose one of the partial derivatives, say $\frac{\partial f}{\partial x}(x, y)$ exists for all $(x, y) \in \mathbf{R}^2$. Then f is integrable on $[0, 1] \times [0, 1]$.*

b.) **Statement.** *Suppose f is integrable on the squares $Q_s = [-s, s] \times [-s, s]$ for all $s > 0$ and $\int_{Q_s} f = 0$ for all s. Then $f \to 0$ as $\|(x, y)\| \to \infty$.*

4.) *Let $R = [0, 1] \times [0, 1]$ be the unit square. Let $E = \{(x, y) \in R : |x - .5| + |y - .5| \le .5\}$ be the dimamond shaped region.*

a.) Choose a partition $\mathcal{G}$ of R consisting of at least nine (9) subrectangles. For your partition find $U(\chi_E, \mathcal{G})$ and the of the boundary $U(\chi_{\partial E}, \mathcal{G})$.

b.) Quickly describe how you would go about showing that E is a Jordan Region. [You don't need to give precise formulas. Only state what conclusions you would expect.]

c.) Find an approximation to $\mathrm{V}(E)$ that is makes an error $< \varepsilon = 4/9$ of the actual value. Prove your estimate. [Hint: think a second!]

5.) *Let $E \subseteq \mathbf{R}^2$ be a Jordan region. Let*

$$f(x) = \begin{cases} 3, & \text{if } x \in E^\circ, \\ 1, & \text{if } x \in E \backslash E^\circ, \end{cases}$$

Show that $f(x)$ is integrable on E and fiind $\int_E f(x)\, dx$.

6.) *Let $E \subseteq \mathbf{R}^2$ be a Jordan region. Let $f : E \to \mathbf{R}$ be given by*

$$f(x) = \begin{cases} \|x\|, & \text{if } x \in E^\circ, \\ 3, & \text{if } x \in E \backslash E^\circ, \end{cases}$$

Show that $f(x)$ is integrable on E.

More problems

7. Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $F(x, y, z) = (x^2 + y^2, xz, y^3 - z^3)$. Assume that there is an open set U about $P_0 = (3, 1, 2)$ and that $V = F(U)$ is an open set about $Q_0 = F(P_0)$ on which F has an inverse function $F^{-1} \in \mathcal{C}^1(V, U)$. Find $D[F^{-1}](Q_0)$.

8. Suppose $G : \mathbf{R}^5 \to \mathbf{R}^3$ is given by $G(p, q, x, y, z) = (px + y^2, q^2 z, py - qz + x)$. Assume that there is an open set U around $(3, 2)$ and a $\mathcal{C}^1$ function $H : U \to \mathbf{R}^3$ so that $H(3, 2) = (1, 5, 4)$ and for all $(p, q) \in U$ we have $G(p, q, H(p, q)) = (28, 16, 8)$. Find $DH(3, 2)$.

9. Let $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbf{R}^2$ be a compact rectangle and $f : \mathcal{R} \to \mathbf{R}$ be continuous. Then $F(y) = \int_a^b f(x, y)\, dx$ exists and is continuous for all $y \in [c, d]$. Is $F(y)$ differentiable?

10. From first principles, show that $\mathcal{B} = \{(x, y) : x^2 + y^2 \leq 1\}$ is a Jordan domain and that $\mathrm{V}(\mathcal{B}) = \pi$.

11. Let $E = [0, 3] \times [0, 4] \subseteq \mathbf{R}^2$. Show that f is integrable on E where
$$f(x, y) = \begin{cases} 2, & \text{if } y = 2x - 1 \text{ and } x \text{ is rational;} \\ 0, & \text{if } y = 2x - 1 \text{ and } x \text{ is irrational;} \\ \sin(x^2 y^4), & \text{otherwise.} \end{cases}$$

12. Prove if true; give a counterexample if false:

    (a) a. **Statement.** Suppose $f : [0, 1] \times [0, 1] \to \mathbf{R}$ is integrable. Then $F(y) = \int_0^1 f(x, y)\, dx$ is a continuous function on $[0, 1]$.

13. Let $D \subseteq \mathbb{R}^n$ be a dense set and $f : D \to \mathbf{R}^m$ be uniformly continuous. Then there is a uniformly continuous function $F : \mathbb{R}^n \to \mathbf{R}^m$ such that $F(x) = f(x)$ for all $x \in D$.

14. Let $R = [0, 1] \times [0, 1]$. Show that $\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \int_0^1 \int_0^1 f(x, y)\, dy\, ds$ where
$$f(x, y) = \begin{cases} x^2 \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

15. Let $Q = [0, 1]^2$ and $\mathbf{y} = (1, 1)$. Find $\displaystyle\int_Q e^{-\mathbf{x} \cdot \mathbf{y}} d\mathbf{x}$.

16. Show that f is not integrable on $R = [0, 1] \times [0, 1]$, where $f(x, y) = \begin{cases} 1, & \text{if } y \in \mathbf{Q}, \\ x, & \text{if } y \notin \mathbf{Q}. \end{cases}$

17. Evaluate the following integrals

    (a) Suppose $0 < a < b$, find $\displaystyle\int_a^b \int_0^x \sqrt{x^2 + y^2}\, dy\, dx$

    (b) Let E be the trapezoid with vertices $(1, 1)$, $(2, 2)$, $(2, 0)$, $(4, 0)$. Find $\displaystyle\int_E \exp\left(\frac{y - x}{y + x}\right) dA$

18. Suppose $V \subseteq \mathbb{R}^n$ is an open set and $\phi : V \to \mathbb{R}^n$ is continuously differentiable with $\Delta_\phi \neq 0$ on V. Prove that for every $\mathbf{x}_0 \in V$,
$$\lim_{r \to 0+} \frac{\mathrm{V}\left(\phi(B_r(\mathbf{x}_0))\right)}{\mathrm{V}(B_r(\mathbf{x}_0))} = |\Delta_\phi(\mathbf{x}_0)|.$$

Solutions

(1.) *Let* $F:\mathbf{R}^5\to\mathbf{R}^2$ *be given by* $F=(f_1,f_2)$ *where*

$$\begin{aligned} f_1(v,w,x,y,z) &= \ v+w^2+x+y,\\ f_2(v,w,x,y,z) &= \ vy+wz. \end{aligned}$$

Show that there is a neighborhood $U\subseteq\mathbf{R}^3$ *containing the point* $(3,4,5)$ *and* $\mathcal{C}^1$ *functions* $G:U\to\mathbf{R}^2$ *where* $G=(g_1,g_2)$ *so that* $g_1(3,4,5)=1$, $g_2(3,4,5)=2$ *and for all* $(x,y,z)\in U$,

$$\begin{aligned} f_1(g_1(x,y,z),g_2(x,y,z),x,y,z)=12,\\ f_2(g_1(x,y,z),g_2(x,y,z),x,y,z)=14. \end{aligned}$$

Find the total derivative $DG(3,4,5)(h,j,k)$.

This is an application of the Implicit Function Theorem which says that if there is enough differentiability, and if the problem can be solved for the linear approximations given by the total derivatives, then, at least in a small neighborhood, the nonlinear problem can be solved as well. The function $F:\mathbf{R}^{2+3}\to\mathbf{R}^2$ is $\mathcal{C}^1$ on $\mathbf{R}^5$ such that $F(1,2,3,4,5)=(12,14)=\mathbf{c}$. To solve for v and w in terms of (x,y,z) we need to be able to solve the linearization. If we put $\mathbf{u}=(v,w)$ and $\mathbf{x}=(x,y,z)$, we are looking for $G:\mathbf{R}^3\to\mathbf{R}^2$ so that $F(G(\mathbf{x}),\mathbf{x})=\mathbf{c}$ and $G(3,4,5)=(1,2)$. Taking $D_{\mathbf{x}}$ gives $D_{\mathbf{u}}F(G(\mathbf{x}),\mathbf{x})\circ DG(\mathbf{x})+D_{\mathbf{x}}F(G(\mathbf{x}),\mathbf{x})=0$ which says that we may solve for the differential $DG(\mathbf{x})$ whenever $D_{\mathbf{u}}F(G(\mathbf{x}),\mathbf{x})$ is invertible and then $DG(\mathbf{x})=-[D_{\mathbf{u}}F(G(\mathbf{x}),\mathbf{x})]^{-1}\circ D_{\mathbf{x}}F(G(\mathbf{x}),\mathbf{x})$. At the center point $(3,4,5)$, the matrix of the transformation is

$$D_{\mathbf{u}}F(G(3,4,5),(3,4,5))=\begin{pmatrix}\frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w}\\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w}\end{pmatrix}\Bigg|_{\mathbf{x}=(3,4,5)}=\begin{pmatrix}1 & 2w\\ y & z\end{pmatrix}\Bigg|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)}=\begin{pmatrix}1 & 4\\ 4 & 5\end{pmatrix}$$

which is invertible. Hence there is an open set $U\subseteq\mathbf{R}^3$ such that $(3,4,5)\in U$ and a $\mathcal{C}^1$ function $G:U\to\mathbf{R}^2$ so that $G(2,3,4)=(1,2)$ and $F(G(\mathbf{x}),\mathbf{x})=\mathbf{c}$ for all $\mathbf{x}\in U$. (Thus, we have checked the differentiability and the solubility of the linearized problem is satisfied. The IFT gives the existence of a $G\in\mathcal{C}^1(U,\mathbf{R}^2)$ so that $G(2,3,4)=(1,2)$ and $F(G(\mathbf{x}),\mathbf{x})=\mathbf{c}$ for all $\mathbf{x}\in U$. You have been given this as a hypothesis. Then take the total derivative of $F(G(\mathbf{x}),\mathbf{x})=\mathbf{c}$ with respect to x and solve for $DG(x)$ at the given point, as above.) By the formula for the differential

$$DG(3,4,5)\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = -[D_{\mathbf{u}}F(G(3,4,5),(3,4,5))]^{-1} \circ D_{\mathbf{x}}F(G(3,4,5),(3,4,5))\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= -\left[\left.\begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\mathrm{w}} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{pmatrix}\right|_{\mathbf{x}=(3,4,5)}\right]^{-1} \left\{\left.\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\mathrm{y}} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}\right|_{\mathbf{x}=(3,4,5)}\right\}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= -\left[\left.\begin{pmatrix} 1 & 2w \\ y & z \end{pmatrix}\right|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)}\right]^{-1} \left\{\left.\begin{pmatrix} 1 & 1 & 0 \\ 0 & v & w \end{pmatrix}\right|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)}\right\}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5 & -4 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} -5 & -1 & 8 \\ 4 & 3 & -2 \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -5h - k + 8\ell \\ 4h + 3k - 2\ell \end{pmatrix}. \quad \square$$

2.)*Let $f(x,y) : [a,b] \times [c,d] \to \mathbf{R}$ be a continuously differentiable ($\mathcal{C}^1$) function. Let $a < b$ be finite numbers. Assume that*

$$F(y) = \int_a^b f(x,y)\,dx$$

exists and is a continuous function for all $y \in [c,d]$. [We showed this in class.] Prove that F is differentiable and that for all $y \in [c,d]$, [Hint: Mean Value Theorem]

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

The method is to verify that the function $G(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$ is the limit of the difference quotient

$$\lim_{\substack{h\to 0 \\ 0<h<b-y}} \frac{F(y+h) - F(y)}{h} = G(y).$$

For simplicity sake, we argue the case that $a \le y < y + h \le b$. (If $h < 0$ then the upper and lower limits of integrals would have to be interchanged.) Since $F \in \mathcal{C}^1([a,b] \times [c,d])$ it follows that the partial derivative $F_y(x,y)$ is uniformly continuous on $[a,b]\times[c,d]$. Thus, for every $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ so that if for $(x_1,y_1),(x,y)\in[a,b]\times[c,d]$ we have $\|(x_1,y_1)-(x,y)\|<\delta(\varepsilon)$ then $|f_y(x_1,y_1)-f_y(x,y)|<\varepsilon/(1+b-a)$. Now, as a function of y we have that f is continuous on $[c,d]$ and differentiable on (c,d), so that we may apply the (one-dimensional) Mean Value Theorem: for every $x\in[a,b]$ and $y<y+h\in[c,d]$ there is $\eta(x,y,h)\in(y,y+h)$ so that $f(x,y+h)-f(x,y)=f_y(x,\eta(x,y,h))h$. If $|h|<\delta(\varepsilon)$ then $\|(x,\eta(x,y,h))-(x,y)\|<\|(x,y+h)-(x,y)\|=|h|<\delta(\varepsilon)$ so that $|f_y(x,\eta(x,y,h))-f_y(x,y)|<\varepsilon/(1+b-a)$. Thus, for arbitrary $\varepsilon>0$, if $0<h<\min\{\delta(\varepsilon),1-y\}$,

$$\left|\frac{1}{h}\left[F(y+h)-F(y)\right]-G(y)\right|=\left|\int_a^b\frac{1}{h}\left[f(x,y+h)-f(x,y)\right]-f_y(x,y)\,dx\right|$$

$$=\left|\int_a^b f_y(x,\eta(x,y,h))-f_y(x,y)\,dx\right|\le\int_a^b\left|f_y(x,\eta(x,y,h))-f_y(x,y)\right|\,dx\le\int_a^b\frac{\varepsilon\,dx}{1+b-a}<\varepsilon.$$

(3.)*Let $f(x,y):\mathbf{R}^2\to\mathbf{R}$. Prove if true; give a counterexample if false:*

a.) **Statement.** *Suppose one of the partial derivatives, say $\frac{\partial f}{\partial x}(x,y)$ exists for all $(x,y)\in\mathbf{R}^2$. Then f is integrable on $Q=[0,1]\times[0,1]$.*

FALSE! Let $f(x,y)=\begin{cases}1, & \text{if } y\in\mathbf{Q},\\ 0, & \text{if } y\notin\mathbf{Q}.\end{cases}$ Then as f is independent of x it follows that $f_x(x,y)=0$ for all (x,y). However f is not Riemann integrable. For any subrectangle R_i we have $m_i=\inf\{f(x,y):(x,y)\in R_i\}=0$ and $M_i=\inf\{f(x,y):(x,y)\in R_i\}=1$. It follows that $\underline{\int}_E f(x,y)\,dA=\sup_{\mathcal{G}}L(f,\mathcal{G})=0$ but $\overline{\int}_E f(x,y)\,dA=\inf_{\mathcal{G}}U(f,\mathcal{G})=\mathrm{V}(Q)=1$, which are not equal, thus f is not integrable on Q.

b.) **Statement.** *Suppose f is integrable on the squares $Q_s=[-s,s]\times[-s,s]$ for all $s>0$ and $\int\limits_{Q_s}f=0$ for all s. Then $f\to 0$ as $\|(x,y)\|\to\infty$.*

FALSE! Let $f(x,y)=\begin{cases}1, & \text{if } xy>0,\\ -1, & \text{if } xy\le 0.\end{cases}$ be 1 on the first and third quadrants, and -1 otherwise. Then $\int_{Q_s}f(x,y)\,dA=0$ for $s>0$, but $f(x,y)\not\to 0$ as $\|(x,y)\|\to\infty$. Along $f(t,t)=1$ where $t>0$ then $\lim_{t\to+\infty}f(t,t)=1$ but $f(t,0)=-1$ so $\lim_{t\to+\infty}f(t,0)=-1$ thus there is no limit for f at infinity.

4.)*Let $R=[0,1]\times[0,1]$ be the unit square. Let $E=\{(x,y)\in R:|x-.5|+|y-.5|\le .5\}$ be the dimamond shaped region.*

a.) Choose a partition $\mathcal{G}$ of R consisting of at least nine (9) subrectangles. For your partition find $U(\chi_E,\mathcal{G})$ and for the boundary $U(\chi_{\partial E},\mathcal{G})$.

Take a partition with 16 uniform squares. The grid square $R_{i,j}=[\frac{i-1}{4},\frac{i}{4}]\times[\frac{j-1}{4},\frac{j}{4}]$, as $i,j=1,\ldots,4$. Then all squares touch $\bar{E}$. For example the point $(\frac14,\frac14)\in R_{11}\cap R_{12}\cap R_{21}\cap R_{22}\cap\partial E$. Thus $U(\chi_{\partial E},\mathcal{G})=1$. But $\partial E\subseteq\bar{E}\subseteq R$ so $U(\chi_E,\mathcal{G})=1$.

b.)*Quickly describe how you would go about showing that E is a Jordan Region. [You don't need to give precise formulas. Only state what conclusions you would expect.]*

To be a Jordan region, two conditions have to be checked. First E is bounded since $\bar{E}\subseteq R=[0,1]^2$. Second, one must also show $\mathrm{V}(\partial E)=0$, *i.e.*that for every $\varepsilon>0$ there is a grid $\mathcal{G}_n$ for R so that $U(\chi_{\partial E},\mathcal{G}_n)<\varepsilon$. For example, one may choose the uniform grid whose squares have side lengths $\frac{1}{2n}$, (so there is an even number $4n^2$ squares all together.) The number of squares t_n that touch ∂E grows much slower than $4n^2$. We can can find t_n exactly for this grid. In the quadrant $0\le x,y,\le\frac12$, the squares on the diagonal and the super and subdiagonals are the only ones that touch ∂E. Thus all together there are $t_n=4(n+2(n-1))=12n-8$. Hence the number of squares times their area gives $U(\chi_{\partial E}, \mathcal{G}_n) = \frac{12n-8}{n^2} \to 0$ as $n \to \infty$. By choosing n large enough, $U(\chi_{\partial E}, \mathcal{G}_n) < \varepsilon$, so ∂E has Jordan Content zero. Thus, E is a Jordan region.

c.)*Find an approximation to* $\mathrm{V}(E)$ *that is makes an error* $< \varepsilon = 4/9$ *of the actual value. Prove your estimate. [Hint: think a second!]*

Using the grid $\mathcal{G}_2$ from (a.) we see that $Q = R_{22} \cup R_{23} \cup R_{32} \cup R_{33} \subseteq E \subseteq R$. Thus if $\mathcal{H}$ is any grid and $\mathcal{K}$ is a common refinement of both $\mathcal{H}$ and $\mathcal{G}_2$ then Q is the union of several rectangles from $\mathcal{K}$ and

$$1 = U(\chi_R, \mathcal{H}) \geq U(\chi_E, \mathcal{H}) \geq U(\chi_E, \mathcal{K}) \geq V(Q) = \frac{1}{4},$$

so that $1 \geq V(E) = \inf_{\mathcal{H}} U(\chi_E, \mathcal{H}) \geq \frac{1}{4}$. Thus the average gives an approximation with $|V(E) - \frac{5}{8}| \leq \frac{3}{8} < \frac{4}{9}$.

5.)*Let* $E \subseteq \mathbf{R}^2$ *be a Jordan region. Let*

$$f(x) = \begin{cases} 3, & \text{if } x \in E^\circ, \\ 1, & \text{if } x \in E \backslash E^\circ, \end{cases}$$

Show that $f(x)$ *is integrable on* E *and fiind* $\int\limits_E f(x)\,dx$.

Let $R \supseteq E$ be a rectangle containing E, and assume that f is extended to be zero off E. A bounded function is integrable if for every $\varepsilon > 0$ there is a grid $\mathcal{G}$ of R so that the upper and lower Riemann sums satisfy $U(f, \mathcal{G}) - L(f, \mathcal{G}) < \varepsilon$. We shall use the fact that E is a Jordan Domain to show this. Choose $\varepsilon > 0$. Since E is a Jordan Domain so $V(\partial E) = 0$, there is a grid $\mathcal{G}$ so that $U(\chi_{\partial E}, \mathcal{G}) < \frac{\varepsilon}{6}$. Since $\mathrm{V}(E) = \inf_{\mathcal{H}} U(\chi_E, \mathcal{H})$, there is a grid $\mathcal{H}$ so that $\mathrm{V}(E) \leq U(\chi_E, \mathcal{H}) < \mathrm{V}(E) + \frac{\varepsilon}{6}$. Let $\mathcal{K}$ be a common refinement of $\mathcal{G}$ and $\mathcal{H}$. Since any rectangle for which $R_i \cap \bar{E} \neq \emptyset$ we have either $R_i \cap \partial E \neq \emptyset$ or $R_i \subseteq E^\circ$ but not both. We let $\mathcal{I}' = \{i : R_i \cap \partial E \neq \emptyset\}$ and $\mathcal{I}'' = \{i : R_i \subseteq E^\circ\}$. Observe that $0 \leq m_i = \inf\{f(x) : x \in R_i\} \leq M_i = \sup\{f(x) : x \in R_i\} \leq 3$ for all i and $m_i = M_i = 3$ for $i \in \mathcal{I}''$ because $f(x) = 3$ for all $x \in R_i \subseteq E^\circ$. Now, the Riemann sums satisfy

$$\begin{aligned} U(f, \mathcal{K}) - L(f, \mathcal{K}) &= \sum_{R_i \cap E \neq \emptyset} (M_i - m_i)\, V(R_i) \leq \sum_{i \in \mathcal{I}'} (M_i - m_i)\, V(R_i) + \sum_{i \in \mathcal{I}''} (M_i - m_i)\, V(R_i) \\ &\leq 3 \sum_{i \in \mathcal{I}'} V(R_i) \leq 3U(\chi_{\partial E}, \mathcal{K}) \leq 3U(\chi_{\partial E}, \mathcal{G}) < 3 \cdot \frac{\varepsilon}{6} < \varepsilon. \end{aligned}$$

Thus the function f is integrable. Moreover, since $R_i \subseteq E^\circ$ for $i \in \mathcal{I}''$, $S = \mathrm{V}\left(\bigcup_{i \in \mathcal{I}''} R_i\right) \leq U(\chi_E, \mathcal{G}')$ for any $\mathcal{G}'$ so $S \leq \mathrm{V}(E)$. Also, $|M_i - 3| \leq 3$.

$$L(f, \mathcal{K}) - \frac{\varepsilon}{2} = \sum_{R_i \cap \bar{E} \neq \emptyset} m_i\, V(R_i) - \frac{\varepsilon}{2} = \sum_{i \in \mathcal{I}'} m_i\, V(R_i) + 3 \sum_{i \in \mathcal{I}''} V(R_i) - \frac{\varepsilon}{2} \leq 3V(\partial E, \mathcal{K}) - 3 \cdot \frac{\varepsilon}{6} + 3\,\mathrm{V}(E)$$

$$\leq 3\,\mathrm{V}(E) \leq 3U(\chi_E, \mathcal{K}) \leq 3 \sum_{R_i \cap \bar{E} \neq \emptyset} V(R_i) = \sum_{i \in \mathcal{I}'} \Big((3 - M_i) + M_i\Big)\, V(R_i) + \sum_{i \in \mathcal{I}''} M_i\, V(R_i)$$

$$\leq 3U(\chi \partial E, \mathcal{K}) + U(f, \mathcal{K}) \leq \frac{\varepsilon}{2} + U(f, \mathcal{K}).$$

Because $\varepsilon > 0$ was arbitrary, $\int_E f\, da = 3\,\mathrm{V}(E)$.

(6.)*Let* $E \subseteq \mathbf{R}^2$ *be a Jordan region. Let* $f : E \to \mathbf{R}$ *be given by*

$$f(x) = \begin{cases} \|x\|, & \text{if } x \in E^\circ, \\ 3, & \text{if } x \in E \backslash E^\circ, \end{cases}$$

Show that $f(x)$ is integrable on E.

Almost the same as the previous problem, so I can do a cut-and-paste job!

Let $R \supseteq E$ be a rectangle containing E, and assume that f is extended to be zero off E. Since E is bounded, there is $3 \le J < \infty$ so that $\|x\| \le J$ for all $x \in E^\circ$. Thus $|f(x)| \le J$ all $x \in \bar{E}$. A bounded function is integrable if for every $\varepsilon > 0$ there is a grid $\mathcal{G}$ of R so that the upper and lower Riemann sums satisfy $U(f,\mathcal{G}) - L(f,\mathcal{G}) < \varepsilon$. We shall use the fact that E is a Jordan Domain to show this.

Choose $\varepsilon > 0$. Since E is a Jordan domain so $\mathrm{V}(\partial E) = 0$, there is a grid $\mathcal{G}$ so that $U(\chi_{\partial E}, \mathcal{G}) < \frac{\varepsilon}{2J}$. Note that the function $x \mapsto \|x\|$ is 1-Lipschitz continuous, thus uniformly continuous. Choose a fine enough grid $\mathcal{H}$ so that $\operatorname{diam}(R_i') = \sup\{\|x-y\| : x, y \in R_i'\} < \frac{\varepsilon}{2V(R)}$ for every R_i', a rectangle of $\mathcal{H}$. Let $\mathcal{K}$ be the common refinement of $\mathcal{H}$ and $\mathcal{G}$. Since any rectangle of $\mathcal{K}$ for which $R_i \cap \bar{E} \ne \emptyset$ we have either $R_i \cap \partial E \ne \emptyset$ or $R_i \subseteq E^\circ$ but not both, we let $\mathcal{I}' = \{i : R_i \cap \bar{E} \ne \emptyset\}$ and $\mathcal{I}'' = \{i : R_i \subseteq E^\circ\}$. Observe that $0 \le m_i = \inf\{f(x) : x \in R_i\} \le M_i = \sup\{f(x) : x \in R_i\} \le J$ for all i. Also, since $f = \|x\|$ is continuous for $x \in R_i \subseteq E^\circ$ whenever $i \in \mathcal{I}''$, and since R_i is compact, by the extreme value theorem, there are $\xi, \eta \in R_i$ so that $m_i = f(\xi)$ and $M_i = f(\eta)$. But, since f is 1-Lipschitz on R_i, we have $M_i - m_i = |f(\eta) - f(\xi)| \le \|\eta - \xi\| \le \operatorname{diam}(R_i) \le \frac{\varepsilon}{2V(R)}$. Now, since the Riemann sums satisfy

$$\begin{aligned}
U(f,\mathcal{K}) - L(f,\mathcal{K}) &= \sum_{R_i \cap \bar{E} \ne \emptyset} (M_i - m_i)\, V(R_i) = \sum_{i \in \mathcal{I}'} (M_i - m_i)\, V(R_i) + \sum_{i \in \mathcal{I}''} (M_i - m_i)\, V(R_i) \\
&\le J \sum_{i \in \mathcal{I}'} V(R_i) + \frac{\varepsilon}{2V(R)} \sum_{i \in \mathcal{I}''} V(R_i) \le J U(\chi_{\partial E}, \mathcal{K}) + \frac{\varepsilon}{2V(R)} V(R) \\
&\le J U(\chi_{\partial E}, \mathcal{G}) + \frac{\varepsilon}{2} < \varepsilon \cdot \frac{\varepsilon}{2J} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus the function f is integrable.

(7.) *(Slight generalization.) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $F(x,y,z) = (x^2 + y^2, xz, y^3 - z^3)$. Show that there is an open set U about $P_0 = (3,1,2)$ so that F is invertible on U and that $F(U)$ is an open set about $Q_0 = F(P_0)$ on which F^{-1} is $\mathcal{C}^1$. Find $D[F^{-1}](Q_0)$. Find $D[F^{-1}](Q)$ where $Q \in F(U)$.*

The function $F(x,y,z)$ polynomial, therefore $\mathcal{C}^1$. We check that $\Delta_F(P_0) \ne 0$ and all of the conclusions follow from the Inverse Function Theorem. The fact that the linearization was invertible at the point enables you to conclude the existence of a local inverse function. Thus there is a neighborhood $P_0 \in U$ such that $V = F(U)$ is open and there is $G \in \mathcal{C}^1(V, \mathbf{R}^3)$ which is the inverse function of $F : U \to V$. (You were given this in the original problem.) Thus in U we have the equation $F(G(x,y,z)) = (x,y,z)$. Apply the chain rule, and solve for DG at the point. Thus $D(F \circ G) = DF(P_0) \circ DG(Q_0) = I$ so $DG(Q_0) = (DF(P_0))^{-1}$. The matrix of $DF(P_0)$ is the Jacobian matrix

$$DF(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \\ z & 0 & x \\ 0 & 3y^2 & -3z^2 \end{pmatrix}; \quad DF(P_0) = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -12 \end{pmatrix};$$

thus $\Delta_F(P_0) = \det(DF_x(P_0)) = -6 \ne 0$. Finally, for Q near $(10, 6, -7) = Q_0 = F(P_0)$, so

$DF^{-1}(Q) =$

$$[DF(F^{-1}(Q))]^{-1} = \frac{1}{6yz^3 - 6x^2y^2}\begin{pmatrix} -3xy^2 & 3z^3 & 3zy^2 \\ 6yz^2 & -6xz^2 & -6xy^2 \\ 2xy & -2x^2 & -2yz \end{pmatrix}; \quad DF^{-1}(Q_0) = \begin{pmatrix} \frac{3}{2} & -4 & -1 \\ -4 & 12 & 3 \\ -1 & 3 & \frac{2}{3} \end{pmatrix}$$

where $(x, y, z) = F^{-1}(Q)$. □

(8.) *(Slight Generalization.) Suppose $G : \mathbf{R}^5 \to \mathbf{R}^3$ is given by $G(p, q, x, y, z) = (px + y^2, q^2 z, py - qz + x)$. Show that there is an open set U around $T_0 = (3, 2)$ and a $\mathcal{C}^1$ function $H : U \to \mathbf{R}^3$ so that $H(3, 2) = (1, 5, 4) = X_0$ and for all $(p, q) \in U$ we have $G(p, q, H(p, q)) = (28, 16, 8)$. Find $DH(3, 2)$. Find $DH(p, q)$.*

The function G is polynomial so $\mathcal{C}^1$. We have to check that the linearization is soluble at $(3, 2, 1, 5, 4)$. This follows if the $D_x G$ part of the Jacobian matrix is invertible.

$$D_x G = \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} & \frac{\partial G_1}{\partial z} \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & \frac{\partial G_2}{\partial z} \\ \frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} & \frac{\partial G_3}{\partial z} \end{pmatrix} = \begin{pmatrix} p & 2y & 0 \\ 0 & 0 & q^2 \\ 1 & p & -q \end{pmatrix}; \quad D_x G(T_0, X_0) = \begin{pmatrix} 3 & 10 & 0 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$$

which is invertible since its determinant is 4. The Implicit Function Theorem applies. There is an open set $T_0 \in U \subseteq \mathbf{R}^2$ and $H \in \mathcal{C}^1(U, \mathbf{R}^3)$ satisfying $F(p, q, H(p, q)) = (28, 16, 8)$ for all $(p, q) \in U$. (You were given this in the original problem.) Find the total derivative of H by differentiating the equation using the chain rule. Think of $\mathcal{H} : U \to \mathbf{R}^5$ is given by $\mathcal{H}(p, q) = (p, q, H(p, q))$, and then differentiate $G \circ \mathcal{H} = \text{const.}$ using the chain rule. The total derivative of G matrix has columns associated to $\mathbf{t} = (p, q)$ and columns associated to $\mathbf{x} = (x, y, z)$ drivatives, $DG = (D_{\mathbf{t}} G, D_{\mathbf{x}} G)$. To find the total derivative of H we need the other part of the Jacobian

$$D_t G = \begin{pmatrix} \frac{\partial G_1}{\partial p} & \frac{\partial G_1}{\partial q} \\ \frac{\partial G_2}{\partial p} & \frac{\partial G_2}{\partial q} \\ \frac{\partial G_3}{\partial p} & \frac{\partial G_3}{\partial q} \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 2qz \\ y & -z \end{pmatrix}; \quad D_t G(T_0, X_0) = \begin{pmatrix} 1 & 0 \\ 0 & 16 \\ 5 & -4 \end{pmatrix}; \quad DH(T_0) = \begin{pmatrix} -3 & -16 \\ -\frac{25}{4} & -25 \\ -10 & 48 \end{pmatrix}$$

since total derivative of implicit function $DH(T) = -[D_x G(T, H(T))]^{-1} D_t G(T, H(T)) =$

$$-\frac{1}{\Delta}\begin{pmatrix} -pq^2 & q^2 & 0 \\ 2yq & -pq & 2y - p^2 \\ 2yq^2 & -pq^2 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 2qz \\ y & -z \end{pmatrix} = -\frac{1}{\Delta}\begin{pmatrix} -xpq^2 & 2q^3 z \\ 2xyq + 2y^2 - yp^2 & p^2 z - 2pq^2 z - 2yz \\ 2xyq^2 & -2pq^3 z \end{pmatrix}$$

where $\Delta = q^2(2y - p^2)$ and $(x, y, z) = H(p, q)$. □

(9.)*Let* $\mathcal{R} = [a,b] \times [c,d] \subseteq \mathbf{R}^2$ *be a compact rectangle and* $f : \mathcal{R} \to \mathbf{R}$ *be continuous. Then* $F(y) = \int_a^b f(x,y)\,dx$ *exists and is continuous for all* $y \in [c,d]$. *Is* $F(y)$ *differentiable?*

Since $f(x,y)$ is continuous on the compact set $\mathcal{R}$, we may suppose that f is uniformly continuous: for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ so that for any $(x,y),(x_0,y_0) \in \mathcal{R}$, if $\|(x,y)-(x_0,y_0)\| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{1+b-a}$. Choose $\varepsilon > 0$. Suppose $c \le y < y_0 \le d$ such that $|y - y_0| < \delta$. Then $\|(x,y) - (x,y_0)\| < \delta(\varepsilon)$ and so $|f(x,y) - f(x,y_0)| < \frac{\varepsilon}{1+b-a}$ for all x. Estimating F,

$$|F(y) - F(y_0)| = \left| \int_a^b [f(x,y) - f(x,y_0)]\,dx \right| \le \int_a^b |f(x,y) - f(x,y_0)|\,dx \le \int_a^b \frac{\varepsilon\,dx}{1+b-a} < \varepsilon,$$

thus, F is continuous on $[c,d]$.

But if there is no differentiability of f with respect to y we can't expect it for F. For example, if $f(x,y) = |y|$ on $[-2,2] \times [-1,1]$ then $F(y) = 4|y|$ is not differentiable on $y \in [-2,2]$.

(10.)*From first principles, show that* $\mathcal{B} = \{(x,y) : x^2 + y^2 \le 1\}$ *is a Jordan domain and that* $\mathrm{V}(\mathcal{B}) = \pi$.

Let $R = [-1,1] \times [-1,1]$. Consider the uniform grid $\mathcal{G}_n$ whose squares have sides $\frac{1}{n}$ so there are a total of $4n^2$ squares in the grid. The diameter of a square R_i of this grid is at most the length of the diagonal $\frac{\sqrt{2}}{n} < \frac{2}{n}$. Let us count the number of squares that touch the circle $\partial\mathcal{B}$. If $R_i \cap \partial\mathcal{B} \ne \emptyset$, then $R_i \subseteq A_n$ where $A_n = \{(x,y) : 1 - \frac{2}{n} < \sqrt{x^2+y^2} < 1 + \frac{2}{n}\}$ is the annulus of width $\frac{4}{n}$ straddling the unit circle. A quickie argument goes as follows (but is circular, since it presumes the area of a circle!) Since $R_i \cap \partial\mathcal{B} \ne \emptyset$ then $R_i \subseteq A_n$ so that $U(\chi_{\partial\mathcal{B}}, \mathcal{G}_n) = \mathrm{V}\left(\bigcup_{R_i \cap \partial\mathcal{B} \ne \emptyset} R_i\right) \le \mathrm{V}(A_n) = \pi\left((1+\frac{2}{n})^2 - (1-\frac{2}{n})^2\right) = \frac{8\pi}{n} \to 0$ as $n \to \infty$.

To make an more honest argument that doesn't beg the question, let's estimate the number of R_i's that touch arc $\partial\mathcal{B}$ in $y \ge |x|$. The function $y = \sqrt{1-x^2}$ has slope bounded by $|y'| \le 1$ over the interval $|x| \le \frac{1}{\sqrt{2}}$. It follows that in any vertical column of squares, the largest number of squares for $|x| < \frac{1}{\sqrt{2}}$ that can touch the segment is three. As there are $2n$ squares between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, then there are at most $3 \cdot 2n = 6n$ squares that covers one of the arcs (of course there are fewer, as the ones near ± 1 miss the arc altogether.) The left and right arcs can be counted in the same way after reversing the roles of x and y. As there are four arcs in $\partial\mathcal{B}$, so there are at most $4 \cdot 6n = 24n$ that touch $\partial\mathcal{B}$. it follows that $U(\chi_{\partial\mathcal{B}}, \mathcal{G}_n) \le \frac{24}{n} \to 0$ as $n \to \infty$, so that $\partial\mathcal{B}$ has content zero and $\mathcal{B}$ is a Jordan region.

To compute the volume, let's compute the volume of $Q_\varepsilon = \{(x,y) : 0 \le x, 0 \le y; \varepsilon^2 \le x^2+y^2 \le 1\}$ using a change of variables. Then $\mathrm{V}(\mathcal{B}) = \lim_{\varepsilon\to 0} 4\,\mathrm{V}(Q_\varepsilon) = \pi$. Use the polar coordinates diffeomorphism. Let $(x,y) = \phi(r,\theta) = (r\cos\theta, r\sin\theta)$ for the region $R_\varepsilon = \{(r,\theta) : \varepsilon \le r \le 1, 0 \le \theta \le \frac{\pi}{4}\}$. Then the Jacobian matrix $D\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ so $\Delta_\phi = \det(D\phi) = r$. Thus

$$\mathrm{V}(Q_\varepsilon) = \int_{Q_\varepsilon} dA(x,y) = \int_{\phi(R_\varepsilon)} dA(x,y) = \int_{R_\varepsilon} r\,dA(r,\theta) = \int_\varepsilon^1 \int_0^{\frac{\pi}{4}} r\,d\theta\,dr = \frac{\pi}{4}\left(1 - \varepsilon^2\right).$$

(11.)*Let* $E = [0,3] \times [0,4] \subseteq \mathbf{R}^2$. *Show that* f *is integrable on* E *where*

$$f(x,y) = \begin{cases} 2, & \textit{if } y = 2x-1 \textit{ and } x \textit{ is rational;} \\ 0, & \textit{if } y = 2x-1 \textit{ and } x \textit{ is irrational;} \\ \sin(x^2y^4), & \textit{otherwise.} \end{cases}$$

The function is continuous on the compact aligned rectangle E except possibly along the diagonal $y = 2x - 1$ which has content zero in E, thus f is integrable on E by a theorem.

Let us give another argument using the fact that continuous functions are integrable. The function $g(x,y) = \sin(x^2y^4)$ is continuous on $R = [0,3] \times [0,4]$ and thus is integrable. Then $f(x,y) = g(x,y) + h(x,y)$ is integrable provided that $h(x,y)$ is integtrable, where

$$f(x,y) = \begin{cases} 2 - \sin(x^2y^4), & \text{if } y = 2x-1 \text{ and } x \text{ is rational;} \\ -\sin(x^2y^4), & \text{if } y = 2x-1 \text{ and } x \text{ is irrational;} \\ 0, & \text{otherwise.} \end{cases}$$

$h(x,y)$ is nonzero on a content zero set, and thus is integrable. To see this more clearly, observe that $|h(x,y)| \le 3$. We show that for all $\varepsilon > 0$ there is a grid $\mathcal{G}_n$ of R so that $U(h,\mathcal{G}_n) - L(h,\mathcal{G}_n) < \varepsilon$. Let $\mathcal{G}_n$ be the grid consisting of squares with $6n$ horizontally and $8n$ vertically. The squares have width and height equal to $\frac{1}{2n}$. The line $\mathcal{L}$ given by $y = 2x-1$ passes through the diagonal of vertical pairs. There are $4n$ such pairs between $0.5 \le x \le 2.5$. In addition, there are $4n$ more squares that touch the line from above and $4n$ more from below. All together, $2\cdot 4n + 4n + 4n = 16n$ squares. Let $\mathcal{I}'$ denote the indices of these squares. Let $\mathcal{I}''$ be the rest of the indices. For $i \in \mathcal{I}'$ we have $-3 \le m_i \le M_1 \le 3$ since $|h| \le 3$ for all of R. But for $i \in \mathcal{I}''$ we have $m_i = M_i = 0$ because such $R_i \subseteq R \backslash \mathcal{L}$. Thus, the difference

$$\begin{aligned} U(h,\mathcal{G}_n) - L(h,\mathcal{G}_n) &= \sum_{R_i \subseteq R} (M_i - m_i)\, V(R_i) \\ &= \sum_{i \in \mathcal{I}'} (M_i - m_i)\, V(R_i) + \sum_{i \in \mathcal{I}''} (M_i - m_i)\, V(R_i) \\ &\le 6 \sum_{i \in \mathcal{I}'} V(R_i) \le \frac{6 \cdot 16n}{4n^2} \end{aligned}$$

tends to zero as $n \to \infty$.

(12.)*Prove if true; give a counterexample if false:*

Statement. *Suppose $f : [0,1] \times [0,1] \to \mathbf{R}$ is integrable. Then $F(y) = \int_0^1 f(x,y)\,dx$ is a continuous function on $[0,1]$.*

FALSE! Let

$$f(x,y) = \begin{cases} 1, & \text{if } y \ge \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f(x,y)$ is integrable because it is the characteristic function of the Jordan region $[0,1] \times [\frac{1}{2}, 1]$ (a rectangle!) Then $F(y) = f(y)$, which is not continuous.

(13.) *Let $D \subseteq \mathbb{R}^n$ be a dense set and $f : D \to \mathbf{R}^m$ be uniformly continuous. Then there is a uniformly continuous function $F : \mathbb{R}^n \to \mathbf{R}^m$ such that $F(x) = f(x)$ for all $x \in D$.*

This is like the homework problem few completed. People left out the (gory bookkeeping) details. First we define a function $F(x)$ for all $x \in \mathbb{R}^n$. Choose $x \in \mathbb{R}^n$. If $x \in D$ then let $F(x) = f(x)$. If $x \notin D$, we construct a sequence $\{x_n\} \subseteq D$ such that $x_n \to x$ and define $F(x) = \lim_{n\to\infty} f(x_n)$.

To see there is such a sequence, since D is dense, for every $n \ge 1$ there is an $x_n \in B_{1/n}(x) \cap D$. Thus $\{x_n\}$ is a Cauchy sequence: for every $\delta > 0$ there is an $N \in \mathbf{N}$ so that $k, \ell \ge N$ implies that $\|x_k - x_\ell\| < \delta$. However, f is uniformly continuous, so for every $\varepsilon > 0$ there is a $\delta > 0$ so that $\|y - z\| < \delta$ for some $y, z \in D$ implies $\|f(y) - f(z)\| < \varepsilon$. Thus taking N corresponding to this δ, for any $k, \ell \ge N$, by taking $y = x_k$ and $z = x_\ell$ we have $\|x_k - x_\ell\| < \delta$ therefore $\|f(x_n) - f(x_\ell)\| < \varepsilon$. Thus we have shown that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbf{R}^m$. Let $F(x) = \lim_{n\to\infty} f(x_n)$. Note that this construction works equally well for $x \in D$ as well as for $x \notin D$. This function may have depended on the choice of sequence converging to x. Suppose that $\{y_n\}$ is another sequence in D converging to x. Then $\{f(y_k)\}$ is also a Cauchy sequences in

$\mathbf{R}^m$ so let $L = \lim_{n\to\infty} f(y_n)$ be its limit. To show $F(x) = L$ we choose $\eta > 0$. By convergence, there are $N_1, N_2 \in \mathbf{N}$ so that $\|f(y_n) - F(x)\| < \eta$ and $\|f(y_k) - L\| < \eta$ whenever $n \ge N_1$ and $k \ge N_2$. By uniform continuity, there is a $\delta > 0$ so that if $\|z_1 - z_2\| < \delta$ for some $z_i \in D$ we have $\|f(z_1) - f(z_2)\| < \eta$. But $x_n \to x$ as $n \to \infty$ so there is an $N_3 \in \mathbf{N}$ so that $n \ge N_3$ implies $\|x - x_n\| < \frac{1}{2}\delta$. Similarly $y_k \to x$ as $k \to \infty$ so there is an $N_4 \in \mathbf{N}$ so that $k \ge N_4$ implies $\|x - x_n\| < \frac{1}{2}\delta$. Hence $\|x_n - y_k\| \le \|x_n - x + x - y_k\| \le \|x_n - x\| + \|x - y_k\| < \frac{1}{2}\delta + \frac{1}{2}\delta$. By uniform continuity, $\|f(x_n) - f(y_k)\| < \eta$. Putting it all together, if $n \ge \max\{N_1, N_3\}$ and $k \ge \max\{N_2, N_4\}$ then

$$\begin{aligned}\|F(x) - L\| =& \|F(x) - f(x_n) + f(x_n) - f(y_k) + f(y_k) - L\|\\ \le& \|F(x) - f(x_n)\| + \|f(x_n) - f(y_k)\| + \|f(y_k) - L\|\\ <& \eta + \eta + \eta.\end{aligned}$$

Since η was arbitrary, $F(x) = L$. Thus the limit does not depend on the sequence $y_k \to x$ and $F(x)$ is a well defined function of x.

Second, $F(x)$ extends f. Indeed, if $x \in D$ then for any $\eta > 0$ there is a $\delta > 0$ so that $\|f(x) - f(x_n)\| < \eta$ whenever $\|x - x_n\| < \delta$ and $x, x_n \in D$. However, since $x_n \to x$ there is an $N \in \mathbf{N}$ so that $n \ge N$ implies $\|x_n - x\| < \delta$. Thus for $n \ge N$ we have $\|f(x) - f(x_n)\| < \eta$. We have shown that $f(x_n) \to f(x)$ but since the limit is unique, $F(x) = f(x)$. (Alternatively, let $y_k = x \in D$ be the constant sequence, $y_k \to x$. Hence $f(y_k) \to f(x)$ but $f(x) = F(x)$ by uniqueness.)

Finally we show that $F(x)$ is uniformly continuous. Pick $\varepsilon > 0$. By uniform continuity of f there is a $\delta > 0$ so that $\|f(x_n) - f(y_k)\| < \frac{1}{3}\varepsilon$ whenever $x_n, y_k \in D$ and $\|y_k - x_n\| < \delta$. Let $\delta_1 = \frac{1}{3}\delta$. We show that whenever $x, y \in \mathbb{R}^n$ such that $\|x - y\| < \delta_1$ then $\|F(x) - F(y)\| < \varepsilon$ hence F is uniformly continuous on $\mathbb{R}^n$. Pick $x, y \in \mathbb{R}^n$ so that $\|x - y\| < \delta_1$. Pick sequences from D so that $x_n \to x$ and $y_k \to y$ as $n, k \to \infty$. Thus for some n, k sufficiently large, we have $\|F(x) - f(x_n)\| < \frac{1}{3}\varepsilon$, $\|x_n - x\| < \frac{1}{3}\delta$, $\|F(y) - f(y_n)\| < \frac{1}{3}\varepsilon$ and $\|y_k - y\| < \frac{1}{3}\delta$. Thus $\|x_n - y_k\| = \|x_n - x + x - y + y - y_k\| \le \|x_n - x\| + \|x - y\| + \|y - y_k\| < \frac{1}{3}\delta + \delta_1 + \frac{1}{3}\delta = \delta$. Hence $\|f(x_n) - f(y_k)\| < \frac{1}{3}\varepsilon$. Finally, $\|F(x) - F(y)\| = \|F(x) - f(x_n) + f(x_n) - f(y_k) + f(y_k) - F(y)\| \le \|F(x) - f(x_n)\| + \|f(x_n) - f(y_k)\| + \|f(y_k) - F(y)\| < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$ and we are done. □

(14.)*Let* $R = [0,1] \times [0,1]$. *Show that* $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \int_0^1 f(x,y)\,dy\,ds$ *where*

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0),\\ 0, & \text{if } (x,y) = (0,0).\end{cases}$$

The function $f(x,y)$ is continuous on R, hence it is integrable on R. The continuity is clear away from $(0,0)$ because the function is a composition of nonzero continuous functions and the denominator avoids zero. At $(0,0)$ the function is continuous because $|f(x,y)| \le x^2 + y^2 \to 0$ as $(x,y) \to (0,0)$. Therefore also, for each $x \in [0,1]$, the function $f(x,\cdot)$ is integrable on $[0,1]$ and for each $y \in [0,1]$, the function $f(\cdot,y)$ is integrable on $[0,1]$. Hence Fubini's Theorem applies. It provides that both iterated integrals agree with the two dimensional integral, answering the question:

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx\right)dy = \int_R f(x,y)\,dA = \int_0^1 \left(\int_0^1 f(x,y)\,dy\right)dx$$

(15.)*Let* $Q = [0,1]^2$ *and* $\mathbf{y} = (1,1)$. *Find* $\int_Q e^{-\mathbf{x}\cdot\mathbf{y}} d\mathbf{x}$.

$f(x_1, x_2) = e^{-\mathbf{x}\cdot\mathbf{y}} = e^{-x_1 - x_2}$ is a continuous function on the rectangle Q, therefore is integrable on Q and for each $z \in [0,1]$, the functions $f(z,\cdot)$ and $f(\cdot, z)$ are integrable on $[0,1]$. Thus Fubini's Theorem applies and we can evaluate the integral using iterated integrals. Thus

$$\int_Q e^{-\mathbf{x}\cdot\mathbf{y}} dA = \int_0^1 \int_0^1 e^{-x_1} e^{-x_2}\, dx_1\, dx_2 = (1 - e^{-1}) \int_0^1 e^{-x_2} dx_2 = (1 - e^{-1})^2.$$

(16.)*Show that* f *is not integrable on* $R = [0,1] \times [0,1]$, *where*

$$f(x,y) = \begin{cases} 1, & \text{if } y \in \mathbf{Q}, \\ x, & \text{if } y \notin \mathbf{Q}. \end{cases}$$

We show that for all partitions $\mathcal{G}$ of R, the upper and lower sums stay apart. Indeed, let $\mathcal{H}$ be the partition $\{R_1', R_2'\}$ where $R_1' = [0, 0.5] \times [0,1]$ and $R_2' = [0.5, 1] \times [0,1]$. Choose a partition $\mathcal{G}$ and let $\mathcal{K}$ be any common refinement of $\mathcal{G}$ and $\mathcal{H}$. For any nondegenerate rectangle R_i of $\mathcal{K}$ such that $R_i \subseteq R_1'$ we have $M_i = 1$ because the rationals are dense in y so R_i contain both rational and irrational points. Also $m_i \le 0.5$ because $x \le 0.5$ on R_1'. It follows that

$$\begin{aligned} U(f,\mathcal{G}) - L(f,\mathcal{G}) &\ge U(f,\mathcal{K}) - L(f,\mathcal{K}) \\ &= \sum_{R_i \subseteq R} (M_i - m_i) V(R_i) \\ &\ge (1 - 0.5) \sum_{R_i \subseteq R_1'} V(R_i) \\ &= 0.5\, \mathrm{V}(R_1') = 0.25, \end{aligned}$$

so the difference cannot approach zero which it must do for f to be integrable.

(17)*Evaluate the following integrals*

(a.)*Suppose* $0 < a < b$, *find* $I = \int_a^b \int_0^x \sqrt{x^2 + y^2}\, dy\, dx$

Let $Q = [a,b] \times [0,1]$. Consider the transformation $\phi(u,v) = (u, uv)$ so $\phi(Q) = \{(x,y) \in \mathbf{R}^2 : a \le x \le b,\ 0 \le y \le x\}$. $D\phi(u,v) = \begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix}$ so $\Delta_\phi = \det(D\phi) = u$. The change of variables formula is

$$\begin{aligned} I &= \int_{\phi(Q)} \sqrt{x^2 + y^2}\, dx\, dy = \int_Q \sqrt{u^2 + u^2 v^2}\, u\, du\, dv \\ &= \int_a^b \int_0^1 u^2 \sqrt{1 + v^2}\, dv\, du = \frac{b^3 - a^3}{3} \left(\frac{1}{\sqrt{2}} + \log(1 + \sqrt{2}) \right). \end{aligned}$$

(b.)*Let* E *be the trapezoid with vertices* $(1,1)$, $(2,2)$, $(2,0)$, $(4,0)$. *Find* $J = \int_E \exp\left(\frac{y - x}{y + x}\right) dA$

Let the diffeomorphism be defined by $\phi(s,t) = (\frac{1}{2}s(1+t), \frac{1}{2}s(1-t))$. Let $D = [2,4] \times [0,1]$. Then $\phi(D) = E$. Also, $D\phi(s,t) = \frac{1}{2} \begin{pmatrix} 1+t & s \\ 1-t & -s \end{pmatrix}$ so $|\Delta_\phi| = |\det(D\phi)| = \frac{s}{2}$. thus, by the change of variables formula $J = \int_{\phi(D)} f(z)\,dV(z) = \int_D f(\phi(\sigma))\,|\Delta_\phi(\sigma)|\,dV(\sigma)$ so

$$J = \int_{\phi(D)} \exp\left(\frac{y-x}{y+x}\right) dx\,dy = \frac{1}{2}\int_D e^{-t} s\,ds\,dt = \frac{1}{2}\int_2^4 \int_0^1 s e^{-t} dt\,ds = 3(1-e^{-1}).$$

(18.)*Suppose* $V \subseteq \mathbb{R}^n$ *is an open set and* $\phi : V \to \mathbb{R}^n$ *is continuously differentiable with* $\Delta_\phi \neq 0$ *on* V*. Prove that for every* $\mathbf{x}_0 \in V$,

$$\lim_{r\to 0+} \frac{\mathrm{V}\left(\phi(B_r(\mathbf{x}_0))\right)}{\mathrm{V}(B_r(\mathbf{x}_0))} = |\Delta_\phi(\mathbf{x}_0)|.$$

We are given that ϕ is continuously differentiable in a neighborhood of zero. Thus for $0 < r$ small so that $B_r(0) \subseteq U$ we have $D\phi(x)$ is continuous in $B_r(0)$ as it is the determinant of the continuous matrix function $D\phi(x)$. By continuity, $\eta(r) = \sup\{|\Delta_\phi(x) - \Delta_\phi(0)| : x \in B_r(0)\} \to 0$ as $r \to 0$. Using the change of variables formula,

$$\begin{aligned}\left|\frac{\mathrm{V}\left(\phi(B_r(\mathbf{x}_0))\right)}{\mathrm{V}(B_r(\mathbf{x}_0))} - |\Delta_\phi(0)|\right| &= \left|\frac{\int_{\phi(B_r(\mathbf{x}_0))} dV(\mathbf{y})}{\int_{B_r(\mathbf{x}_0)} dV(\mathbf{x})} - |\Delta_\phi(0)|\right| = \frac{\left|\int_{B_r(\mathbf{x}_0)} (|\Delta_\phi(\mathbf{x})| - |\Delta_\phi(0)|)\,dV(\mathbf{x})\right|}{\int_{B_r(\mathbf{x}_0)} dV(\mathbf{x})} \\ &\le \frac{\int_{B_r(\mathbf{x}_0)} \Big||\Delta_\phi(\mathbf{x})| - |\Delta_\phi(0)|\Big|\,dV(\mathbf{x})}{\int_{B_r(\mathbf{x}_0)} dV(\mathbf{x})} \le \frac{\int_{B_r(\mathbf{x}_0)} \eta(r)\,dV(\mathbf{x})}{\int_{B_r(\mathbf{x}_0)} dV(\mathbf{x})} = \eta(r)\end{aligned}$$

which tends to zero as $r \to 0$.