Math 3220 § 2. **Third Midterm Exam** Name:____________________
Treibergs April 3, 2013

(1.) *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function and $\mathbf{a}\in\mathbb{R}^2$ a point. State the definition: f is a differentiable at $\mathbf{a}$. Determine whether f is differentiable at $(0,0)$ and prove your answer, where*

$$f(x,y)=\begin{cases}\dfrac{(x+y)^4}{x^2+y^2}, & \text{if } (x,y)\neq(0,0);\\[2ex] 0, & \text{if } (x,y)=(0,0).\end{cases}$$

Definition: f is *differentiable* at $\mathbf{a}\in\mathbb{R}^2$ if there is a 1×2 real matrix L such that

$$\lim_{\mathbf{h}\to(0,0)}\frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-L\mathbf{h}}{\|\mathbf{h}\|}=0.$$

For the given function we observe that $f(t,0)=\dfrac{t^4}{t^2}=t^2$ so that $f_x(0,0)=0$ and that $f(0,t)=\dfrac{t^4}{t^2}=t^2$ so that $f_y(0,0)=0$. It follows that if f were differentiable at $\mathbf{a}=(0,0)$, then the differential would have to be given by the Jacobiabn matrix $L=\left[f_x(0,0),f_y(0,0)\right]=[0,0]$. Hence for $\mathbf{h}=(x,y)\neq(0,0)$ and $\mathbf{a}=(0,0)$,

$$\begin{aligned}\frac{|f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-L\mathbf{h}|}{\|\mathbf{h}\|}&=\frac{|f(x,y)-f(0,0)-[0,0][x,y]^T|}{\|(x,y)\|}=\frac{\left|\dfrac{(x+y)^4}{x^2+y^2}-0-0\right|}{\sqrt{x^2+y^2}}\\&=\frac{|x+y|^4}{(x^2+y^2)^{3/2}}\le\frac{(2\sqrt{x^2+y^2})^4}{(x^2+y^2)^{3/2}}=16\sqrt{x^2+y^2}=16\|\mathbf{h}\|\to 0\end{aligned}$$

as $\mathbf{h}=(x,y)\to(0,0)$. In this estimate we used $|x+y|\le|x|+|y|\le\sqrt{x^2+y^2}+\sqrt{x^2+y^2}$.

(2.) *Let f be a real valued function defined on $\mathbb{R}^p$. Suppose that all first partial derivatives of f exist and are differentiable at all points of $\mathbb{R}^p$. Assume that at some point $\mathbf{a}\in\mathbb{R}^p$, the differential vanishes, $df(\mathbf{a})=0$, and that the matrix of second derivatives $d^2f(\mathbf{x})$ is positive definite at all points. Show that if $\mathbf{x}\neq\mathbf{a}$ then $f(\mathbf{x})>f(\mathbf{a})$.*

The given conditions are exactly those required to represent $f(\mathbf{x})$ by the first order Taylor's formula for f at $\mathbf{a}$. Thus if $\mathbf{x}\neq\mathbf{a}$ then

$$f(\mathbf{x})=f(\mathbf{a})+df(\mathbf{a})(\mathbf{x}-\mathbf{a})+\frac12 d^2f(\mathbf{c})(\mathbf{x}-\mathbf{a})^2>f(\mathbf{a})+0+0$$

where $\mathbf{c}$ is in interior point on the line segment $[\mathbf{a},\mathbf{x}]$ from $\mathbf{a}$ to $\mathbf{x}$, $df(\mathbf{a})=\mathbf{0}$ by hypothesis and at the point $\mathbf{c}\in\mathbb{R}^p$, $df^2(\mathbf{c})$ is positive definite. This means that for $\mathbf{h}=\mathbf{x}-\mathbf{a}\neq\mathbf{0}$ we have $df^2(\mathbf{c})(\mathbf{h})^2=\mathbf{h}^Td^2f(\mathbf{a})\mathbf{h}>0$.

(3.) *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

1. **Statement.** *Let L be a $q \times p$ real matrix, M be a $q \times q$ real matrix and $F(\mathbf{x}, \mathbf{y}) = L\mathbf{x} + M\mathbf{y}$. Then there is a function $g : \mathbb{R}^p \to \mathbb{R}^q$ such that if $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $F(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, then $\mathbf{y} = g(\mathbf{x})$.*

   FALSE. Suppose M is not invertible such as $M = 0$, then we may find two unequal vectors $\mathbf{y} \neq \mathbf{z}$ such that $M\mathbf{y} = M\mathbf{z} = \mathbf{0}$ and let $\mathbf{x} = \mathbf{0}$. We have $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{z}) = \mathbf{0}$ but there is no function because $g(\mathbf{x})$ cannot take both valves $\mathbf{y}$ and $\mathbf{z}$.

2. **Statement.** *Suppose $F \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. Then F is an open mapping.*

   FALSE. For example $f(x, y) = (x^2, y^2)$ is $\mathcal{C}^1$ because it is a polynomial, but it doesn't take open sets to open sets. It takes the open unit ball $B_1(0,0)$ to the set $\{(u, v) \in \mathbb{R}^2 : 0 \le u,\ 0 \le v \text{ and } u + v < 1\}$ which is not open.

3. **Statement.** *Suppose that $f \in \mathcal{C}^2(\mathbb{R}^2)$. Then $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$.*

   TRUE. We have a theorem that if first partial derivatives and $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$ exist in a neighborhood of $(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$ is continuous at $(0,0)$, then the other cross partial exists and is equal: $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$. In this case, the hypothesis gives us that $f \in \mathcal{C}^2(\mathbb{R}^2)$, which means that all partial derivatives up to second order exist and are continuous at all points.

(4.) *Show that if $F = (f_1, f_2) : \mathbb{R}^3 \to \mathbb{R}^2$ is a $\mathcal{C}^1$ function and $\mathbf{a} \in \mathbb{R}^3$ is a point where $dF(\mathbf{a})$ has rank 2, then there is a $\mathcal{C}^1$ function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ has a $\mathcal{C}^1$ inverse near $\mathbf{a}$.*

Since the differential

$$df(\mathbf{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \frac{\partial f_1}{\partial x_3}(\mathbf{a}) \\ \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \frac{\partial f_2}{\partial x_3}(\mathbf{a}) \end{pmatrix} = \begin{pmatrix} \nabla f_1(\mathbf{a}) \\ \\ \nabla f_2(\mathbf{a}) \end{pmatrix}$$

is rank two, the rows are independent. So there is a vector $\mathbf{c} \in \mathbb{R}^3$ which completes two independent vectors to a basis $\{\nabla f_1(\mathbf{a}), \nabla f_2(\mathbf{a}), \mathbf{c}\}$ (for example $\mathbf{c} = \nabla f_1(\mathbf{a}) \times \nabla f_2(\mathbf{a})$). Let $f_3(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$. Then $f_3 \in \mathcal{C}^1$ as it is polynomial and $\nabla f_3(\mathbf{a}) = \mathbf{c}$. Let $\Phi = (f_1, f_2, f_3)$ which is $\mathcal{C}^1$ since all f_i are. The differential

$$d\Phi(\mathbf{a}) = \begin{pmatrix} \nabla f_1(\mathbf{a}) \\ \\ \nabla f_2(\mathbf{a}) \\ \\ \mathbf{c} \end{pmatrix}$$

is invertible since the rows are independent. The conditions for the inverse function theorem are satisfied. Hence there is open set $U \in \mathbb{R}^3$ such that $\mathbf{a} \in U$ and $V = \Phi(U)$ is an open set, and there is a function $G \in \mathcal{C}^1(V, U)$ such that G is a local inverse of Φ: $G \circ \Phi(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in U$ and $\Phi \circ G(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in V$.

(5.) *Using Lagrange Multipliers, find where the function $f(x,y,z) = x+y+z$ attains its maximum on the set S.*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 5 \text{ and } x - z = 1\right\}.$$

Observe that S is the intersection of a circular cylinder with a slanted plane so it is an elliptical curve in three space, thus a compact smooth parameterized one dimensional "surface." Hence the smooth function f takes its maximum value on S and the point where the maximum occurs will solve the Lagrange Multiplier equation.

The necessary conditions for the coordinates of the extreme values are first, that the constraint equations be satisfied

$$\begin{aligned} g_1(x,y,z) &= x^2 + y^2 = 5 \\ g_2(x,y,z) &= x - z = 1 \end{aligned}$$

and second, that the Lagrange Multiplier equation holds for the unknown constants λ and μ

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g_1(x,y,z) + \mu \nabla g_2(x,y,z) \\ (1,1,1) &= \lambda(2x, 2y, 0) + \mu(1, 0, -1) \end{aligned}$$

or, equivalently,

$$\begin{aligned} 1 &= 2\lambda x + \mu \\ 1 &= 2\lambda y \\ 1 &= -\mu. \end{aligned}$$

Thus $\lambda \neq 0$ and

$$\mu = -1, \qquad x = \frac{1}{\lambda}, \qquad y = \frac{1}{2\lambda}.$$

Inserting into the first constraint,

$$5 = x^2 + y^2 = \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = \frac{5}{4\lambda^2}$$

so

$$\lambda = \pm\frac{1}{2}.$$

If $\lambda = +\frac{1}{2}$ then from the second constraint equation

$$x = \frac{1}{\lambda} = 2; \qquad y = \frac{1}{2\lambda} = 1, \qquad z = x - 1 = 1, \qquad f(2,1,1) = 4.$$

If $\lambda = -\frac{1}{2}$ then

$$x = \frac{1}{\lambda} = -2; \qquad y = \frac{1}{2\lambda} = -1, \qquad z = x - 1 = -3, \qquad f(-2,-1,-3) = -6.$$

We have found all solutions of the necessary conditions. Since the maximum of f on S is one of these points, it is the solution corresponding to the greatest f, namely, when $\lambda = \frac{1}{2}$ at the point $(2,1,1) \in S$ where $f = 4$.