Math 3220 § 1. **First Midterm Exam** Name: Solutions
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1. *Using just the definition of convergence in* $\mathbf{R}^2$*, show that the limit exists:* $\lim_{n\to\infty}\left(\frac{\sin n}{\sqrt{n}}, \frac{1}{2^n}\right)$.

Proof. First, observe that each component converges to zero so set $\mathbf{x} = (0, 0)$. Using $\sin^2 n \le 1$ and $2^{2n} \ge n$ for all $n \in \mathbb{N}$ we get

$$\|\mathbf{x}_n - \mathbf{x}\| = \left(\frac{\sin^2 n}{n} + \frac{1}{2^{2n}}\right)^{\frac{1}{2}} \le \left(\frac{1}{n} + \frac{1}{n}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{n}}. \tag{1}$$

To see that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$, choose $\varepsilon > 0$. Let $N = \frac{2}{\varepsilon^2}$. For any $n > N$, by (1), $\|\mathbf{x}_n - \mathbf{x}\| \le \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} = \varepsilon$ thus convergence is proved. □

2. *Let* $\{\mathbf{x}_k\}$ *and* $\{\mathbf{y}_k\}$ *be sequences in* $\mathbf{R}^n$ *and* $\mathbf{x}$ *and* $\mathbf{y}$ *be points in* $\mathbf{R}^n$*. Suppose* $\mathbf{x}_k \to \mathbf{x}$*, and* $\mathbf{y}_k \to \mathbf{y}$ *as* $k \to \infty$*. Show* $\|\mathbf{x}_k\|\,\mathbf{y}_k \to \|\mathbf{x}\|\,\mathbf{y}$ *as* $k \to \infty$.

Proof. The first argument uses the Computation Theorem from the chapter: Since $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ in $\mathbf{R}^n$, then the norm $\|\mathbf{x}_k\| \to \|\mathbf{x}\|$ as $k \to \infty$. Also, whenever there is a sequence of constants $c_k \to c$ and vectors in $\mathbf{y}_k \to \mathbf{y}$ as $k \to \infty$ in $\mathbf{R}^n$ then $c_k\mathbf{y}_k \to c\mathbf{y}$ as $k \to \infty$. Hence, taking $c_k = \|\mathbf{x}_k\|$ yields the result. □

Many arguments are acceptable. The other extreme is just to use the definition.

Proof. First, the convergence $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ implies that $\|\mathbf{x}_k\|$ is bounded. Choose $\varepsilon = 1$. There is N_1 so that if $k > N_1$ then $\|\mathbf{x}_k - \mathbf{x}\| < 1$. For these k,

$$\|\mathbf{x}_k\| = \|\mathbf{x}_k - \mathbf{x} + \mathbf{x}\| \le \|\mathbf{x}_k - \mathbf{x}\| + \|\mathbf{x}\| < 1 + \|\mathbf{x}\|.$$

Adding and subtracting the intermediate term as for product problems and using boundedness and the reverse triangle inequality,

$$\begin{aligned}
\big\|\|\mathbf{x}_k\|\,\mathbf{y}_k - \|\mathbf{x}\|\,\mathbf{y}\big\| &= \big\|\|\mathbf{x}_k\|\,\mathbf{y}_k - \|\mathbf{x}_k\|\,\mathbf{y} + \|\mathbf{x}_k\|\,\mathbf{y} - \|\mathbf{x}\|\,\mathbf{y}\big\| \\
&= \big\|\|\mathbf{x}_k\|\,(\mathbf{y}_k - \mathbf{y}) + (\|\mathbf{x}_k\| - \|\mathbf{x}\|)\,\mathbf{y}\big\| \\
&\le \big\|\|\mathbf{x}_k\|\,(\mathbf{y}_k - \mathbf{y})\big\| + \big\|(\|\mathbf{x}_k\| - \|\mathbf{x}\|)\,\mathbf{y}\big\| \\
&= \|\mathbf{x}_k\|\,\|\mathbf{y}_k - \mathbf{y}\| + \big|\|\mathbf{x}_k\| - \|\mathbf{x}\|\big|\|\mathbf{y}\| \\
&\le \big(1 + \|\mathbf{x}\|\big)\,\|\mathbf{y}_k - \mathbf{y}\| + \|\mathbf{x}_k - \mathbf{x}\|\|\mathbf{y}\|.
\end{aligned}$$

Using the convergence $\mathbf{x}_k \to \mathbf{x}$ and $\mathbf{y}_k \to \mathbf{y}$ as $k \to \infty$ for any $\varepsilon > 0$ there is an N_2 so that if $k > N_2$, $\|\mathbf{x}_k - \mathbf{x}\| < \frac{\varepsilon}{1+\|\mathbf{x}\|+\|\mathbf{y}\|}$. Also there is an N_3 so that if $k > N_2$, $\|\mathbf{y}_k - \mathbf{y}\| < \frac{\varepsilon}{1+\|\mathbf{x}\|+\|\mathbf{y}\|}$. Put $N = \max\{N_1, N_2, N_3\}$. Now for any $k > N$,

$$\big\|\|\mathbf{x}_k\|\,\mathbf{y}_k - \|\mathbf{x}\|\,\mathbf{y}\big\| < \frac{\big(1+\|\mathbf{x}\|\big)\varepsilon}{1+\|\mathbf{x}\|+\|\mathbf{y}\|} + \frac{\|\mathbf{y}\|\varepsilon}{1+\|\mathbf{x}\|+\|\mathbf{y}\|} = \varepsilon. \quad \square$$

3. *State the definition:* (X, δ) *is a* metric space. *Let* $\|\mathbf{u} - \mathbf{v}\|$ *be the metric for* $\mathbf{R}^n$ *as usual. Show that* $\hat\delta$ *is another metric, where* $\hat\delta(\mathbf{u}, \mathbf{v}) = \frac{\|\mathbf{u}-\mathbf{v}\|}{1+\|\mathbf{u}-\mathbf{v}\|}$.

A *metric space* is a set X and a function $\delta : X \times X \to \mathbf{R}$ such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ all three conditions hold:

a. $\delta(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y}, \mathbf{x})$

b. $\delta(\mathbf{x}, \mathbf{y}) \ge 0$ and $\delta(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

c. $\delta(\mathbf{x},\mathbf{z}) \le \delta(\mathbf{x},\mathbf{y}) + \delta(\mathbf{y},\mathbf{z})$.

Proof. We check that all three properties hold for $\hat\delta$. The first condition follows from $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{y}-\mathbf{x}\|$: $\hat\delta(\mathbf{x},\mathbf{y}) = \frac{\|\mathbf{x}-\mathbf{y}\|}{1+\|\mathbf{x}-\mathbf{y}\|} = \frac{\|\mathbf{y}-\mathbf{x}\|}{1+\|\mathbf{y}-\mathbf{x}\|} = \hat\delta(\mathbf{y}-\mathbf{x})$.

The function $f(s) = \frac{s}{1+s}$ is stictly increasing on $0 \le s < \infty$. The second condition follows for $s \ge 0$ from $f(s) \ge 0$ and $f(s) = 0$ if and only if $s = 0$, and properties of $\|\mathbf{x}-\mathbf{y}\|$. Let $s = \|\mathbf{x}-\mathbf{y}\| \ge 0$ by positivity of norm. Then $\hat\delta(\mathbf{x},\mathbf{y}) = f(s) \ge 0$ and if $0 = \hat\delta(\mathbf{x},\mathbf{y}) = f(s)$ then $s = \|\mathbf{x}-\mathbf{y}\| = 0$ which implies $\mathbf{x} = \mathbf{y}$ by positive definiteness.

By the usual triangle inequality, $\|\mathbf{x}-\mathbf{z}\| \le \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|$ so by monotonicity of f, $\hat\delta(\mathbf{x},\mathbf{z}) = f(\|\mathbf{x}-\mathbf{z}\|) \le f(\|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|)$. Hence

$$\begin{aligned}\hat\delta(\mathbf{x},\mathbf{z}) &\le \frac{\|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|}{1+\|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|} = \frac{\|\mathbf{x}-\mathbf{y}\|}{1+\|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|} + \frac{\|\mathbf{y}-\mathbf{z}\|}{1+\|vx-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|}\\ &\le \frac{\|\mathbf{x}-\mathbf{y}\|}{1+\|\mathbf{x}-\mathbf{y}\|} + \frac{\|\mathbf{y}-\mathbf{z}\|}{1+\|\mathbf{y}-\mathbf{z}\|} = \hat\delta(\mathbf{x},\mathbf{y}) + \hat\delta(\mathbf{y},\mathbf{z}).\end{aligned}$$

□

4. *Let $\{\mathbf{x}_k\}$ be a sequence in $\mathbf{R}^n$ and $M < \infty$, $r < 1$ be constants such that the norm $\|\mathbf{x}_k\| \le Mr^k$ for all k. Show that the infinite sum $\sum_{k=1}^\infty \mathbf{x}_k$ converges.*

Proof. The infinte sum converges provided that the sequence of partial sums converge. Let $\mathbf{S}_n = \sum_{k=1}^n \mathbf{x}_k$. In $\mathbf{R}^n$, since a Cauchy sequence is convergent, it suffices to show that $\{\mathbf{S}_n\}$ is a Cauchy sequence. Choose $\varepsilon > 0$. Let $N = \log(\varepsilon(1-r)/M)/\log r$. Suppose that both $n, m > N$. If $n = m$ then $\|\mathbf{S}_n - \mathbf{S}_m\| = 0 < \varepsilon$. Hence, after swapping if necessary, we may suppose $n > m$. Thus, using the triangle inequality with many terms, the hypothesis and the formula for a geometric sum,

$$\begin{aligned}\|\mathbf{S}_n - \mathbf{S}_m\| &= \|\textstyle\sum_{k=1}^n \mathbf{x}_k - \sum_{k=1}^m \mathbf{x}_k\| = \left\|\sum_{k=m+1}^n \mathbf{x}_k\right\| \le \sum_{k=m+1}^n \|\mathbf{x}_k\|\\ &\le \textstyle\sum_{k=m+1}^n Mr^k = \frac{Mr^{m+1}\left(1-r^{n-m}\right)}{1-r} < \frac{Mr^{m+1}}{1-r} < \frac{Mr^N}{1-r} = \varepsilon.\end{aligned}$$

□

5. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample. For these problems, $\mathbf{R}^2$ is endowed with the usual real vector space structure.*

a. *The function $\mathbf{u} \,\tilde{\bullet}\, \mathbf{v} = u_1v_1 - u_2v_2$ provides another inner-product for $\mathbf{R}^2$.*

FALSE. The function $\tilde{\bullet}$ is not positive definite. For $\mathbf{u} = (1,3)$ we get $\mathbf{u} \,\tilde{\bullet}\, \mathbf{u} = 1\cdot 1 - 3\cdot 3 = -8$ which should be positive for an inner-poduct.

b. *The function $\|\!|\mathbf{u}|\!\| = |u_1| + 2|u_2|$ provides another norm for $\mathbf{R}^2$.*

TRUE. The function satisfies the three conditions:

It is positively multiplicative: for all $\mathbf{u} \in \mathbf{R}^2$ and $\alpha \in \mathbf{R}$, $\|\!|\alpha\mathbf{u}|\!\| = |\alpha u_1| + 2|\alpha u_2| = |\alpha|(|u_1| + 2|u_2|) = |\alpha|\|\!|\mathbf{u}|\!\|$.

It is positive definite: for all $\mathbf{u} \in \mathbf{R}^2$, $\|\!|\mathbf{u}|\!\| = |u_1| + 2|u_2| \ge 0$ and $\|\!|\mathbf{u}|\!\| = 0$ if and only if $|u_1| + 2|u_2| = 0$ if and only if $|u_1| = 0$ and $|u_2| = 0$ if and only if $\mathbf{u} = (u_1, u_2) = (0,0)$.

It satisfies the triangle inequality: for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$, $\|\!|\mathbf{u}+\mathbf{v}|\!\| = |u_1 + v_1| + 2|u_2 + v_2| \le |u_1| + |v_1| + 2|u_2| + 2|v_2| = (|u_1| + 2|u_2|) + (|v_1| + 2|v_2|) = \|\!|\mathbf{u}|\!\| + \|\!|\mathbf{v}|\!\|$

c. *The function $\tilde\delta(\mathbf{u},\mathbf{v}) = |u_1 - v_1| + |u_2 - v_2|^2$ provides another metric for $\mathbf{R}^2$.*

FALSE. The triangle inequality fails. For example if $\mathbf{u} = (1,1)$, $\mathbf{v} = (3,7)$ and $\mathbf{w} = (2,4)$ then $\tilde\delta(\mathbf{u},\mathbf{v}) = 2 + 36 = 38$, $\tilde\delta(\mathbf{u},\mathbf{w}) = 1 + 9 = 10$ and $\tilde\delta(\mathbf{w},\mathbf{v}) = 1 + 9 = 10$ and so $38 = \tilde\delta(\mathbf{u},\mathbf{v}) \not\le \tilde\delta(\mathbf{u},\mathbf{w}) + \tilde\delta(\mathbf{w},\mathbf{v}) = 20$.