Math 3220 § 1. **Sample Problems for Third Midterm** Name: Problems With Solutions
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Questions 1–6 appeared in my Fall 2000 and Fall 2001 Math 3220 exams.

(1.) Let $E = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 2\}$. Using only the definition of connectedness, the fact that intervals are the only connected sets connected in $\mathbf{R}^1$, and properties of continuous functions, show that E is a connected subset of $\mathbf{R}^2$.

(2.) For each part, determine whether the statement is TRUE or FALSE. Give a reason if true, a counterexample if false.

(a.) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $G \subseteq \mathbf{R}^m$ be open. Then for any point $x \in f^{-1}(G)$ there is a $\delta > 0$ so that the open δ-ball about x, $B_\delta(x) \subseteq f^{-1}(G)$.

(b.) Let $\Omega \subseteq \mathbf{R}^n$ be open and $f : \Omega \to \mathbf{R}^m$ be continuous. Then $f(\Omega)$ is open.

(c.) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $E \subseteq \mathbf{R}^m$. Suppose E is connected in $\mathbf{R}^m$. Then $f^{-1}(E)$ is connected in $\mathbf{R}^n$.

(3.) Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable at $a \in \mathbf{R}^n$.

(a.) Let $g : \mathbf{R}^n \to \mathbf{R}$ be defined by $g(x) = x \bullet f(x)$, (dot product.) Find the total derivative (differential) $Dg(a)(h)$ where $h \in \mathbf{R}^n$.

(b.) Without using the product theorem, prove your answer.

(4.) For each part, determine whether the statement is TRUE or FALSE. If the statement is true, give a justification. If the statement is false, give a counterexample. You may use theorems.

(a.) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous. Suppose that for all $(x, y) \in \mathbf{R}^2$ both

$$\lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad \text{and} \qquad \lim_{k\to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

exist. Then f is differentiable at $(1, 2)$.

(b.) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$ function for which the third partial derivatives $f_{xxy}(x, y)$ exist for all $(x, y) \in \mathbf{R}^2$ such that $f_{xxy}(x, y)$ is continuous at $(0, 0)$. Then $f_{xyx}(0, 0)$ and $f_{yxx}(0, 0)$ exist and are equal $f_{xxy}(0, 0) = f_{xyx}(0, 0) = f_{yxx}(0, 0)$.

(5.) Let $K \subseteq \mathbf{R}^2$ be a compact set. Suppose $\{\mathbf{x}_n\}_{n\in\mathbf{N}} \subseteq K$ is a sequence in K which is a Cauchy sequence in $\mathbf{R}^2$. Then there is a point $\mathbf{k} \in K$ so that $\mathbf{x}_n \to \mathbf{k}$ as $n \to \infty$.

(6.) Suppose $\mathbf{f}, \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$. Assume that $\mathbf{g}$ is differentiable at $\mathbf{x}_0 \in \mathbf{R}^2$ and that for some $\alpha > 1$ and $M < \infty$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\| \le M\|\mathbf{x} - \mathbf{x}_0\|^\alpha$$

for all $\mathbf{x} \in \mathbf{R}^2$. Show that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ and that $D\mathbf{f}(\mathbf{x}_0) = D\mathbf{g}(\mathbf{x}_0)$.

(7.) Suppose $S_i \subseteq \mathbf{R}^n$ are closed nonempty sets which are contained in the compact set K. Assume that the subsets form a decreasing sequence $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Then they have a nonempty intersection $\bigcap_{i\in\mathbf{N}} S_i \ne \emptyset$.

(8.) $E = [0, 1] \cap \mathbf{Q}$, the set of rational points between zero and one, is not compact.

(9.) Theorem. Suppose $E \subseteq \mathbf{R}^n$ is bounded and $f : E \to \mathbf{R}^m$ is uniformly continuous. Then $f(E)$ is bounded. This would not be true if "uniformly continuous" were replaced by "continuous."

(10.) Theorem. Let $\mathcal{S} = [0, 1] \times [0, 1] \subseteq \mathbf{R}^2$ and $F : \mathcal{S} \to \mathbf{R}$ be continuous. Then F is not one to one.

(11.) TRUE or FALSE? If true, give a justification. If false, give a counterexample. You may use theorems.

(a.) Let $f : \mathbf{R}^2 \to \mathbf{R}$. Suppose that both iterated limits exist and are equal. The the limit exists.

$$L = \lim_{x\to 0} \lim_{y\to 0} f(x, y) = \lim_{y\to 0} \lim_{x\to 0} f(x, y) \qquad \Longrightarrow \qquad L = \lim_{(x,y)\to(0,0)} f(x, y)$$

(b.) Suppose $f : \mathbf{R}^3 \to \mathbf{R}^2$ is given below. Then f is differentiable on $\mathbf{R}^3$.

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy + x^2z^3 \\ x^4 + y + y^5z^6 \end{pmatrix}.$$

Solutions.

(1.) Let $E = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 2\}$. Using only the definition of connectedness, the fact that intervals are the only connected sets connected in $\mathbf{R}^1$, and properties of continuous functions, show that E is a connected subset of $\mathbf{R}^2$.

The set E is path connected. For example if $x, y \in E$ then $\sigma : [0, 1] \to E$ given by $\sigma(t) = (1-t)x + ty$ is a continuous path in E. In fact, for $0 \le t \le 1$ and using the Schwarz Inequality, $\|f(t)\|^2 = (1-t)^2\|x\|^2 + 2t(1-t)x \cdot y + t^2\|y\|^2 \le (1-t)^2\|x\|^2 + 2t(1-t)\|x\|\|y\| + t^2\|y\|^2 = ((1-t)\|x\| + t\|y\|)^2 < (2(1-t) + 2t)^2 = 4$ so $f(t) \in E$. The components of σ are polynomial so σ is continuous.

Since E is path connected, it is connected. If not there are relatively open sets A_1, A_2 in E so that $A_1 \ne \emptyset$, $A_2 \ne \emptyset$, $A_1 \cap A_2 = \emptyset$ and $E = A_1 \cup A_2$. Choose $x \in A_1$ and $y \in A_2$ and a path $\sigma : [0, 1] \to E$ so that $\sigma(0) = x$ and $\sigma(1) = y$. $\sigma^{-1}(A_1)$ and $\sigma^{-1}(A_2)$ are relatively open in $[0, 1]$, are disjoint because $A_1 \cap A_2 = \emptyset$ implies $\sigma^{-1}(A_1) \cap \sigma^{-1}(A_2) = \sigma^{-1}(A_1 \cap A_2) = \emptyset$, are nonempty because there are $x \in \sigma^{-1}(A_1)$ and $y \in \sigma^{-1}(A_2)$ and $[0, 1] \subseteq \sigma^{-1}(A_1) \cup \sigma^{-1}(A_2) = \sigma^{-1}(A_1 \cup A_2) = \sigma^{-1}(E)$. Thus $\sigma^{-1}(A_1)$ and $\sigma^{-1}(A_2)$ disconnect $[0, 1]$, which is a contradiction because $[0, 1]$ is connected.

(2.) For each part, determine whether the statement is TRUE or FALSE.

(2a.) Statement. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $G \subseteq \mathbf{R}^m$ be open. Then for any point $x \in f^{-1}(G)$ there is a $\delta > 0$ so that the open δ-ball about x, $B_\delta(x) \subseteq f^{-1}(G)$.

TRUE! Since G is open, there is $\varepsilon > 0$ so that $B_\varepsilon(f(x)) \subseteq G$. But, since f is continuous, for all positive numbers, such as this $\varepsilon > 0$, there is a $\delta > 0$ so that for all $z \in \mathbf{R}^n$, if $\|z - x\| < \delta$ then $\|f(z) - f(x)\| < \varepsilon$. We claim that for this $\delta > 0$, $B_\delta(x) \subseteq f^{-1}(G)$. To see it, choose $z \in B_\delta(x)$ to show $f(z) \in G$. But such z satisfies $\|z - x\| < \delta$ so that $\|f(z) - f(x)\| < \varepsilon$ or in other words, $f(z) \in B_\varepsilon(f(x)) \subseteq G$.

(2b.) Statement. Let $\Omega \subseteq \mathbf{R}^n$ be open and $f : \Omega \to \mathbf{R}^m$ be continuous. Then $f(\Omega)$ is open.

FALSE! Counterexample: the constant function $f(x) = c$ is continuous but $f(\Omega) = \{c\}$ is a singleton set which is not open.

(2c.) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $E \subseteq \mathbf{R}^m$. Suppose E is connected in $\mathbf{R}^m$. Then $f^{-1}(E)$ is connected in $\mathbf{R}^n$.

FALSE! Counterexample: $f(x) = x^2$ is continuous from $\mathbf{R}$ to $\mathbf{R}$ but $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$.

(3.) Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable at $a \in \mathbf{R}^n$. Let $g : \mathbf{R}^n \to \mathbf{R}$ be defined by $g(x) = x \bullet f(x)$, (dot product.) Find the total derivative (differential) $Dg(a)(h)$ where $h \in \mathbf{R}^n$. Without using the product theorem, prove your answer.

The product rule gives $Dg(x)(h) = D(x \bullet f(x))(h) = h \bullet g(x) + x \bullet Df(x)(h)$. This is the differential because

$$\begin{aligned}
\lim_{h \to 0} \frac{\|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} &= \lim_{h \to 0} \frac{\|(x+h) \bullet f(x+h) - x \bullet f(x) - h \bullet f(x) - x \bullet Df(x)(h)\|}{\|h\|} \\
&= \lim_{h \to 0} \frac{\|x \bullet (f(x+h) - f(x) - Df(x)(h)) + h \bullet (f(x+h) - f(x))\|}{\|h\|} \\
&\le \lim_{h \to 0} \left\{ \|x\| \frac{\|f(x+h) - f(x) - Df(x)(h)\|}{\|h\|} + \frac{\|h\|}{\|h\|} \|f(x+h) - f(x)\| \right\} = \|x\| \cdot 0 + 1 \cdot 0 = 0.
\end{aligned}$$

(4a.) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous. Suppose that for all $(x, y) \in \mathbf{R}^2$ both

$$\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \qquad \text{and} \qquad \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

exist. Then f is differentiable at $(1, 2)$.

FALSE! The two limits are nothing more than $f_x(x, y)$ and $f_y(x, y)$, the partial derivatives. There are functions where the partial derivatives exist at all points, but the function is not differentiable. (If it were known that f_x and f_y are continuous at some point $a \in \mathbf{R}^2$, then our theorem says that the function would be differentiable at a.) An example of such a function is

$$f(x, y) = \begin{cases} \frac{(x-1)^2(y-2)}{(x-1)^2 + (y-2)^2}, & \text{if } (x, y) \ne (1, 2); \\ 0. & \text{if } (x, y) = (1, 2). \end{cases}$$

Away from $(1,2)$, the denominator avoids zero, so the partial derivatives exist and are continuous, hence f is differentiable. Also, observe that $f(1,y) = f(x,2) = 0$ for all x, y. Hence $f_y(1,y) = f_x(x,2) = 0$ so the partial derivatives exist at $(1,2)$. If the function were differentiable at $(1,2)$, then the vanishing of the partial derivatives implies that the differential would have to be $T(h,k) = 0$ all h, k. But the limit

$$\lim_{(h,k)\to(1,2)} \frac{\|f(1+h,2+k) - f(1,2) - T(h,k)\|}{\|(h,k)\|} = \lim_{(h,k)\to(1,2)} \frac{\|f(1+h,2+k)\|}{\|(h,k)\|}$$

does not exist. To see this, consider the first approach $(h,k) = (t,0)$ as $t \to 0$. The numerator vanishes so along this approach the limit would be zero. Then consider the second approach $(h,k) = (t,t)$ for $t > 0$. Then $f(1+t,2+t) = t/2$ and $\|(t,t)\| = \sqrt{2}|t|$. Then the difference quotient tends to $1/\sqrt{2}$. Since the two approaches have different limits, there is no two dimensional limit: the function is not differentiable at $(1,2)$.

(4b.) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$ function for which the third partial derivatives $f_{xxy}(x,y)$ exist for all $(x,y) \in \mathbf{R}^2$ such that $f_{xxy}(x,y)$ is continuous at $(0,0)$. Then $f_{xyx}(0,0)$ and $f_{yxx}(0,0)$ exist and are equal $f_{xxy}(0,0) = f_{xyx}(0,0) = f_{yxx}(0,0)$.

TRUE! This is just an application of the equality of cross partials theorem to f_x and f_y which are $\mathcal{C}^1$ by assumption, since f is $\mathcal{C}^2$. We are given that $(f_x)_{xy}$ exists and is continuous at $(0,0)$. But, this is sufficient to be able to assert the existence of the other mixed partial derivative, and that it is equal to the first at the point $(f_x)_{xy}(0,0) = (f_x)_{yx}(0,0)$. But since $f \in \mathcal{C}^2(\mathbf{R}^2)$, we also have that $f_{xy} = f_{yx}$ for all of $\mathbf{R}^2$. Hence, all third derivaives exist and are equal $f_{xxy}(0,0) = (f_x)_{xy}(0,0) = (f_x)_{yx}(0,0) = (f_{xy})_x(0,0) = (f_{yx})_x(0,0) = f_{yxx}(0,0)$.

(5.) Let $K \subseteq \mathbf{R}^2$ be a compact set. Suppose $\{\mathbf{x}_n\}_{n\in\mathbf{N}} \subseteq K$ is a sequence in K which is a Cauchy sequence in $\mathbf{R}^2$. Then there is a point $\mathbf{k} \in K$ so that $\mathbf{x}_n \to \mathbf{k}$ as $n \to \infty$.

Since $\{x_n\}$ is Cauchy, it is convergent in $\mathbf{R}^2$: there is a $\mathbf{k} \in \mathbf{R}^2$ so that $x_n \to \mathbf{k}$ as $n \to \infty$. But as K is compact it is closed. But a closed set contains its limit points, so $\mathbf{k} \in K$.

(6.) Suppose $\mathbf{f}, \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$. Assume that $\mathbf{g}$ is differentiable at $\mathbf{x}_0 \in \mathbf{R}^2$ and that for some $\alpha > 1$ and $M < \infty$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\| \le M\|\mathbf{x} - \mathbf{x}_0\|^\alpha$$

for all $\mathbf{x} \in \mathbf{R}^2$. Show that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ and that $D\mathbf{f}(\mathbf{x}_0) = D\mathbf{g}(\mathbf{x}_0)$.

It suffices to show that the difference quotient limits to zero. Now,

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Dg(x)(h)\|}{\|h\|} = \\
&= \lim_{h\to 0} \frac{\|f(x+h) - g(x+h) - f(x) + g(x) + g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} \\
&\le \lim_{h\to 0} \frac{\|f(x+h) - g(x+h)\| + \|g(x) - f(x)\| + \|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} \\
&\le \lim_{h\to 0} \left\{ \frac{\|h\|^\alpha + \|0\|^\alpha}{\|h\|} + \frac{\|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} \right\} \\
&= \lim_{h\to 0} \|h\|^{\alpha-1} + \lim_{h\to 0} \frac{\|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} = 0 + 0.
\end{aligned}$$

(7.) Theorem. Suppose $S_i \subseteq \mathbf{R}^n$ are closed nonempty sets which are contained in the compact set K. Assume that the subsets form a decreasing sequence $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Then they have a nonempty intersection $\bigcap_{i\in\mathbf{N}} S_i \ne \emptyset$.

Proof.Suppose it is false. Then $\bigcap_{i\in\mathbf{N}} S_i = \emptyset$. Let $U_i = \mathbf{R}^n \backslash S_i$ which are open since S_i are closed. By deMogran's formula, $\cup_i U_i = \cup_i (\mathbf{R}^n \backslash S_i) = \mathbf{R}^n \backslash (\cap_i S_i) = \mathbf{R}^n \backslash \emptyset = \mathbf{R}^n$. Thus $\{U_i\}$ is an open cover of K. Since K is compact, there are finitely many $i_1, i_2, \ldots, i_n$ so that $K \subseteq U_{i_1} \cup \cdots \cup U_{i_n} = (\mathbf{R}^n \backslash S_{i_1}) \cup \cdots \cup (\mathbf{R}^n \backslash S_{i_n}) = \mathbf{R}^n \backslash (S_{i_1} \cap \cdots \cap S_{i_n}) = \mathbf{R}^n \backslash S_p$ where $p = \max\{i_1, \ldots, i_n\}$ since the S_i's are nested. But this says $K \cap S_p = \emptyset$ which contradicts the fact that S_p is a nonempty subset of K.

(8.) $E = [0,1] \cap \mathbf{Q}$, the set of rational points between zero and one, is not compact.

Proof. We find an open cover without finite subcover. Let $c = 1/\sqrt{2}$ or any other irrational number $c \in [0,1]$. Consider the sets $U_0 = (c, \infty)$ and $U_i = (-\infty, c - 1/i)$ for $i \in \mathbf{N}$. Then $\mathcal{C} = \{U_i\}_{i=0,1,2,\ldots}$ is an open cover.

For if $x \in E$, since x is rational, $x \neq c$. If $x > c$ then $x \in U_0$. If $x < c$, by the Archimidean property, there is an $i \in \mathbf{N}$ so that $1/i < c - x$. It follows that $c - 1/i > x$ so $x \in U_i$. On the other hand no finite collection will cover. Indeed, if we choose any finite cover it would have to include U_0 to cover $1 \in E$ and therefore take the form $\{U_0, U_{i_1}, \ldots, U_{i_J}\}$ for a finite set of numbers $i_1, \ldots, i_J \in \mathbf{N}$. Hence if $K = \max\{i_1, \ldots, i_J\}$ then $U_0 \cup U_{i_1} \cup \ldots \cup U_{i_J} = (-\infty, c - 1/K) \cup (c, \infty)$. But in the gap $[c - 1/K, c]$ there are rational numbers, by the density of rationals. Thus $E \not\subseteq U_0 \cup U_{i_1} \cup \ldots \cup U_{i_J}$. (Of course the easy argument is to observe that E is not closed so can't be compact.)

(9.) Theorem. Suppose $E \subseteq \mathbf{R}^n$ is bounded and $f : E \to \mathbf{R}^m$ is uniformly continuous. Then $f(E)$ is bounded. This would not be true if "uniformly continuous" were replaced by "continuous."

Proof. One idea is to divide E into finitely many little pieces so that f doesn't vary very much on any one of them. Then the bound on f is basically the max of bounds at one point for each little piece. f is uniformly continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $x, y \in E$ such that $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \varepsilon$. Fix an $\varepsilon_0 > 0$ and let uniform continuity give $\delta_0 > 0$. Since E is bounded, there is an $R < \infty$ so that $E \subseteq B_R(0)$. Finitely many $\delta_0/2$ balls are required to cover $B_R(0)$, that is, there are points $x_i \in \mathbf{R}^n$ so that $B_R(0) \subseteq \cup_{i=1}^J B_{\delta_0/2}(x_i)$. This can be accomplished by chopping the ball into small enough cubes and taking x_i's as the centers of the cubes. e.g., the cube $[-\delta_0/4\sqrt{n}, \delta_0/4\sqrt{n}] \times \cdots \times [-\delta_0/4\sqrt{n}, \delta_0/4\sqrt{n}] \subseteq B_{\delta_0/2}(0)$. Choose points of E in those balls that meet E. Let $\mathcal{I} = \{i \in \{1, \ldots, J\} : B_{\delta_0/2}(x_i) \cap E \neq \emptyset\}$ and choose $y_i \in B_{\delta_0/2}(x_i) \cap E$ if $i \in \mathcal{I}$. Let $M = \max\{\|f(y_i)\| : i \in \mathcal{I}\}$ be the largest norm among the points y_i in E. Then the claim is that $f(E) \subseteq B_{M+\varepsilon_0}(0)$. To see this, choose $z \in E$. Since E is in the union of little balls, there is an index $j \in \mathcal{I}$ so that $z \in B_{\delta_0/2}(x_j)$. Since $y_j \in B_{\delta_0/2}(x_j)$ also, it follows that $\|z - y_j\| = \|z - x_j + x_j - y_j\| \leq \|z - x_j\| + \|x_j - y_j\| < \delta_0/2 + \delta_0/2 = \delta_0$. By the uniform continuity, $\|f(y_j) - f(z)\| < \varepsilon_0$. It follows that $\|f(z)\| = \|f(z) - f(y_j) + f(y_j)\| \leq \|f(z) - f(y_j)\| + \|f(y_j)\| < \varepsilon_0 + M$ and we are done.

The result doesn't hold if f is not uniformly continuous. Let $E = B_1(0) \backslash \{0\}$ and f the function from problem (4.). By (4a.) f is continuous on E but $f(E) = (1, \infty)$ is unbounded.

(10.) Theorem. Let $\mathcal{S} = [0,1] \times [0,1] \subseteq \mathbf{R}^2$ and $F : \mathcal{S} \to \mathbf{R}$ be continuous. Then F is not one to one.

Proof. Consider the circle $\sigma(t) = (\frac{1}{2} + \frac{1}{2}\sin t, \frac{1}{2} + \frac{1}{2}\cos t) \in \mathcal{S}$ as $t \in [0, 2\pi]$. Then $f(t) = F(\sigma(t))$ is a periodic continuous function. If f is constant then $F(\sigma(0)) = F(\sigma(\pi))$ so F is not $1-1$. Since $[0, 2\pi]$ is compact, there are points $\theta_0, \theta_1 \in [0, 2\pi]$ where $f(\theta_0) = \inf\{f(t) : t \in [0, 2\pi]\}$ and $f(\theta_1) = \sup\{f(t) : t \in [0, 2\pi]\}$. Also $f(\theta_0) < f(\theta_1)$. For convenience, suppose $\theta_0 < \theta_1$. The point is that the curves $\sigma((\theta_0, \theta_1))$ and $\sigma((\theta_1, \theta_0 + 2\pi))$ are two opposite arcs of the circle running from the minimum of f on the circle to the maximum. And any intermediate value gets taken on at least once in each arc, thus there are two point where f is equal and F is therefore not $1-1$. More precisely, choose any number $f(\theta_0) < y < f(\theta_1)$. By the intermediate value theorem applied to $f : [\theta_0, \theta_1] \to \mathbf{R}$, there is $\theta_3 \in (\theta_0, \theta_1)$ so that $f(\theta_3) = y$. Also by the intermediate value theorem applied to $f : [\theta_1, \theta_0 + 2\pi] \to \mathbf{R}$, there is $\theta_4 \in (\theta_1, \theta_0 + 2\pi)$ so that $f(\theta_4) = y$. Since $\sigma(\theta_3) \neq \sigma(\theta_4)$ because $0 = \theta_1 - \theta_1 < \theta_4 - \theta_3 < \theta_1 + 2\pi - \theta_1 = 2\pi$, it follows that F is not $1-1$ since $F(\sigma(\theta_3)) = F(\sigma(\theta_4))$. The case $\theta_0 > \theta_1$ is similar.

(11a.) Let $f : \mathbf{R}^2 \to \mathbf{R}$. Suppose that both iterated limits exist and are equal. The the limit exists.

$$L = \lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y) \qquad \Longrightarrow \qquad L = \lim_{(x,y) \to (0,0)} f(x, y)$$

FALSE! Consider

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

For $x \neq 0$ fixed, the function $f(x, y) \to 0$ as $y \to 0$. Thus $\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} 0 = 0$. For $y \neq 0$ fixed, the function $f(x, y) \to 0$ as $x \to 0$. Thus $\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} 0 = 0$. Compare to the two dimensional limit. For $(x, y) = (0, t)$, $f(0, t) = 0$ so that along this path, $f(0, t) \to 0$ as $t \to 0$. Now consider $(x, y) = (t^2, t)$ for $t > 0$. Now $f(t^2, t) = \frac{1}{2}$ so that along this path, $f(t^2, t) \to \frac{1}{2}$ as $t \to 0$. Being inconsistent along path limits, there is no limit for the function.

(b.) Suppose $f : \mathbf{R}^3 \to \mathbf{R}^2$ is given by $f(x, y, z) = (xy + x^2 z^3, x^4 + y + y^5 z^6)$. Then f is differentiable on $\mathbf{R}^3$.

TRUE! The partial derivatives are

$$\frac{\partial f}{\partial x}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y + 2xz^3 \\ 4x^3 \end{pmatrix}, \qquad \frac{\partial f}{\partial y}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 1 + 5y^4 z^6 \end{pmatrix}, \qquad \frac{\partial f}{\partial z}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x^2 z^2 \\ 6y^5 z^5 \end{pmatrix}$$

Since f is a polynomial function, its first partial derivatives exist at all points and polynomial functions. But by Theorem 11.5, since the partial derivatives are continuous at all points, the function is differentiable at all points.