Math 3210 § 1. **Third Midterm Exam** Name: Solutions
Treibergs April 3, 2019

1. *Let $f : \mathbf{R} \to \mathbf{R}$. State the definition: f is* uniformly continuous on $\mathbf{R}$. *Using just the definition, prove $f(x) = \sqrt{4+x^2}$ is uniformly continuous on $\mathbf{R}$.*

   $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous on $\mathbf{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

   $$|f(x) - f(y)| < \epsilon \qquad \text{whenever } x, y \in \mathbf{R} \text{ and } |x - y| < \delta.$$

   To see that $f(x) = \sqrt{4+x^2}$ is uniformly continuous, choose $\epsilon > 0$. Let $\delta = \epsilon$. Then for every $x, y \in \mathbf{R}$ such that $|x - y| < \delta$ we have

   $$\begin{aligned} |f(x) - f(y)| &= \left|\sqrt{4+x^2} - \sqrt{4+y^2}\right| \\ &= \left| \frac{\left(\sqrt{4+x^2} - \sqrt{4+y^2}\right)\left(\sqrt{4+x^2} + \sqrt{4+y^2}\right)}{\sqrt{4+x^2} + \sqrt{4+y^2}} \right| \\ &= \frac{\left|(4+x^2) - (4+y^2)\right|}{\sqrt{4+x^2} + \sqrt{4+y^2}} \\ &= \frac{|x^2 - y^2|}{\sqrt{4+x^2} + \sqrt{4+y^2}} \\ &= \frac{|(x+y)(x-y)|}{\sqrt{4+x^2} + \sqrt{4+y^2}} \\ &= \frac{|x+y|}{\sqrt{4+x^2} + \sqrt{4+y^2}}|x-y| \\ &\le \frac{|x|+|y|}{\sqrt{4+x^2} + \sqrt{4+y^2}}|x-y| \\ &\le \frac{\sqrt{4+x^2} + \sqrt{4+y^2}}{\sqrt{4+x^2} + \sqrt{4+y^2}}|x-y| \\ &= |x-y| < \delta = \epsilon. \quad \square \end{aligned}$$

2. *Let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions. State the definition: the sequence of functions $\{f_n\}$* converges uniformly *on $\mathbf{R}$ to a function f. Let $f_n(x) = \dfrac{1}{1+(x-n)^2}$. Determine whether there is a function $f(x)$ such that $\{f_n\}$ converges uniformly to f, converges pointwise but not uniformly to f or does not converge to any f. Prove your result.*

   A sequence of functions $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ on $\mathbf{R}$ if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that

   $$|f_n(x) - f(x)| < \epsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } n > N.$$

   In this case $f_n(x) \to 0$ pointwise but not uniformly. To see it, for any $x \in \mathbf{R}$,

   $$\lim_{n\to\infty} \frac{1}{1+(x-n)^2} = \lim_{n\to\infty} \frac{\dfrac{1}{(n-x)^2}}{\dfrac{1}{(n-x)^2} + 1} = \frac{0}{0+1} = 0.$$

   so $f_n(x) \to 0$ pointwise in $\mathbf{R}$.

On the other hand, we prove the negation of uniform convergence:

$(\exists\epsilon_0 > 0)(\forall N \in \mathbf{R})(\exists n \in \mathbb{N})(\exists x_n \in \mathbf{R})(x_n > N \text{ and } |f_n(x_n) - f(x_n)| \geq \epsilon_0)$.

Let $\epsilon_0 = 1$. Choose $N \in \mathbf{R}$. By the Archimedean Property, there is $n \in \mathbb{N}$ such that $n > N$. Put $x_n = n$. Then $f_n(x_n) = 1$ and $|f_n(x) - f(x)| = |1 - 0| = 1 \geq \epsilon_0 = 1$.

Another way to see it, for the sequence $x_n = n$ we have $f_n(x_n) = 1$ for all n so that $f_n(x_n)$ doesn't converge to zero, as it must do for the convergence to be uniform.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT.*Suppose that $f_n : \mathbf{R} \to \mathbf{R}$ are contiuous functions and $f(x) = \lim_{n\to\infty} f_n(x)$ for all x then f is continuous on $\mathbf{R}$.*

   FALSE. Here is a counterexample. Define the sequence of functions by

$$f_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ x^n, & \text{if } 0 \leq x \leq 1; \\ 1, & \text{if } 1 < x. \end{cases}$$

   Then $f_n(x)$ is continuous on $\mathbf{R}$ for every n, but the pointwise limit is

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x < 1; \\ 1, & \text{if } 1 \leq x. \end{cases}$$

   which is not continuous at $x = 1$. Since uniform limits of continuous functions are continuous, the convergence of this sequence couldn't have been uniform on $\mathbf{R}$.

   (b) STATEMENT *Suppose $f : [a, b] \to \mathbf{R}$ has the property that for any k between $f(a)$ and $f(b)$ there is a $c \in [a, b]$ such that $f(c) = k$. Then f is continuous on $[a, b]$.*

   FALSE. Here is a counterexample. Let $[a, b] = [0, 1]$ and

$$f(x) = \begin{cases} 2x, & \text{if } x \leq \frac{1}{2}; \\ 2x - 1, & \text{if } \frac{1}{2} < x. \end{cases}$$

   Then f is not continuous at $x = \frac{1}{2}$ because it jumps there. However, for any $0 = f(0) \leq k \leq f(1) = 1$ there is a $c \in [0, 1]$ such that $f(c) = k$, namely $c = \dfrac{k}{2}$.

   (c) STATEMENT.*Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function such that $f(x) \geq 0$ for all $x \neq 0$ and $\lim_{x\to 0} f(x) = L$ exists, where $L \in \mathbf{R}$. Then $L \geq 0$.*

   TRUE. Since the limit exists $\lim_{x\to 0} f(x) = L$, suppose for contradiction that the limit were negative $L < 0$. Let $\epsilon = -\dfrac{L}{2}$. By the definition of limit, there is a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon = -\frac{L}{2} \qquad \text{whenever } 0 < |x - 0| < \delta.$$

   Choose an x_0 so that $0 < |x_0 - 0| < \delta$. For this x_0, by assumption

$$0 \leq f(x_0) = L + (f(x_0) - L) \leq L + |f(x_0) - L| < L - \frac{L}{2} = \frac{L}{2} < 0,$$

   which is a contradiction. Hence $L < 0$ is false and $f(0) \geq 0$ follows.

4. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function and $a \in \mathbf{R}$ a point. State the definition: f is* differentiable *at a. Using just the definition of differentiable, show that if there are real constants b, c, and k such that $|f(x) - b - cx| \le kx^2$ for all x then $f(x)$ is differentiable at $0 \in \mathbf{R}$.*

   $f : \mathbf{R} \to \mathbf{R}$ is differentiable at a point $a \in \mathbf{R}$ if the difference quotient have a finite limit as $x \to a$ at that point
$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$
   Observe that when $x = 0$ the inequality implies $f(x) = b$. It also says $f'(0) = c$, which we'll show. This is equivalent to showing that the limit of the difference at $a = 0$ is zero:
$$\begin{aligned}\lim_{x\to 0} \left| \frac{f(x) - f(0)}{x - 0} - c \right| &= \lim_{x\to 0} \left| \frac{f(x) - b}{x} - c \right| = \lim_{x\to 0} \left| \frac{f(x) - b}{x} - \frac{cx}{x} \right| \\ &= \lim_{x\to 0} \frac{|f(x) - b - cx|}{|x|} \le \lim_{x\to 0} \frac{k|x|^2}{|x|} = \lim_{x\to 0} k|x| = 0.\end{aligned}$$
   Hence
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = c.$$

5. *State the definition: the real sequence $\{a_n\}$ is a* Cauchy Sequence. *Does the following series converges to a real number? Prove your answer. [Hint: consider partial sums.]*
$$S = \sum_{k=1}^{\infty} \frac{\sin k}{(k^2)!}$$
   Consider the sequence of partial sums
$$S_n = \sum_{k=1}^{n} \frac{\sin k}{(k^2)!}.$$
   Note that for $k \in \mathbf{N}$ we have $k^2 \ge k$ so
$$(k^2)! \ge k! = 1 \cdot 2 \cdot 3 \cdots k \ge 1 \cdot 2 \cdot 2 \cdots 2 = 2^{k-1}. \tag{1}$$
   We show that $\{S_n\}$ is a Cauchy Sequence, hence converges to a real number $S = \lim_{n\to\infty} S_n$. Choose $\epsilon > 0$. Let $N = 1 - \dfrac{\log \epsilon}{\log 2}$. For any $m, n > N$ we may have $m = n$ so $|S_n - S_m| = 0 < \epsilon$ or we may have $m \ne n$. By swapping roles of m and n if necessary, we may assume that $m > n$. Then, using $|\sin k| \le 1$ and (1),
$$\begin{aligned}|S_m - S_n| &= \left| \sum_{k=1}^{m} \frac{\sin k}{(k^2)!} - \sum_{k=1}^{n} \frac{\sin k}{(k^2)!} \right| = \left| \sum_{k=n+1}^{m} \frac{\sin k}{(k^2)!} \right| \le \sum_{k=n+1}^{m} \frac{|\sin k|}{(k^2)!} \\ &\le \sum_{k=n+1}^{m} \frac{1}{2^{k-1}} = \frac{1}{2^n} \sum_{\ell=0}^{m-n-1} \frac{1}{2^\ell} = \frac{1}{2^n} \cdot \frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} \le \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} = \epsilon,\end{aligned}$$
   where we substituted the dummy index $k = n + 1 + \ell$. □