Math 3210 § 1. **Second Midterm Exam** Name: Solutions
Treibergs February 27, 2019

1. $\mathbb{Q}$ *denotes the rational numbers. Consider the subset* $E \subset \mathbf{R}$ *given by*

$$E = \left\{ x \in \mathbf{R} : \ 1 < x < 2 \text{ and } x = \sqrt{2}\, r \text{ for some } r \in \mathbb{Q} \right\}$$

State the definition: $\ell = \operatorname{glb} E$. *Find* $\operatorname{glb} E$. *Using just the definition, prove that your answer is correct.*

$\operatorname{glb} E$ is the greatest lower bound of E. The real number ℓ satisfies (1) ℓ is a lower bound for E, that is, $(\forall e \in E)(\ell \le e)$, and (2) ℓ is the largest of all lower bounds, namely, no larger number is a lower bound for E, that is, $(\forall b > \ell)(\exists e \in E)(e < b)$.

For this problem, $\ell = \inf E = 1$. We check the two conditions. (1) By the definition of the set E we have $x \in E$ implies $x > 1$, thus $\ell = 1$ is a lower bound for E. (2) For any $b > 1$, by the density of rationals there is $r \in \mathbb{Q}$ such that

$$\frac{1}{\sqrt{2}} < r < \frac{\min\{b, 2\}}{\sqrt{2}}.$$

It follows that $x = r\sqrt{2}$ satisfies $1 < x < 2$ so that $x \in E$ and $x < b$ so that b is not a lower bound of E.

2. *Suppose that* $\{a_n\}$ *is a real sequence. State the definition:* $L \in \mathbf{R}$ *is the limit of the sequence* $L = \lim_{n\to\infty} a_n$. *Determine the limit* $a_n \to L$ *as* $n \to \infty$. *Show using just the definition of limit, and not the Main Limit Theorem, that the sequence converges to your limit.*

$$a_n = \frac{\sqrt{4n^2+3}}{n}$$

We say that $L \in \mathbb{R}$ is the limit $a_n \to L$ as $n \to \infty$ if for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that

$$|a_n - L| < \epsilon \qquad \text{whenever } n > N.$$

The limit of this particular sequence is $L = 2$. Choose $\epsilon > 0$. Let $N = \sqrt{\dfrac{3}{4\epsilon}}$. Then for any $n \in \mathbb{N}$ such that $n > N$ we have

$$\begin{aligned} |a_n - L| &= \left| \frac{\sqrt{4n^2+3}}{n} - 2 \right| = \left| \frac{\sqrt{4n^2+3} - 2n}{n} \right| = \left| \frac{(\sqrt{4n^2+3} - 2n)}{n} \cdot \frac{(\sqrt{4n^2+3} + 2n)}{(\sqrt{4n^2+3} + 2n)} \right| \\ &= \left| \frac{(4n^2+3) - 4n^2}{n(\sqrt{4n^2+3} + 2n)} \right| = \frac{3}{n(\sqrt{4n^2+3} + 2n)} < \frac{3}{n(\sqrt{4n^2} + 2n)} = \frac{3}{4n^2} < \frac{3}{4N^2} = \epsilon. \end{aligned}$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement. *Suppose the real sequence* $a_n \to a$ *as* $n \to \infty$. *Then* $\lfloor a_n \rfloor \to \lfloor a \rfloor$ *(floor function).*

   False. Consider $a_n = \dfrac{n}{n+1}$ then $\lfloor a_n \rfloor = 0$ so that $\lfloor a_n \rfloor \to 0$ as $n \to \infty$ but $a_n \to a = 1$ as $n \to \infty$ which has $\lfloor a \rfloor = 1$.

(b) STATEMENT. *Let $I_n = (a_n, b_n)$ be nonempty, bounded, open and nested intervals: $I_n \supset I_{n+1}$ for all n. Then $\bigcap_{n\in\mathbf{N}} I_n \neq \emptyset$.*

FALSE. Let $I_n = \left(0, \dfrac{1}{n}\right)$ be open, nested intervals. Then $\bigcap_{n\in\mathbf{N}} I_n = \emptyset$. To see it, for contradiction, if there were $x \in \bigcap_{n\in\mathbf{N}} I_n$ then $x \in I_1$ so that $0 < x$. But for n large enough, $\dfrac{1}{n} < x$ so $x \notin I_n$ which implies $x \notin \bigcap_{n\in\mathbf{N}} I_n$, a contradiction.

(c) STATEMENT. *Suppose that the real sequence $\{a_n\}$ is unbounded and $a_n \neq 0$. Then $\dfrac{1}{a_n} \to 0$ as $n \to \infty$.*

FALSE. For $\{a_n\}$ to be unbounded above does not mean that $a_n \to \infty$. For example let $a_n = n$ if n is odd and $a_n = 1$ if n is even. Then $\{a_n\}$ is unbounded since the sequence includes arbitrarily large members, but every other term is unity. Hence the odd terms of $\dfrac{1}{a_n}$ tend to zero but the even terms are all one which don't tend to zero. Thus there is no limit to $\dfrac{1}{a_n}$ since all of its subsequences don't have the same limit.

4. *Define a sequence recursively by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n \geq 1$. Prove that $\{x_n\}$ is monotone and bounded. Show that there the real limit $L = \lim_{n\to\infty} x_n$ exists. Find L.*

We show that the sequence is decreasing and bounded below, hence the limit exists by the Monotone Sequences Theorem.

All terms are bounded below by zero. This can be seen by induction. We have the base case $x_1 = 5 \geq 0$. For the induction case, assume that $x_n \geq 0$ for some n. Then the next term $x_{n+1} = \sqrt{4 + x_n} \geq \sqrt{4 + 0} \geq 0$ as well because the function $f(x) = \sqrt{4 + x}$ is strictly increasing for $x \geq 0$. Hence $x_n \geq 0$ for all $n \in \mathbb{N}$ by induction.

The terms of the sequence are decreasing. For the base case, we have $x_2 = \sqrt{4 + x_1} = \sqrt{4 + 5} = 3$, thus $x_2 < x_1$. Now for the induction case, assume for some n that $x_{n+1} < x_n$. It follows that $x_{n+2} = \sqrt{4 + x_{n+1}} < \sqrt{4 + x_n} = x_{n+1}$ since f is strictly increasing. Thus the induction case holds and so we have $x_{n+1} < x_n$ for all $n \in \mathbb{N}$ by induction.

To find the value of L, we observe by the Main Limit Theorem,

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{4 + x_n} = \sqrt{4 + L}.$$

Squaring yields $L^2 = L + 4$. By the quadratic formula

$$L = \frac{1 \pm \sqrt{1 + 4 \cdot 4}}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

But we know that $x_n \geq 0$ so that $L \geq 0$. Thus the only root satisfying this inequality is

$$L = \frac{1 + \sqrt{17}}{2}.$$

5. *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are real sequences. Show using just the definition of limit that if* $a = \lim_{n\to\infty} a_n$ *and* $|a_n - b_n| \le \dfrac{1}{n}$ *for all* n*, then* $a = \lim_{n\to\infty} b_n$.

To show that $b_n \to a$ as $n \to \infty$, choose $\epsilon > 0$. By the assumption that $a_n \to a$ as $n \to \infty$, there is $N_1 \in \mathbb{R}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \qquad \text{whenever } n > N_1.$$

Also let $N_2 = \dfrac{2}{\epsilon}$ so that if $n > N_2$ then $\dfrac{1}{n} < \dfrac{\epsilon}{2}$. Put $N = \max\{N_1, N_2\}$. If $n > N$ we get by adding and subtracting a_n, using $n > N_1$ for the first term and the assumption $|a_n - b_n| \le \dfrac{1}{n}$ on the second term,

$$\begin{aligned} |a - b_n| &= |(a - a_n) + (a_n - b_n)| \le |a - a_n| + |a_n - b_n| \\ &< \frac{\epsilon}{2} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{1}{N_2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$