

Homework for Math 3210 §3, Fall 2011

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Our texts are the manuscripts by Joseph L. Taylor, “Foundations of Analysis,” 2010 and Anne Roberts, “Basic Logic Concepts,” 2005. These PDF Notes available for download from

<http://www.math.utah.edu/~taylor/foundations.html>
<http://www.math.utah.edu/%7Eaeroberts/M3210-1d.pdf>

Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on Dec. 9, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Please hand in problems A1 – A4 on Friday, August 26.

A1. Truth table. Construct a truth table for the following statement.

$$[P \wedge (P \Rightarrow Q)] \Rightarrow Q.$$

A2. Equivalent Statements. Verify using truth tables that

$$P \wedge [\sim (Q \wedge R)]$$

is equivalent to

$$(P \wedge [\sim Q]) \vee (P \wedge [\sim R]).$$

A3. Quantified Statements. Determine the truth value of each statement assuming that x , y , z are real numbers.

$$\begin{aligned} &(\exists x)(\forall y)(\exists z)(x + y = z); \\ &(\exists x)(\forall y)(\forall z)(x + y = z); \\ &(\forall x)(\forall y)(\exists z)[(z > y) \Rightarrow (z > x + y)]; \\ &(\forall x)(\exists y)(\forall z)[(z > y) \Rightarrow (z > x + y)]. \end{aligned}$$

A4. Negate and Interpret. Write formally, with quantifiers in the right order. Negate the sentence and interpret.

“Everybody doesn’t like something but nobody doesn’t like Sara Lee.”

Please hand in problems B1 – B2 on Friday, Sept. 2.

B1. Functions and sets. Please hand in the following exercises from Taylor's *Foundations of Analysis*

7[2, 5, 9, 11]

B2. Image and intersection. Let A and B be sets and $f : A \rightarrow B$ be a function. Show that f is one-to-one if and only if

$$f(E \cap F) = f(E) \cap f(F)$$

for all subsets $E, F \subset A$.

Please hand in problems C1 on Friday, Sept. 9.

C1. Peano's Axioms and Induction. Please hand in the following exercises from Taylor's *Foundations of Analysis*

15[8, 14, 16, 17]

Please hand in problems D1 – D3 on Friday, September 16.

D1. Problems from Taylor's *Foundations of Analysis*.

- 15[2] Prove that if $m, n \in \mathbb{N}$, then $m + n \neq n$. [Hint: use induction on n .]
- 15[3] Use the preceding exercise to prove that if $m, n \in \mathbb{N}$ such that $m \leq n$ and $n \leq m$ then $m = n$.
- 15[7] Assuming addition has been defined, give an inductive definition for product on \mathbb{N} . Use it to prove if $m, n \in \mathbb{N}$ then $n \leq nm$.
- 22[4] Assuming that \mathbb{Z} satisfies the axioms of a commutative ring, show that the rationals \mathbb{Q} satisfy the additive and multiplicative associative axioms **A2** and **M2** of a field.
- 22[8] Assuming only the axioms of a field, show that $xy = 0$ implies $x = 0$ or $y = 0$.

D2. In a commutative ring $(R, +, \times)$, show that

$$-(-x) = x \quad \text{for all } x \in R.$$

D3. Determine whether the following relations are equivalence relations:

1. In the natural numbers \mathbb{N} let $m \sim n$ if m and n are relatively prime (have no common integer factor other than one).
2. In $\mathbb{N} \times \mathbb{N}$, let $(a, b) \sim (c, d)$ if $a + d = b + c$.
3. In the real numbers \mathbb{R} , let $x \sim y$ if $xy \geq 0$.

Please hand in problems E1 on Friday, September 23.

E1. Problems from Taylor's *Foundations of Analysis*.

22[11] Assume that F is an ordered field. For all $x, y \in F$, show that $0 < x < y$ implies that $0 < y^{-1} < x^{-1}$.

22[13] Generalize Theorem 1.3.9 by showing that every rational solution of a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

with integer coefficients a_k is an integer solution.

28[2] For each of the following sets, find the least upper bound of the set, if it exists and prove your answer.

- (a) the set of odd integers;
- (b) $\{1 - 1/n : n \in \mathbb{N}\}$;
- (c) $\{r \in \mathbb{Q} : r^3 < 8\}$;
- (d) $\{\sin x : x \in \mathbb{R}\}$.

28[8] Prove that if x is irrational and r is a non-zero rational, then $x+r$ and rx are irrational.

28[9] We know that $\sqrt{2}$ is irrational. Use this fact and the previous exercise to show that if $r < s$ are rational numbers, then there is an irrational number x with $r < x < s$.

Please hand in problems F1 – F4 on Friday, September 30.

F1. Problems from Taylor's *Foundations of Analysis*.

34[2] Find the supremum and infimum of the following sets. Tell whether each set has a maximum or minimum.

- (a) $(1, 8]$;
- (b) $\{n^2 : n \in \mathbb{N}\}$;
- (c) $\left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\}$.

34[8a] Let A and B be two nonempty sets of real numbers. Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

34[9c] Find $\sup_I f$ and $\inf_I f$ where $f(x) = 2x - x^2$ and $I = [0, 1]$. Which of these are the maximum and the minimum of the function f on I ?

34[12a] Let f and g be defined on a set containing A as a subset. Show

$$\sup_A (f + g) \leq \sup_A f + \sup_A g.$$

Give an example that shows that “ $<$ ” is possible.

43[8] Guess the limit and prove using the definition of limit that your guess is correct.

$$\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n}.$$

F2. The Well Ordering Principle for the natural numbers says that every nonempty subset $S \subset \mathbb{N}$ has a least element. It is a consequence of the Peano axioms (see 15[17]). Show that for every real number $x > 1$ there is a natural number $n \in \mathbb{N}$ such that $n < x \leq n + 1$.

F3. Find the supremum and infimum of the real set $E = \left\{ \frac{n^2 - 5n + 26}{n^2 - 6n + 10} : n \in \mathbb{N} \right\}$.

F4. Prove that if a, b, x, y are real numbers that satisfy the inequalities

$$|x - a| < 1, \quad |y - b| < 2, \quad |a - b| > 7$$

then $|x - y| > 4$.

Please hand in problems G1 on Friday, October 21.

G1. Problems from Taylor's *Foundations of Analysis*.

50[1] Use the Main Limit Theorem to find

$$\lim_{n \rightarrow \infty} \frac{2n^3 - n + 1}{3n^3 + n^2 + 6}.$$

50[8] Prove that if $\{b_n\}$ is a sequence of positive terms and $b_n \rightarrow b > 0$ as $n \rightarrow \infty$, then there is a number $m > 0$ such that $b_n \geq m$ for all n .

50[11] For each number n , let $b_n = n^{1/n} - 1$. Then b_n is positive and $n = (1 + b_n)^n$. Use the Binomial Theorem (Theorem 1.2.12) to prove that

$$n \geq \frac{n(n-1)}{2} b_n^2$$

and, hence, that

$$b_n \leq \sqrt{\frac{2}{n-1}}.$$

50[12] Prove that

$$\lim_{n \rightarrow \infty} n^{1/n} = 1.$$

[Hint: Use the result of the previous exercise.]

55[3] If $a_1 = 1$ and $a_{n+1} = (1 - 2^{-n})a_n$, prove that a_n converges.

55[4] Let $\{d_n\}$ be a sequence of 0's and 1's and define the sequence $\{a_n\}$ by

$$a_n = d_1 2^{-1} + d_2 2^{-2} + \cdots + d_n 2^{-n}.$$

Prove that the sequence a_n converges to a number between 0 and 1 (inclusive!)

Please hand in problems H1 – H2 on Friday, October 28.

H1. Problems from Taylor's *Foundations of Analysis*.

60[1] Give an example of a nested sequence of bounded open intervals that does not have a point in its intersection.

60[2] Give an example of a nested sequence of unbounded intervals that does not have a point in its intersection.

60[6c] Find a subsequence which converges. Justify your answer.

$$a_n = \frac{n}{2^k} - 1 \quad \text{with } k \text{ an integer chosen so that } 2^k \leq n < 2^{k+1}.$$

60[12] Given the series $\sum_{k=1}^{\infty} a_n$, set $t_n = \sum_{k=1}^n |a_k|$. Prove that the series converges if $\{t_n\}$ is bounded.

70[3] We know that \sqrt{x} is continuous on $[0, \infty)$ by Theorem 3.1.6. Give another proof of this fact using only the definition of continuity (Definition 3.1.1.)

70[11] Prove that the function is not continuous at 0:

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

70[12] Prove that the function is continuous at 0:

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

H2. Show that the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges.

Please hand in problems I1 – I2 on Friday, November 4.

I1. Problems from Taylor's *Foundations of Analysis*.

75[3] Prove that if f is a continuous function on a closed, bounded interval $[a, b]$ and if (x_0, y_0) is any point in the plane, then there is a closest point to (x_0, y_0) on the graph of f in the plane.

75[8] Let f be a continuous function from $[0, 1]$ to $[0, 1]$. Prove that there is a point $c \in [0, 1]$ such that $f(c) = c$. Such a point is called a *fixed point*.

75[9] Use the intermediate value theorem to prove that if n is a natural number, then every positive number has an n th root.

75[10] Prove that a polynomial of odd degree has at least one real root.

80[1] Is the function $f(x) = x^2$ uniformly continuous on $(0, 1)$? Justify your answer.

80[3] Is the function $f(x) = x^2$ uniformly continuous on $(0, \infty)$? Justify your answer.

80[4] Using only the ε - δ definition of uniform continuity, prove that the function

$$f(x) = \frac{x}{x+1}$$

is uniformly continuous on $[0, \infty)$.

85[2] Prove that the sequence of functions $f_n(x) = \frac{1}{n+x^2}$ converges uniformly to 0 on \mathbb{R} .

I2. Let $(a, b) \subset \mathbb{R}$ be an open interval, $c \in (a, b)$ and let $f : (a, b) \rightarrow \mathbb{R}$. Show that $f(x)$ is continuous at $c \in (a, b)$ if and only if $f(c) = \lim_{x \rightarrow c} f(x)$.

The limit of a function is defined similarly to the limit of a sequence. Note that the limit of a function does not involve the value of the function at c .

Definition. Let $c \in [a, b]$ and $f : (a, b) \rightarrow \mathbb{R}$. We say that $L \in \mathbb{R}$ is the *limit of $f(x)$ as $x \rightarrow c$* if for every $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \epsilon \quad \text{whenever } x \in (a, b), x \neq c \text{ and } |x - c| < \delta.$$

To say that the limit of f as $x \rightarrow c$ exists and equals L we write $L = \lim_{x \rightarrow c} f(x)$.

Please hand in problems J1 on Friday, November 11.

J1. Problems from Taylor's *Foundations of Analysis*.

85[2] Prove that the sequence of functions $f_n(x) = \frac{1}{n+x^2}$ converges uniformly to 0 on \mathbb{R} .

85[7] Prove Theorem 3.4.7: Let $\{f_n\}$ be a sequence of functions defined on a set D . If $\{f_n\}$ converges uniformly in D to 0, then $f_n(x_n)$ converges to 0 for every sequence $\{x_n\}$ of points in D .

85[12] Prove that if $\{a_n\}$ is a bounded sequence of numbers and a sequence $\{s_n\}$ of functions is defined on $(-1, 1)$ by

$$s_n(x) = \sum_{k=0}^n a_k x^k,$$

then $\{s_n\}$ converges to a continuous function on $(-1, 1)$. [Hint: Prove that the sequence is uniformly Cauchy on each subinterval $[-r, r]$ for $0 < r < 1$.]

93[7] If $f(x) = \frac{\sin x}{|x|}$, find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

93[12] Let $f(x)$ be a nonnegative function on (a, b) . If $\lim_{x \rightarrow b^-} f(x)$ exists, prove that it is a nonnegative number.

Please hand in problems K1 on Friday, November 18.

K1. Problems from Taylor's *Foundations of Analysis*.

85[8] Prove that if $\{f_n\}$ be a sequence uniformly continuous functions on a set D and if this sequence converges uniformly to f on D , then f is also uniformly continuous.

97[4] Using the theorems of section 4.2, find the derivative of $f(x) = \tan\left(\frac{x}{1+x^2}\right)$.

97[9] Prove that if f is defined on an open interval I and has positive derivative at a point $a \in I$, then there is an open interval J containing a and contained in I , such that $f(x) < f(a) < f(y)$ whenever $x, y \in J$ and $x < a < y$. [Hint: see Exercise 4.1.11.]

97[11] Is the function $f(x)$ differentiable at 0? How about the function $g(x)$? Where

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0; & \text{if } x = 0; \end{cases} \quad g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0; & \text{if } x = 0. \end{cases}$$

Please hand in problems L1 on Wednesday, November 23.

L1. Problems from Taylor's *Foundations of Analysis*.

102[3] If $r > 0$, prove that $\ln y - \ln x \leq \frac{y-x}{r}$ if $r \leq x \leq y$.

102[5] Suppose that f is a differentiable function on $(0, \infty)$. If both f and f' have finite limits at ∞ , then $\lim_{x \rightarrow \infty} f'(x) = 0$. [Hint: Apply the Mean Value Theorem to f for large values of a and b .]

102[7] Prove that $\ln x \leq x - 1$ for all $x > 0$. [Hint: Analyze where $x - 1 - \log x$ is increasing and where it is decreasing.]

102[8] Find where $e^{-x}x^e$ is increasing and where it is decreasing. Which is bigger: e^π or π^e ?

107[2] Prove that $|\sin x - x| \leq \frac{1}{6}|x|^3$.

107[4] If f is a function which is differentiable on the open interval I containing 0 and if $f(0) = 0$, then prove that there is a c between 0 and $x \in I$ at which

$$f(x) = \frac{f'(c)x^n}{c^{n-1}n}.$$

[Hint: Apply the Cauchy Mean Value Theorem to $f(x)$ and $g(x) = x^n$.]

1. 107[6–13] For any two of the following problems, find the limit if it exists.

6. $\lim_{x \rightarrow \infty} \frac{\ln x}{x^r}$ where $r > 0$.

7. $\lim_{x \rightarrow 0} x \ln x$.

8. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.

9. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ and $\lim_{x \rightarrow 0} \frac{1 + \cos x}{x^2}$.

10. $\lim_{x \rightarrow 0} x^x$.

11. $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

12. $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$.

13. $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}}$.

Please hand in problems M1 on Friday, Dec. 2.

M1. Problems from Taylor's *Foundations of Analysis*.

116[3] Using induction, prove that $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

116[4] Prove that $\int_0^b x^2 dx = \frac{b^3}{3}$ by expressing the integral as the limit of Riemann sums and finding the limit.

116[8] Suppose that m and M are lower and upper bounds for f on $[a, b]$. Prove that

$$m(b-a) \leq \int_a^b f(x) dx \leq \int_a^b f(x) dx \leq M(b-a).$$

If f were integrable, what conclusion do you draw from this about $\int_a^b f(x) dx$?

116[10] Suppose that f is any non-decreasing function on the bounded interval $[a, b]$. If \mathcal{P}_n is the partition into n equal subintervals, show that

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = (f(b) - f(a)) \frac{b-a}{n}.$$

What do you conclude about the integrability of f ?

123[5] Prove:

Corollary 5.2.8. *Suppose $a \leq b \leq c$ and f is a bounded function on $[a, c]$. Then f is integrable on $[a, c]$ if and only if it is integrable on $[a, b]$ and on $[b, c]$. In this case,*

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

Please hand in problems N1 on Friday, Dec. 9. All outstanding homework is due on this day.

N1. Problems from Taylor's *Foundations of Analysis*.

123[6] Prove for all $n \in \mathbb{N}$ that

$$1 \leq \int_{-1}^1 \frac{dx}{1+x^{2n}} \leq 2.$$

123[9] Prove that if f is integrable on $[a, b]$ then so is f^2 . [Hint: If $|f(x)| \leq L$ for all $x \in [a, b]$, then $|f^2(x) - f^2(y)| \leq 2L|f(x) - f(y)|$ for all $x \in [a, b]$.]

123[10] Prove that if f and g are integrable on $[a, b]$ then so is fg . [Hint: Polarize! Write fg as the difference of two squares of functions you know to be integrable, and use Ex. 9.]

123[13] Let $\{f_n\}$ be sequence of integrable functions defined on the closed, bounded interval $[a, b]$. Suppose that $\{f_n\}$ converges uniformly to a function f on $[a, b]$. Then f is integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

129[4] Find

$$\frac{d}{dx} \int_{1/x}^x e^{-t^2} dt.$$

129[5] If $f(x) = -1/x$ then $f'(x) = 1/x^2$. Thus, Theorem 5.3.1 seems to imply

$$\int_{-1}^1 \frac{dx}{x^2} = f(1) - f(-1) = -1 - 1 = -2.$$

However, $1/x^2$ is a positive function, so its integral over $[-1, 1]$ should be positive. What is wrong?

129[8] For arbitrary integer n , find

$$\int_0^x t^n \ln t dt.$$

129[9] Suppose that f is integrable on $[a, b]$, $c \in [a, b]$ and $z \in \mathbb{R}$. Define a new function by

$$g(x) = \begin{cases} f(x), & \text{if } x \in [a, b] \text{ and } x \neq c; \\ z, & \text{if } x = c. \end{cases}$$

Then g is integrable on $[a, b]$ and

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

The FINAL EXAM is Wednesday, Dec. 14 at 10:30 AM in the usual classroom, WBB 207.