Math 3210 § 2.	**Final Exam**	Name: Sample Exam
Treibergs		December 10, 2009

Final Given Dec. 18, 2008.

1. Let $\mathcal{F}$ be an ordered field and let $A \subset \mathcal{F}$.
   (a) Define what it means for $m \in \mathcal{F}$ to be an *upper bound* of A.
   (b) Define what it means for $m \in \mathcal{F}$ to be the *least upper bound* of A.
   (c) Define what it means for $\mathcal{F}$ to be *complete.*
   (d) Let $A = \{x \in \mathbb{Q} : x < \pi\} \subset \mathbb{R}$. Find the least upper bound of A, and prove your answer.

2a. Let $D \subset \mathbb{R}$, let $a \in D$ and let $f : D \to \mathbb{R}$. Define what it means for f to be *continuous* at a.

b. Let $D = \mathbb{R}$ and $f : D \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \ge 3, \\ 1, & \text{if } x < 3. \end{cases}$$

Show directly from the definition that f is not ontinuous at 3.

3. Let $f : [0, 2] \to \mathbb{R}$ be a bounded nonnegative function ($f(x) \ge 0$ for all x) that is integrable on $[0, 2]$. Suppose that $\lim\limits_{x\to 1} f(x) = 3$. Show that

$$\int_0^2 f(t)\, dt > 0.$$

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$0 \le f(x) \le (x - 1)^2$$

for all $x \in \mathbb{R}$. Show that f is differentiable at $x = 1$ and find $f'(1)$.

5a. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$. State the definition: f is a *uniformly continuous* on D.

b. Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, where $f(x) = \dfrac{x}{1 + x^2}$.

6a. Let $\{S_n\}$ be a sequence of real numbers. State the definition: $\{S_n\}$ is a *Cauchy Sequence..*

b. For each $n \in \mathbb{N}$ let $a_n \in \mathbb{R}$ and define

$$S_n = a_1 + a_2 + \cdots + a_n,$$
$$T_n = |a_1| + |a_2| + \cdots + |a_n|.$$

Suppose that $T = \lim\limits_{n\to\infty} T_n$ exists and is finite. Show that $S = \lim\limits_{n\to\infty} S_n$ exists and is finite. [Hint: you can use part (a.) This shows that if $\sum_{i=1}^{\infty} |a_i|$ converges then so does $\sum_{i=1}^{\infty} a_i$.]

7. Determine whether the improper integral exists. If it does, find its value.

$$I = \int_{-1}^{1} \frac{\sin t}{|t|^{3/2}}\, dt$$

8. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that for all $x \in [0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$. Then $\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = 0$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of differentiable functions such that $f_n \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$. Then f is differentiable on $\mathbb{R}$.

   (c) Let $f : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b). Suppose that the derivative function has a finite limit $L = \lim_{x\to a+} f'(x)$. Then f is differentiable at a and $L = f'(a)$.

9a. Let $f : [1,10] \to \mathbb{R}$ be a bounded function. State the **definition:** f is *integrable* on $[0,10]$.

b. Fill in the blank. [There is more than one answer but don't write the definition again. Your statement must be an "if and only if" statement to receive credit. See problem (*c.*)]

   Theorem B.

   Let $f : [0,10] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[0,10]$ **if and only if**

?

c. Using just the definition or just your Theorem B above, show that $f(t) = \sin t$ is integrable on $[0,10]$.

Final Given December 15, 2004.

1. Let $E = \left\{ \frac{p}{q} : p, q \in \mathbf{N} \right\}$. Find the infimum $\inf E$. Prove your answer.

2. Using only the definition of integrability, prove that $f(x)$ is integrable on $[0,1]$, where

$$f(x) = \begin{cases} 0, & \text{if } x = \frac{1}{3}; \\ 1, & \text{if } x = \frac{2}{3}; \\ 2, & \text{otherwise.} \end{cases}$$

3. Let $f : [0,1] \to \mathbf{R}$ be continuous on $[0,1]$ and suppose that $f(x) = 0$ for each rational number x in $[0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

4. Determine whether the statements are true or false. If the statement is true, give the reason. If the statement is false, provide a counterexample.

   (a) **Statement.** Let f be differentiable at a. Then $\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$.

   (b) **Statement.** Let $f : [0,1] \to \mathbf{R}$ be such that $|f(x)|$ is Riemann integrable on $[0,1]$. Then $f(x)$ is Riemann Integrable on $[0,1]$.

   (c) **Statement.** If $f : [a,b] \to \mathbf{R}$ is differentiable on $[a,b]$ then $F(x)$ is continuous on $[a,b]$, where $F(x) = \int_a^x f(t)\,dt$.

5. Suppose that f and g are continuous functions on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) = g(0)$ and that $f'(x) \leq g'(x)$ for all $x \in (a, b)$. Show that $f(x) \leq g(x)$ for all $x \in [0, 1]$.

6. Let $E \subseteq \mathbf{R}$ and $f : E \to \mathbf{R}$.

   (a) State the definition: f is uniformly continuous on E.

   (b) Let $f(x)$ be uniformly continuous on $\mathbf{R}$. Prove that

$$\lim_{t \to 0} \left\{ \sup_{x \in \mathbf{R}} |f(x) - f(x+t)| \right\} = 0.$$

7. Show that $\{z_n\}_{n \in \mathbf{N}}$ is Cauchy, where $z_n = \displaystyle\int_n^{n+1} \frac{\sin t}{1+t}\, dt.$

8. Let $x_1 = 0$ and $x_{n+1} = \frac{1}{2} + \sin(x_n)$ for all $n > 1$. Prove that $\{x_n\}_{n \in \mathbf{N}}$ converges.