Math 3210 § 2. **A Final & Some Solved Problems** Name: SAMPLE
Treibergs $\sigma\tau$ December 1, 2009

Final Given Dec. 15, 2000. (That course covered Chapter 7 instead of Chapter 5.)

(1.) *Using only the definition of differentiability and limit theorems, show that* $f(x) = \frac{x}{1+x}$ *is differentiable at* $x = 2$ *and that* $f'(2) = \frac{1}{9}$.

(2.) *Let* $E = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \dots\right\} = \left\{\frac{(-1)^n \cdot n}{n+1} : n = 1, 2, 3, \dots\right\}$. *Find* $\inf E$. *Prove your answer.*

(3.) *Define a sequence by* $x_1 = 10$ *and* $x_{n+1} = 2 + \frac{1}{3}x_n$ *for* $n \ge 1$.
i. Show that x_n *is decreasing.*
ii. Show that x_n *is bounded below.*
iii. Show that $x_n \to 3$ *as* $n \to \infty$.

(4.) *Let* $\{x_n\}_{n\in\mathbf{N}}$ *be a seqence of real numbers.*
i. State the definition: $\{x_n\}_{n\in\mathbf{N}}$ *is a Cauchy Sequence.*
ii. Show that the sequence $\{y_n\}_{n\in\mathbf{N}}$ *is a Cauchy Sequence, where*
$y_1 = \frac{1}{1}$, $y_2 = \frac{1}{1} - \frac{1}{1\cdot 2}$, $y_3 = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3}$, $y_4 = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4}, \dots$ *In general,* $y_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$.
[Hint: You may first wish to prove that $\frac{1}{k!} \le \frac{1}{2^{k-1}}$*.]*

(5.) *Let* $E \subset \mathbf{R}$ *be a subset,* $a \in E$ *be a point and* $f : E \to \mathbf{R}$ *be a function.*
i. State the definitions: f is *continuous at* a. f is *continuous on* E.
ii. Define $f(x) = x + \frac{1}{x}$. Show directly from the definition that f is continuous on $(0, \infty)$.

(6.) *Let* $f : \mathbf{R} \to \mathbf{R}$ *be defined by* $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbf{Q} \text{ (} x \text{ is a rational number,)} \\ 0, & \text{if } x \notin \mathbf{Q} \text{ (} x \text{ is not a rational number.)} \end{cases}$
i. Show that f *is differentiable at* $x = 0$ *and find* $f'(0)$.
ii. If $a \ne 0$*, is* f *differentiable at* a*? Why?*

(7.) *Assume that the function* $f : \mathbf{R} \to \mathbf{R}$ *is differentiable on* $\mathbf{R}$ *and satisfies* $f(x) > 0$ *for all* $x \in \mathbf{R}$. *Using only the definition of differentiability and limit theorems (and not the quotient rule for derivatives,) show that the reciprocal* $g(x) = \frac{1}{f(x)}$ *is differentiable for all* $a \in \mathbf{R}$ *and that* $g'(a) = -\frac{f'(a)}{f(a)^2}$.

(8.) *Define a function* $f : \mathbf{R} \to \mathbf{R}$ *by* $f(x) = \frac{x^5 + x^2 - 2x}{x^4 + 1}$. *Give an argument if true; give a counterexample if false:*
i. $f : \mathbf{R} \to \mathbf{R}$ *is onto.*
ii. $f : \mathbf{R} \to \mathbf{R}$ *is one-to-one.*
iii. There is a number $x_0 \in \mathbf{R}$ *so that* $f'(x_0) = 0$.

(9.) *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a differentiable function. Suppose there is a constant* $0 \le M < \infty$ *so that* $|f'(\xi)| \le M$ *for all* $\xi \in \mathbf{R}$.
i. Prove that for all $x, y \in \mathbf{R}$ *there holds* $|f(x) - f(y)| \le M|x - y|$.
ii. Assuming (i.), prove that f *is uniformly continuous on* $\mathbf{R}$.

(10.) *Define the sequence of functions by* $f_n(x) = \frac{1}{n^2} + \frac{x}{n}$. *Find* $\lim_{n\to\infty} f_n(x)$ *and show that the convergence is uniform on* $[-3, 3]$.

Final Given Dec. 15, 2004.

(1.) *Let* $E = \left\{ \dfrac{p}{q} : p, q \in \mathbf{N} \right\}$. *Find the infimum,* $\inf E$. *Prove your answer.*

(2.) *Using only the definition of integrability, prove that* $f(x)$ *is integrable on* $[0,1]$, *where*

$$f(x) = \begin{cases} 0, & \text{if } x = \frac{1}{3}; \\ 1, & \text{if } x = \frac{2}{3}; \\ 2, & \text{otherwise.} \end{cases}$$

(3.) *Let* $f : [0,1] \to \mathbf{R}$ *be continuous on* $[0,1]$ *and suppose that* $f(x) = 0$ *for each rational number* x *in* $[0,1]$. *Prove that* $f(x) = 0$ *for all* $x \in [0,1]$.

(4.) *Determine whether the statements are true or false. If the statement is true, give the reason. If the statement is false, provide a counterexample.*

i. **Statement.** *Let* $f : \mathbf{R} \to \mathbf{R}$ *be differentiable at* a. *Then* $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$

ii. **Statement.** *Let* $f : [0,1] \to \mathbf{R}$ *be such that* $|f(x)|$ *is Riemann integrable on* $[0,1]$. *Then* $f(x)$ *is Riemann Integrable on* $[0,1]$.

iii. **Statement.** *If* $f : [a,b] \to \mathbf{R}$ *is differentiable on* $[a,b]$. *Then* $F(x) = \displaystyle\int_a^x f(t)\,dt$ *is continuous on* $[a,b]$.

(5.) *Suppose that* f *and* g *are continuous functions on* $[0,1]$ *and differentiable on* $(0,1)$. *Suppose that* $f(0) = g(0)$ *and that* $f'(x) \le g'(x)$ *for all* $x \in (0,1)$. *Show that* $f(x) \le g(x)$ *for all* $x \in [0,1]$.

(6.) *Let* $E \subseteq \mathbf{R}$ *and* $f : E \to \mathbf{R}$.

i. State the definition: f *is uniformly continuous on* E.

ii. Let $f(x)$ *be uniformly continuous on* $\mathbf{R}$. *Prove that* $\displaystyle\lim_{t\to 0} \left\{ \sup_{x\in\mathbf{R}} |f(x) - f(x+t)| \right\} = 0.$

(7.) *Show that* $\{z_n\}_{n\in\mathbf{N}}$ *is a Cauchy sequence, where* $z_n = \displaystyle\int_n^{n+1} \frac{\sin t}{1+t}\,dt.$

(8.) *Let* $x_1 = 0$ *and* $x_{n+1} = \frac{1}{2} + \sin(x_n)$ *for all* $n > 1$. *Prove that* $\{x_n\}_{n\in\mathbf{N}}$ *converges.*

Some solved problems from the last quarter of the semester (since the last midterm).

(1.) *Show that $f(x) = [3x]$ is integrable on $[0,1]$, where $[x]$ is the greatest integer function. What is $\int_0^1 f(x)\,dx$?*

To show f is integrable on $[0,1]$ we need to show that for every $\varepsilon > 0$ there is a partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0,1]$ so that that the upper sum minus the lower sum satisfies $U(f,P) - L(f,P) < \varepsilon$.

Choose $\varepsilon > 0$ and let η be any number such that $0 < \eta < \min\{\varepsilon, 1\}$. Because $f(x) = 0$ on the interval $[0, \frac{1}{3})$, $f = 1$ on $[\frac{1}{3}, \frac{2}{3})$, $f = 2$ on $[\frac{2}{3}, 1)$ and $f(1) = 3$, we shall choose a partition that has narrow intervals near the jumps of the function. Let

$$P = \left\{ x_0 = 0 < x_1 = \frac{1-\eta}{3} < x_2 = \frac{1}{3} < x_3 = \frac{2-\eta}{3} < x_4 = \frac{2}{3} < x_5 = \frac{3-\eta}{3} < x_6 = 1 \right\}$$

The sups and infs on the intervals are computed as follows

$$\begin{aligned} M_1 &= \sup_{x\in[x_0,x_1]} f(x) = \sup_{x\in\left[0,\frac{1-\eta}{3}\right]} 0 = 0, \quad M_2 = \sup_{x\in\left[\frac{1-\eta}{3},\frac{1}{3}\right]} f(x) = 1,\ M_3 = 1,\ M_4 = 2,\ M_5 = 2,\ M_6 = 3; \\ m_1 &= \inf_{x\in[x_0,x_1]} f(x) = \inf_{x\in\left[0,\frac{1-\eta}{3}\right]} 0 = 0, \quad m_2 = \inf_{x\in\left[\frac{1-\eta}{3},\frac{1}{3}\right]} f(x) = 0,\ m_3 = 1,\ m_4 = 1,\ m_5 = 2,\ m_6 = 2; \end{aligned}$$

Hence, since $M_1 - m_1 = M_3 - m_3 = M_5 - m_5 = 0$, $M_2 - m_2 = M_4 - m_4 = M_6 - m_6 = 1$ and $x_{2j} - x_{2j-1} = \dfrac{\eta}{3}$,

$$U - L = \sum_{i=1}^{6} (M_i - m_i)(x_i - x_{i-1}) = \frac{\eta}{3}\left[(M_2 - m_2) + (M_4 - m_4) + (M_6 - m_6)\right] = \frac{3\eta}{3} < \varepsilon.$$

Thus f is integrable on $[0,1]$. We may compute the upper sum using $x_{2j+1} - x_{2j} = \dfrac{1-\eta}{3}$

$$\begin{aligned} U = \sum_{i=1}^{6} M_i(x_i - x_{i-1}) &= 0 + 1\cdot(x_2 - x_1) + 1\cdot(x_3 - x_2) + 2\cdot(x_4 - x_3) + 2\cdot(x_5 - x_4) + 3\cdot(x_6 - x_5) \\ &= \frac{\eta}{3} + \frac{1-\eta}{3} + 2\cdot\frac{\eta}{3} + 2\cdot\frac{1-\eta}{3} + 3\cdot\frac{\eta}{3} = 1 + \eta \end{aligned}$$

Similarly $L(f,P) = 1$. We deduce the value of the integral from the fact that for integrable functions, for any partition the upper and lower sums bracket the integral. Thus for our partition above,

$$1 = L(f,P) \le \int_0^1 f(x)\,dx \le U(f,P) < 1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\int_0^1 f(x)\,dx = 1$.

(2.) *Suppose that $f : [-a,a] \to \mathbf{R}$ is integrable and odd ($f(x) = -f(x)$ for all $x \in [-a,a]$.) Show that $\int_{-a}^{a} f(x)\,dx = 0$.*

One solution is to use Riemann's point sum to approximate the integral. Since f is integrable, if $I = \int_{-a}^{a} f(x)\,dx$ then $S(f,P,T) \to I$ as $\|P\| \to 0$ (see problem 10). This means that for every $\varepsilon > 0$ there is a partition P_ε such that for any refinement $Q \supseteq P_\varepsilon$ where $Q = \{-a = x_0 < x_1 < \cdots < x_n = a\}$ and for any choice of sampling points $T = (t_1, t_2, \ldots, t_n)$ where $t_i \in (x_{i-1}, x_i)$, we have that the Riemann Point Sum satisfies $|S(f,Q,T) - I| < \varepsilon$.

Choose any $\varepsilon > 0$ and let P_ε be the corresponding partition. Throw in the reflections of all division points of P_ε and zero to get a refined symmetric partition $Q = P_\varepsilon \cup (-P_\varepsilon) \cup \{0\}$, where $-P_\varepsilon = \{-t : t \in P_\varepsilon\}$. (So $Q = -Q$.) Hence we may use an unusual numbering with $2n$ intervals for Q so that $x_{-i} = -x_i$,

$$Q = \{-a = x_{-n} < x_{1-n} < \cdots < x_{-1} < x_0 = 0 < x_1 < \cdots < x_n = a\}.$$

We also choose symmetric sample points $T = (t_{1-n}, t_{2-n}, \ldots, t_n)$ where $t_i \in (x_{i-1}, x_i)$ and $t_{1-j} = -t_j \in (x_{-j}, x_{1-j})$ for $i, j = 1, \ldots, n$. Now since f is odd, this implies that $f(t_{1-k}) = -f(t_k)$ for all k. Thus after changing dummy index $j = 1 - k$, the Riemann sum is

$$\begin{aligned}
S(f, Q, T) &= \sum_{j=1-n}^{n} f(t_j)(x_j - x_{j-1}) \\
&= \sum_{j=1-n}^{0} f(t_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \\
&= \sum_{k=1}^{n} f(t_{1-k})(x_{1-k} - x_{-k}) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \\
&= \sum_{k=1}^{n} -f(t_k)(-x_{k-1} + x_k) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \\
&= 0.
\end{aligned}$$

Thus we have shown that for all $\varepsilon > 0$, $|0 - I| = |S(f, Q, T) - I| < \varepsilon$ so $I = 0$.

Another method is to split the integral as the sum $I = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ and then change variables $x = \varphi(\xi) = -\xi$ in the first integral $\int_{-a}^{0} f(x)\,dx = \int_a^0 f(\varphi(\xi))\,\varphi'(\xi)\,d\xi = -\int_a^0 f(-\xi)\,d\xi = -\int_0^a f(\xi)\,d\xi$ which cancels the second integral.

(3.) *Suppose $a < b$, $0 < k$ and $f : [a, b] \to \mathbf{R}$ is an integrable function such that $f(x) > k$ for all $x \in [a, b]$. Show that $h(x) = \frac{1}{f(x)}$ is integrable on $[a, b]$.*

We are to show that $\frac{1}{f}$ is bounded and for every $\varepsilon > 0$ there is a partition P such that the upper sum minus the lower sum $U(\frac{1}{f}, P) - L(\frac{1}{f}, P) < \varepsilon$.

As f is integrable, it is bounded: there is $M \in \mathbf{R}$ so that , $k \le f(x) \le M$ for all $x \in [a, b]$. But as $k > 0$ we conclude that $\frac{1}{M} \le \frac{1}{f(x)} \le \frac{1}{k}$ for all $x \in [a, b]$, thus $\frac{1}{f}$ is bounded.

Choose $\varepsilon > 0$. As f is integrable, there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ so that $U(f, P) - L(f, P) < k^2\varepsilon$. Consider the sup and inf over each of the subintervals

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Since $k \le m_i \le f(x) \le M_i$ for $x \in [x_{i-1}, x_i]$, sup and inf for $\frac{1}{f}$ satisfy on this interval

$$\frac{1}{M_i} \le \frac{1}{f(x)} \le \frac{1}{m_i} \quad \Longrightarrow \quad \frac{1}{M_i} \le m_i' = \inf_{x \in [x_{i-1}, x_i]} \frac{1}{f(x)}, \qquad M_i' = \sup_{x \in [x_{i-1}, x_i]} \frac{1}{f(x)} \le \frac{1}{m_i}.$$

Thus estimating the upper and lower sums for $\frac{1}{f}$, since all $(x_i - x_{i-1}) \ge 0$,

$$\begin{aligned} U\left(\frac{1}{f}, P\right) - L\left(\frac{1}{f}, P\right) &= \sum_{j=1}^{n} (M_i' - m_i')(x_i - x_{i-1}) \\ &\le \sum_{j=1}^{n} \left(\frac{1}{m_i} - \frac{1}{M_i}\right)(x_i - x_{i-1}) \\ &= \sum_{j=1}^{n} \frac{(M_i - m_i)}{m_i M_i}(x_i - x_{i-1}) \\ &\le \sum_{j=1}^{n} \frac{(M_i - m_i)}{k^2}(x_i - x_{i-1}) \\ &= \frac{1}{k^2}(U(f,P) - L(f,P)) < \frac{k^2 \varepsilon}{k^2} = \varepsilon. \end{aligned}$$

(4.) *Suppose that $f, g : \mathbf{R} \to \mathbf{R}$ are differentiable functions such that $g(x) \ne 0$ and $g'(x) \ne 0$ for all $x \ne 0$. Suppose that $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \pi$. Show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \pi$.*

This is just a particular case of l'Hôpital's Rule. So we may follow Proof (4.18). First notice that $g(x) - g(0) \ne 0$ and $g'(x) \ne 0$ if $x \ne 0$, so we may divide by them. By the sequential characterization of limits, it suffices to show that $\lim_{k\to\infty} \frac{f(x_i)}{g(x_i)} = \pi$ for every sequence $\{x_i\}_{i\in\mathbf{N}}$ such that $x_i \ne 0$ for all i and $x_i \to 0$ as $i \to \infty$. Similarly by the equivalence of the existence of a limit to the existence of equal left and right limits, we may assume $x_i > 0$ for all i or $x_i < 0$ for all i. As f and g are continuous at 0, we have $f(0) = g(0) = 0$. We may suppose that $0 < x_i$ for all i. Now, since f is assumed to be continuous on $[0, x_i]$ and differentiable on $(0, x_i)$ (actually we assumed more,) by the Generalized Mean Value Theorem there is a $c_i \in (0, x_i)$ so that

$$\frac{f(x_i)}{g(x_i)} = \frac{f(x_i) - f(0)}{g(x_i) - g(0)} = \frac{f'(c_i)}{g'(c_i)} \to \pi \quad \text{as } i \to \infty.$$

Now, letting $i \to \infty$, we have $x_i \to 0$. Since $0 < c_i < x_i$, by the Squeeze Theorem, $c_i \to 0$, so the conclusion follows since the sequential characterization implies $\lim_{i\to\infty} \frac{f'(c_i)}{g'(c_i)} = \pi$. A similar argument on the left side gives the same conclusion.

(5.) *Let f be continuous on $[a, b]$ and that $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for all $x \in [a, b]$. Then $f(x) = 0$ for all $x \in [a, b]$.*

Observe that if $x \in [a, b]$ then

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt \implies \int_a^x f(t)\,dt = \tfrac{1}{2}\int_a^b f(t)\,dt = \text{const.}$$

Thus, since f is continuous, by the Fundamental Theorem of Calculus, for all $x \in [a, b]$, the primitive function is differentiable and equals

$$f(x) = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\text{const.} = 0.$$

(6.) *Let $f(x) = x + x^3 + x^5$. Show f has a continuous inverse function $f^{-1} : \mathbf{R} \to \mathbf{R}$. Find $(f^{-1})'(3)$ if possible.*

First observe that since f is a polynomial, it is differentiable on $\mathbf{R}$. Second, observe that f is strictly increasing on $\mathbf{R}$. (This is like Theorem 4.24.) To see this, choose any numbers $a < b$ and since f is continuous on $[a,b]$ and differentiable on (a,b), by the Mean Value Theorem there is a $c \in (a,b)$ so that $f(b)-f(a) = f'(c)(b-a)$. However $f'(c) = 1+3c^2+5c^4 > 0$, so this implies $f(b) > f(a)$. Third we apply the Inverse Function Theorem 4.26. As f is strictly increasing (so one to one,) and continuous on $\mathbf{R}$, there is a continuous, strictly increasing inverse function on $f(\mathbf{R}) = \mathbf{R}$ (since f is not bounded above and not bounded below.) By Theorem 4.27, since f is one to one, continuous and differentiable at x_0 where $f'(x_0) = 1+3x_0^2+5x_0^4 \neq 0$, the inverse function is differentiable. If $f(x_0) = y_0$ (for example $f(1) = 3$) then

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} \qquad \Longrightarrow \qquad \left(f^{-1}\right)'(3) = \frac{1}{f'(1)} = \frac{1}{1+3(1^2)+5(1^4)} = \frac{1}{9}.$$

(7.) *Suppose that $x > -1$. Show that $\frac{1}{\sqrt{1+x}} \geq 1 - \frac{x}{2}$.*

This is an application of the Mean Value Theorem. Let $h(x) = (1+x)^{-1/2} - 1 + \frac{1}{2}x$. This function is differentiable for $x > -1$. Also notice that $h(0) = 0$. Then for any $x > 0$ there is a $c \in (0,x)$ so that $h(x) = h(x) - h(0) = h'(c)(x-0)$. But $h'(c) = \frac{1}{2}\left(1-(1+c)^{-3/2}\right) > 0$ for $c > 0$ because $(1+c)^{-3/2} < 1$. Hence $h(x) > 0$. Similarly, if $-1 < x < 0$ then there is a $c \in (x,0)$ so that $-h(x) = h(0) - h(x) = h'(c)(0-x)$. This time $h'(c) < 0$ because $(1+c)^{-3/2} > 0$. Hence also $h(x) > 0$. Putting inequalities on the two intervals and at zero together, $h(x) \geq 0$ for all $x > -1$.

(8.) *Calculate* $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} e^{t^2}\,dt.$

Let $F(x) = \int_0^x e^{t^2}\,dt$. Since $f(t) = e^{t^2}$ is continuous as it is the composition of continuous functions, $F(x)$ is differentiable and $F' = f$, by the Fundamental Theorem of Calculus. The limit becomes the limit of a difference quotient

$$\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}\,dt = \lim_{h\to 0}\frac{F(3+h)-F(3)}{h} = F'(3) = f(3) = e^9.$$

(9.) *Suppose $g : \mathbf{R} \to \mathbf{R}$ is a continuous function. Find* $\displaystyle\frac{d}{dt}\int_0^t g(x-t)\,dx.$

We do not yet have the machinery to pass derivatives through integrals. Therefore we shall change variables in the integral to put the dependence on t in the limits of integration. To that end, choose t and x_0 so that $x_0 < -|t|$. Let $F(t) = \int_{x_0}^t g(\xi)\,d\xi$. Since t is constant as far as the integration is concerned, change variables according to x=$\varphi(\xi) = \xi + t$. Then since φ is continuously differentiable on $\mathbf{R}$, and g is continuous on $\varphi(\mathbf{R}) = \mathbf{R}$, we have by the change of variables formula,

$$\int_0^t g(x-t)\,dx = \int_{\varphi(-t)}^{\varphi(0)} g(x-t)\,dx = \int_{-t}^0 g(\varphi(\xi)-t)\varphi'(\xi)\,d\xi = \int_{-t}^0 g(\xi)\cdot 1\,d\xi = F(0) - F(-t).$$

Since g is continuous on $\mathbf{R}$, by the Fundamental Theorem of Calculus, F is differentiable and $F'(\xi) = g(\xi)$. Now, by the chain rule,

$$\frac{d}{dt}\int_0^t g(x-t)\,dx = \frac{d}{dt}\left(F(0)-F(-t)\right) = 0 - F'(-t)\cdot(-1) = g(-t).$$

(10.) *Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. The mesh $\|P\| = \max\{x_i - x_{i-1} : i = 1,\ldots,n\}$ is the length of the largest subinterval. Let $f : [a,b] \to \mathbf{R}$ be a bounded function. Show that f is integrable on $[a,b]$ if and only if*

(†) For all $\varepsilon > 0$ there is $\delta > 0$ so that $U(f,P) - L(f,P) < \epsilon$ whenever $\|P\| < \delta$.

You may use the theorem that f is integrable on $[a,b]$ if and only if

(‡.) *For all $\varepsilon > 0$ there is a partition P so that $U(f,P) - L(f,P) < \epsilon$.*

(†.) $\implies$ (‡.): Choose ϵ and let δ be given by (†). Then any partition such that $\|P\| < \delta$ works: $U(f,P) - L(f,P) < \epsilon$ for that P. Such a partition may be taken to be the one with equal subinternals, namely, $P = P_n = \{x_i\}_{i=1,\dots,n}$ where $x_i = a + \frac{i}{n}(b-a)$.

(‡.) $\implies$ (†.): Choose $\epsilon > 0$. By (‡), there is a partition $Q = \{a = y_1 < y_2 < \cdots < y_m = b\}$ such that $U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}$. The idea is to choose the mesh size δ much finer than Q so that only a few subintervals of a partition $P = \{a = x_i < x_2 < \cdots < x_n = b\}$ such that $\|P\| < \delta$ straddle the y_i's. Let $\eta = \min\{y_i - y_{i-1} : i = 1,\dots,m\}$ be the size of the smallest interval in Q. Since f is bounded there is $K < \infty$ so that $|f(x)| < K$ for all $x \in [a,b]$. Let $\delta = \min\left\{\dfrac{\eta}{2}, \dfrac{\varepsilon}{4mK}\right\}$ and P any partition of $[a,b]$ such that $\|P\| < \delta$.

Let $M_i = \sup\limits_{[x_{i-1},x_i]} f$, $m_i = \inf\limits_{[x_{i-1},x_i]} f$, $M'_j = \sup\limits_{[y_{j-1},y_j]} f$ and $m'_j = \inf\limits_{[y_{i-1},y_i]} f$.

Note that if $[x_{i-1},x_i] \subset [y_{j-1},y_j]$ then $M_i - m_i \le M'_j - m'_j$. And always we have $M_i - m_i \le 2K$.

Split the sum into the sum over those subintervals $[x_{i-1},x_i] \subset [y_{j-1},y_j]$ and the sum over those few i's with $y_j \in (x_{i-1},x_i)$ when y_j are not in P. Since $\|P\| < \eta$, no two y_j's are ever in the same subinterval $[x_{i-1},x_i]$.

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{j=1}^{m}\left(\sum_{[x_{i-1},x_i]\subset[y_{j-1},y_j]}(M_i - m_i)(x_i - x_{i-1}) + \sum_{y_j\in(x_{i-1},x_i)}(M_i - m_i)(x_i - x_{i-1})\right) \\
&\le \sum_{j=1}^{m}\left(\sum_{[x_{i-1},x_i]\subset[y_{j-1},y_j]}(M'_j - m'_j)(x_i - x_{i-1}) + \sum_{y_j\in(x_{i-1},x_i)} 2K\delta\right) \\
&\le \sum_{j=1}^{m}(M'_j - m'_j)(y_j - y_{j-1}) + 2Km\delta = U(f,Q) - L(f,Q) + 2Km\delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

(11.) *Suppose $f, f_n : [a,b] \to \mathbf{R}$ are functions such that f_n is integrable and $f_n \to f$ uniformly on $[a,b]$. Then f is integrable on $[a,b]$ and*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx. \tag{✠}$$

We show f is bounded and that for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$, hence f is integrable. The idea is similar to proving that the uniform limit of continuous functions is continuous, namely, to approximate the limit f by an f_n with large enough n. Choose $0 < \varepsilon < 5(b-a)$. By uniform convergence, there is $R \in \mathbf{R}$ so that $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$ whenever $n > R$ and $x \in [a,b]$. Choose $n \in \mathbb{N}$ so that $n > R$. Since f_n is integrable it is bounded: for some $K_n < \infty$, $|f_n(x)| < K_n$ for all $x \in [a,b]$. Thus by uniform convergence, $|f(x)| = |f(x) - f_n(x) + f_n(x)| \le |f(x) - f_n(x)| + |f_n(x)| \le \frac{\varepsilon}{5(b-a)} + K_n \le 1 + K_n$ since $\varepsilon < 5(b-a)$, hence f is bounded. Since f_n is integrable, there is a partition $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ so that $U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{2}$. We use the same partition for f.

Let $M_i = \sup\limits_{[x_{i-1},x_i]} f_n$, $m_i = \inf\limits_{[x_{i-1},x_i]} f_n$. By uniform continuity, for $x \in [x_{i-1},x_i]$,

$$m_i - \frac{\varepsilon}{5(b-a)} < f_n(x) - |f_n(x) - f(x)| \le f(x) \le f_n(x) + |f(x) - f_n(x)| < M_i + \frac{\varepsilon}{5(b-a)}$$

Let $M_i' = \sup\limits_{[x_{i-1},x_i]} f$ and $m_i' = \inf\limits_{[x_{i-1},x_i]} f$. Thus $M_i' \le M_i + \frac{\varepsilon}{5(b-a)}$ and $m_i' \ge m_i - \frac{\varepsilon}{5(b-a)}$. Estimating,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^m (M_i' - m_i')(x_i - x_{i-1}) \\ &\le \sum_{i=1}^m \left(M_i + \frac{\varepsilon}{5(b-a)} - m_i + \frac{\varepsilon}{5(b-a)} \right)(x_i - x_{i-1}) \\ &= U(f_n,P) - L(f_n,P) + \frac{2\varepsilon}{5(b-a)} \sum_{i=1}^m (x_i - x_{i-1}) \\ &\le \frac{\varepsilon}{2} + \frac{2\varepsilon(b-a)}{5(b-a)} < \varepsilon. \end{aligned}$$

So f is integrable on $[a,b]$. By uniform convergence, for all $n > R$ and all $x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{5(b-a)}$ implies

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \frac{\varepsilon}{5(b-a)}\,dx = \frac{\varepsilon(b-a)}{5(b-a)} < \varepsilon.$$

As ε was arbitrary, this implies (✠).

(12.) *Let $p > 0$. Show that the improper integral of $f(x) = \dfrac{\cos x}{(\ln x)^p}$ exists on (e,∞).*

Since f is continuous, on each interval $[e,R]$ the function f is integrable. The improper integral $\int_e^\infty f(x)\,dx$ exists provided $\lim\limits_{R\to\infty} \int_e^R f(x)\,dx$ exists as a finite number.

The idea is to split the integral $I(R) = \int_e^R f(x)\,dx$ into three parts, each of which converges as $R \to \infty$. Let $\alpha = \frac{7\pi}{2} > e$, $\beta(R) = 2\pi k(R) + \frac{3\pi}{2}$ where $k(R) \in \mathbb{N}$ such that $2\pi k(R) + \frac{3\pi}{2} \le R < 2\pi k(R) + \frac{7\pi}{2}$ for $R > \frac{11\pi}{2}$, $I_1 = \int_e^\alpha f(x)\,dx$, $I_2 = \int_\alpha^{\beta(R)} f(x)\,dx$ and $I_3 = \int_{\beta(R)}^R f(x)\,dx$. Then $I(R) = I_1 + I_2 + I_3$. I_1 is constant for all R. Since $(\ln x)^p$ is increasing for $x \ge e$ and tends to infinity as $x \to \infty$, and $\beta(R) \to \infty$ as $R \to \infty$,

$$|I_3(R)| \le \int_{\beta(R)}^R \frac{|\cos x|}{(\ln x)^p}\,dx \le \int_{\beta(R)}^R \frac{dx}{(\ln \beta(R))^p} = \frac{R - \beta(R)}{(\ln \beta(R))^p} \le \frac{2\pi}{(\ln \beta(R))^p} \to 0 \text{ as } R \to \infty.$$

Finally, since $\cos(x) = -\cos(x+\pi)$,

$$I_2 = \sum_{j=2}^{k(R)+1} \int_{2\pi j - \frac{\pi}{2}}^{2\pi j + \frac{3\pi}{2}} \frac{\cos x\,dx}{(\ln x)^p} = \sum_{j=2}^{k(R)+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{[\ln(2\pi j + x)]^p} - \frac{1}{[\ln(2\pi j + \pi + x)]^p} \right) \cos x\,dx.$$

$I_2(k(R))$ converges as $R \to \infty$ if and only if $I_2(k)$ converges as $k \to \infty$. But the parenthesis term and $\cos x$ are both positive, so that $I_2(k)$ is an increasing sequence in k, and hence converges if and only if it is bounded. But using $\cos x \le 1$ and $(\ln s)^p$ is increasing,

$$I_2 \le \pi \sum_{j=2}^{k(R)+1} \left(\frac{1}{[\ln(2\pi j - \frac{\pi}{2})]^p} - \frac{1}{[\ln(2\pi j + \frac{3\pi}{2})]^p} \right) = \frac{\pi}{[\ln(\frac{7\pi}{2})]^p} - \frac{\pi}{[\ln(2\pi k(R) + \frac{7\pi}{2})]^p} \le \frac{\pi}{[\ln(\frac{7\pi}{2})]^p}$$

since the sum telescopes.

(13.) *Evaluate the improper integral* $\mathcal{I} = \displaystyle\int_0^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}}\,dx$, *if it exists.*

The function is continuous on $(0, \pi/2]$ so it is integranble on every interval $[\varepsilon, \pi/2]$ where $0 < \varepsilon < \pi/2$. Thus the improper integral exists if $\displaystyle\lim_{\varepsilon\to 0+} \int_\varepsilon^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}}\,dx$ has a finite limit. But changing variables $u = \varphi(x) = \sin x$,

$$\int_\varepsilon^{\pi/2} \frac{\cos x\,dx}{(\sin x)^{1/3}} = \int_\varepsilon^{\pi/2} \frac{\varphi'(x)\,dx}{(\varphi(x))^{1/3}} = \int_{\sin\varepsilon}^{1} \frac{du}{u^{1/3}} = \frac{3}{2}u^{2/3}\Big|_{u=\sin\varepsilon}^{1} = \frac{3}{2}\left(1^{2/3} - (\sin\varepsilon)^{2/3}\right) \to \frac{3}{2}$$

as $\varepsilon \to 0+$, so the improper integral exists and $\mathcal{I} = 3/2$.