Math 3210 § 2. **First Midterm Exam** Name: Solutions
Treibergs September 16, 2009

1. Prove that $n! > 2^n$ for all natural numbers $n \ge 4$.

   Proof. Use induction on n. In the base case $n = 4$, then
   $\text{Lhs} = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4 = \text{Rhs}$.

   Induction case. Assume that for any $n \ge 4$ we have $n! > 2^n$. Then by the induction hypothesis and $n + 1 \ge 2$,

   $$(n+1)! = (n+1)n! > (n+1)2^n \ge 2 \cdot 2^n = 2^{n+1}.$$

2. Using only the axioms for the field $(F, +, \cdot)$, show that for all $x, y \in F$ such that $x \ne 0$, $y \ne 0$ and $xy \ne 0$ we have $(xy)^{-1} = y^{-1}x^{-1}$.

   Proof. We first prove the following Lemma.

   Lemma 1. *If $p, q \in F$ such that $q \ne 0$ and $pq = 1$ then $p = q^{-1}$.*

   Proof of Lemma. Since $q \ne 0$ there is $q^{-1} \in F$ by multiplicative inverse axiom (M4).

   $$\begin{aligned} pq = 1 &\implies (pq)q^{-1} = 1 \cdot q^{-1} && \text{Multiply by } q^{-1};\\ &\implies p(qq^{-1}) = 1 \cdot q^{-1} && \text{Associativity of multiplication (M2);}\\ &\implies p(q^{-1}q) = 1 \cdot q^{-1} && \text{Commutativity of multiplication (M1);}\\ &\implies p \cdot 1 = 1 \cdot q^{-1} && \text{Multiplicative inverse (M4);}\\ &\implies 1 \cdot p = 1 \cdot q^{-1} && \text{Commutativity of multiplication (M1);}\\ &\implies p = q^{-1} && \text{Multiplicative identity (M3).} \end{aligned}$$

   Proof. $x \ne 0$ and $y \ne 0$ so x^{-1} and y^{-1} exist by the multiplicative inverse axiom (M4). Let $p = y^{-1}x^{-1}$ and $q = xy \ne 0$ by assumption. Then

   $$\begin{aligned} pq &= (y^{-1}x^{-1})(xy) \\ &= y^{-1}\big(x^{-1}(xy)\big) && \text{Associativity of multiplication (M2);}\\ &= y^{-1}\big((x^{-1}x)y\big) && \text{Associativity of multiplication (M2);}\\ &= y^{-1}(1 \cdot y) && \text{Multiplicative inverse (M4);}\\ &= y^{-1}y && \text{Multiplicative identity (M3);}\\ &= 1 && \text{Multiplicative inverse (M4).} \end{aligned}$$

   Hence, by the lemma, $y^{-1}x^{-1} = p = q^{-1} = (xy)^{-1}$, as to be shown.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) **Statement.** If $A, B, C \subset X$ are subsets then $A \backslash B = C$ implies $A = B \cup C$.
   False. *e.g.,* take $X = \mathbb{R}$, $A = [0, 2]$, $B = [1, 3]$ so $C = [0, 1)$. But then $A \ne B \cup C = [0, 3]$.

   (b) **Statement.** Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions such that the composite $g \circ f : X \to Z$ is *one-to-one.* Then $f : X \to Y$ is *one-to-one.*
   True. Choose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Apply g to both sides $g \circ f(x_1) = g \circ f(x_2)$. But $g \circ f$ is one-to-one so $x_1 = x_2$. Thus f is one-to-one.

(c) **Statement.** If $f : X \to Y$ is *onto*, then for all subsets $A, B \subset X$ we have $f(A \cap B) = f(A) \cap f(B)$.

FALSE. *e.g.,* take $X = Y = \mathbb{R}$ and $f(x) = x^2(x-1)$ which is onto. But for $A = (-1, 0)$ and $B = (0, 1)$ we have $A \cap B = \emptyset$ so $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = (-2, 0) \cap [-\frac{4}{27}, 0) = [-\frac{4}{27}, 0) \neq \emptyset = f(A \cap B)$.

4. Let $f : X \to Y$ be a function and $V_\alpha \subset Y$ be a subset for each $\alpha \in A$. Show

$$f^{-1}\Big(\bigcap_{\alpha \in A} V_\alpha \Big) = \bigcap_{\alpha \in A} f^{-1}(V_\alpha).$$

Proof. We show x is in the left set iff x is in the right set.

$$\begin{aligned} x \in f^{-1}\Big(\bigcap_{\alpha \in A} V_\alpha \Big) &\iff f(x) \in \bigcap_{\alpha \in A} V_\alpha \\ &\iff (\forall \alpha \in A)\ (f(x) \in V_\alpha) \\ &\iff (\forall \alpha \in A)\ \big(x \in f^{-1}(V_\alpha)\big) \\ &\iff x \in \bigcap_{\alpha \in A} f^{-1}\Big(V_\alpha\Big). \end{aligned}$$

5. The text describes the rational numbers as equivalence classes of symbols

$$\mathbb{Q} = \left\{ \tfrac{p}{q} : p, q \in \mathbb{Z} \text{ such that } q \neq 0 \right\} \Big/ \sim$$

where $\frac{p}{q} \sim \frac{n}{m}$ if and only if $pm = nq$. In order to construct a function on the rationals, the following rule is proposed: for each $\left[\frac{a}{b}\right] \in \mathbb{Q}$, let $f\left(\left[\frac{a}{b}\right]\right) = \left[\frac{a^2}{a^2+b^2}\right]$ Determine whether this rule actually defines a function $f : \mathbb{Q} \to \mathbb{Q}$. If f is well-defined, prove it. If not, explain why not.

f IS WELL-DEFINED. Note that since $b \neq 0$ then $a^2 + b^2 \neq 0$ so that the symbol $\frac{a^2}{a^2+b^2}$ represents an equivalence class in $\mathbb{Q}$. Choose another representative $\frac{p}{q} \in \left[\frac{a}{b}\right]$ to show that f computed from $\frac{p}{q}$ is equivalent to f computed from $\frac{a}{b}$. As $\frac{p}{q} \in \left[\frac{a}{b}\right]$ we have $aq = bp$. But then

$$p^2(a^2 + b^2) = a^2p^2 + (bp)^2 = a^2p^2 + (aq)^2 = a^2(p^2 + q^2).$$

Thus we have shown that $\frac{a^2}{a^2+b^2} \sim \frac{p^2}{p^2+q^2}$ so f is well-defined.