Math 3070 § 1. **First Midterm Exam** Name: Solutions
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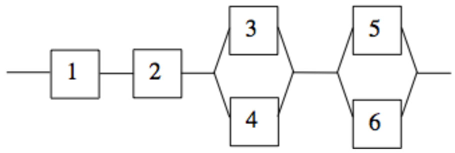
(1.) *Consider the system of components connected as in the accompanying picture. Components 1 and 2 are connected in series, so that subsystem works iff both 1 and 2 work; since 3 and 4 are connected in parallel, that subsystem works iff either 3 or 4 works. If components work independently of one another and $P(\{component\ works\}) = .7$, calculate $P(\{system\ works\})$.*



Let A_i denote the event that the ith component works and $p = P(A_i) = .7$ the probability that it works. There are three subsystems in series which work iff all of them work. Two of the subsystems are components in parallel, which work if either of them work. Thus the event that the system works is

$$\text{“System works”} = A_1 \cap A_2 \cap (A_3 \cup A_4) \cap (A_5 \cup A_6).$$

Since the A_i's are mutually independent,

$$P(\text{“System works”}) = P(A_1) \cdot P(A_2) \cdot P(A_3 \cup A_4) \cdot P(A_5 \cup A_6).$$

By the union formula,

$$\begin{aligned} P(A_3 \cup A_4) &= P(A_3) + P(A_4) - P(A_3 \cap A_4) \\ &= P(A_3) + P(A_4) - P(A_3) \cdot P(A_4) \\ &= 2p - p^2. \end{aligned}$$

Hence

$$P(\text{“System works”}) = p^2(2p - p^2)^2 = p^4(2 - p)^2 = (.7)^4(2 - .7)^2 = \boxed{.406}.$$

(2.) *A box of bolts contains 8 thick bolts, 5 medium bolts and 3 thin bolts. A box of nuts contains 6 that fit the thick bolts, 4 that fit the medium bolts and 2 that fit the thin bolts. One bolt and one nut are chosen at random. Let $\mathcal{A}$ be the event that the nut fits the bolt. What is the probability $P(\mathcal{A})$? Let $\mathcal{B}$ be the event that the nut is larger than the bolt. What is the probability $P(\mathcal{B})$? Are the events $\mathcal{A}$ and $\mathcal{B}$ independent? Give a mathematical explanation.*

The sample space is the set of ordered pairs of a bolt and a nut. The number of outcomes is the number of bolts times the number of nuts $(8+5+3) \cdot (6+4+2) = 16 \cdot 12 = 196$. The event that the nut fits the bolt is

$$\begin{aligned} \mathcal{A} = \{(b,n) : \text{both } b \text{ and } n \text{ are thick}\} &\cup \{(b,n) : \text{both } b \text{ and } n \text{ are medium}\} \\ &\cup \{(b,n) : \text{both } b \text{ and } n \text{ are thin}\} \end{aligned}$$

There are $8 \cdot 6 + 5 \cdot 4 + 3 \cdot 2 = 48 + 20 + 6 = 74$ outcomes in $\mathcal{A}$ so that, assuming each pair is equally likely, $P(\mathcal{A}) = \frac{74}{196} = \boxed{.378}$.

The event that the nut is larger than the bolt is

$$\begin{aligned} \mathcal{B} = \{(b,n) : b \text{ is medium and } n \text{ is thick}\} &\cup \{(b,n) : b \text{ is thin and } n \text{ is thick}\} \\ &\cup \{(b,n) : b \text{ is thin and } n \text{ is medium}\} \end{aligned}$$

There are $5 \cdot 6 + 3 \cdot 6 + 3 \cdot 4 = 30 + 18 + 12 = 60$ outcomes in $\mathcal{B}$ so that, assuming each pair is equally likely, $P(\mathcal{B}) = \frac{60}{196} = \boxed{.306}$.

Since the nut can't fit the bolt and be larger than the bolt at the same time, the events $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive: $\mathcal{A} \cap \mathcal{B} = \emptyset$. It follows that $\mathcal{A}$ and $\mathcal{B}$ are $\boxed{\text{not independent}}$:

$$.116 = (.378)(.306) = P(\mathcal{A}) \cdot P(\mathcal{B}) \neq P(\mathcal{A} \cap \mathcal{B}) = P(\emptyset) = 0.$$

(3.) *Suppose that the Paradise City Council consists of eight women and six men. A five member budget subcommittee was chosen at random from all members of the council. What is the probability that there are exactly three women on the budget subcommittee? What is the probability that there are more women than men on the budget subcommittee?*

There are $8 + 6 = 14$ in the Council. There are $\binom{14}{5} = 2002$ combinations of council members taken five at a time. (Budget subcommittees of size five where order is not important). The number of ways of choosing five person subcommittees with three women is the number of ways to choose combinations of three women from eight, $\binom{8}{3} = 42$, times the number of ways to choose the remaining two men from the six, $\binom{6}{2} = 15$. Thus the probability of exactly three women on the subcommittee is

$$\frac{\#\{\text{subcommittees with three women and two men}\}}{\#\{\text{subcommittees of five}\}} = \frac{\binom{8}{3}\binom{6}{2}}{\binom{14}{5}} = \frac{56 \cdot 15}{2002} = \boxed{.420}$$

There are more women than men if there are three, four or five women on the subcommittee. Adding the numbers gives the probability that there are more women than men on the subcommittee

$$\frac{\binom{8}{3}\binom{6}{2} + \binom{8}{4}\binom{6}{1} + \binom{8}{5}\binom{6}{0}}{\binom{14}{5}} = \frac{56 \cdot 15 + 70 \cdot 6 + 56 \cdot 1}{2002} = \frac{1316}{2002} = \boxed{.657}$$

(4.) *Santaquin Circuits produces circuit boards that are rated excellent, acceptable or unacceptable. Suppose that 30% of all boards are excellent, 60% are acceptable and 10% are unacceptable. Further, suppose that 10% of the excellent boards fail, 20% of the acceptable boards fail and 100% of the unacceptable boards fail (the unacceptable boards are discarded without being used). What is the probability that a board is rated excellent and fails? What is the probability that a board fails? Given that a board fails, what is the probability it was rated excellent?*

Let $\mathcal{A}_1$ be the event that the board is rated excellent, $\mathcal{A}_2$ that it is rated acceptable and $\mathcal{A}_3$ that it is rated unacceptable. The events $\mathcal{A}_i$ are mutually exclusive and exhaustive. We are given the probabilities $P(\mathcal{A}_1) = .3$, $P(\mathcal{A}_2) = .6$ and $P(\mathcal{A}_3) = .1$. Let $\mathcal{F}$ be the event that the board fails. We are given the conditional probabilities of failure for each type of board $P(\mathcal{F}|\mathcal{A}_1) = .1$, $P(\mathcal{F}|\mathcal{A}_2) = .2$ and $P(\mathcal{F}|\mathcal{A}_3) = 1.0$. The probability that that a board is rated excellent and fails is

$$P(\mathcal{A}_1 \cap \mathcal{F}) = P(\mathcal{A}_1) \cdot P(\mathcal{F}|\mathcal{A}_1) = (.3)\,(.1) = \boxed{.03}.$$

The probability that that a board fails is, from the total probability formula,

$$\begin{aligned} P(\mathcal{F}) &= P(\mathcal{A}_1 \cap \mathcal{F}) + P(\mathcal{A}_2 \cap \mathcal{F}) + P(\mathcal{A}_3 \cap \mathcal{F}) \\ &= P(\mathcal{A}_1) \cdot P(\mathcal{F}|\mathcal{A}_1) + P(\mathcal{A}_2) \cdot P(\mathcal{F}|\mathcal{A}_2) + P(\mathcal{A}_3) \cdot P(\mathcal{F}|\mathcal{A}_3) \\ &= (.3)\,(.1) + (.6)\,(.2) + (.1)\,(1.0) = .03 + .12 + .10 = \boxed{.25}. \end{aligned}$$

Given that a board fails, the probability it was rated excellent is

$$P(\mathcal{A}_1|\mathcal{F}) = \frac{P(\mathcal{A}_1 \cap \mathcal{F})}{P(\mathcal{F})} = \frac{.03}{.25} = \boxed{.12}.$$

(5.) *A bag contains 26 scrabble tiles, each labeled by a different letter of the alphabet. Three tiles are randomly selected in order from the bag without replacement. For* $i = 1, 2, 3$*, let* $\mathcal{A}_i$ *be the event that the ith letter is a vowel (A, E, I, O, U). What is the sample space? How many outcomes are in the sample space? Express the event that the word has no vowels in terms of the* A_i*'s. What it the probability that the word has no vowels? Express the event that the word has at most one vowel in terms of the* A_i*'s. What it the probability that the word has at most one vowel? [Hint: Be careful. The events* $\mathcal{A}_i$ *are not independent.]*

The sample space $\mathcal{S}$ is the set of permutations of 26 letters taken three at a time. In other words it is the set of three letter words chosen from 26 tiles without replacement and where order is important. $\#\mathcal{S} = P_{3,26} = 26 \cdot 25 \cdot 24 = \boxed{15600}$.

The event that the word has no vowels is the same as the event that the first, the second and the third are all not vowels:

$$\text{“No vowels”} = \mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'.$$

There are $26 - 5 = 21$ ways to choose the first non-vowel, 20 remaining ways to choose the second non-vowel and 19 ways to choose the last non-vowel, so the probability that there are no vowels is

$$P(\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3') = \frac{21 \cdot 20 \cdot 19}{15600} = \boxed{.512}.$$

The event that the word has at most one vowel is the same as the event that there are no vowels or the first is a vowel and the second and third are not, or the second is a vowel and the first and third are not or that the third is a vowel and the first and second are not:

$$\text{“At most one vowel”} = \left(\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'\right) \cup \left(\mathcal{A}_1 \cap \mathcal{A}_2' \cap \mathcal{A}_3'\right) \cup \left(\mathcal{A}_1' \cap \mathcal{A}_2 \cap \mathcal{A}_3'\right) \cup \left(\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3\right).$$

These are pairwise disjoint, so that we can add their counts. For the last three, there are five choices for the vowel, 21 choices for the first non-vowel and 20 for the second non-vowel. The probability is thus

$$P(\text{At most one vowel}) = \frac{21 \cdot 20 \cdot 19 + 3 \cdot 5 \cdot 21 \cdot 20}{26 \cdot 25 \cdot 24} = \frac{7980 + 3 \cdot 2100}{15600} = \boxed{.915}.$$