| Math 3070 § 1. | **Hand Calculator Example:** | Name: | $\mathfrak{Example}$ |
| Treibergs | **Derenzo's Formula for** $1 - \Phi(z)$. | June 8, 2011 | |

In Problem 4.58, Devore describes an approximation to the complementary cumulative normal distribution function $1 - \Phi(z)$. This function is computed via W. D. Cody's Algorithm in **R**, and for the purposes of this demonstration, I shall assume that it is given exactly there by

$$1 - \Phi(z) = \texttt{pnorm}(z, \texttt{lower.tail} = \texttt{FALSE}).$$

The approximating function given by Stephen E. Derenzo

$$\mathcal{FP}(z) = \frac{1}{2} \exp\left(-\frac{(83z + 351)z + 562}{\dfrac{703}{z} + 165} \right).$$

It was found by fitting best coefficients to function of the form exponential of rational function and then approximating the coefficients by integers. Its purpose is to have an accurate formula that is easy for a hand calculator to handle. The canned functions in hand held calculators do not work well above a couple of standard deviations.

The approximation makes less than a .042% relative error over the range $0 < z \leq 5$.

Since $1 - \Phi(z)$ is a very small positive number, by taking logs we will see finer details. For the record,

$$\log(\mathcal{FP}(z)) = -\log(2) - \frac{(83z + 351)z + 562}{\dfrac{703}{z} + 165}.$$

R Session:

```
R version 2.11.1 (2010-05-31)
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Type 'q()' to quit R.

[R.app GUI 1.34 (5589) i386-apple-darwin9.8.0]

[Workspace restored from /home/1004/ma/treibergs/.RData]
```

```
> #################### DEFINE & PLOT DERENZO'S FUNCTION ####################
> q <- sqrt(5.5);q
[1] 2.345208
> x <- seq(0,q,.01)^2

> FP <- function(z){.5*exp(-((83*z+351)*z+562)/(703/z+165))}

> plot(x,pnorm(x,lower.tail=FALSE,log=FALSE),col=2,type="l",ylab="Probability",xlab="z")
> title(expression(paste("Derenzo's Approximation of ",1-Phi(z))))
> abline(h=0,col="gray20")
> abline(v=0,col="gray20")
> abline(h=seq(.1,.5,.1),col="gray20",lty=3)
> abline(v=1:5,col="gray20",lty=3)
> lines(x,FP(x),col=4,lty=2)
> legend(3.5,.45,legend=c(expression(1-Phi(z)),"Derenzo"),fill=c(2,4),bg="white")
> # M3074CalcApprox1.pdf
```
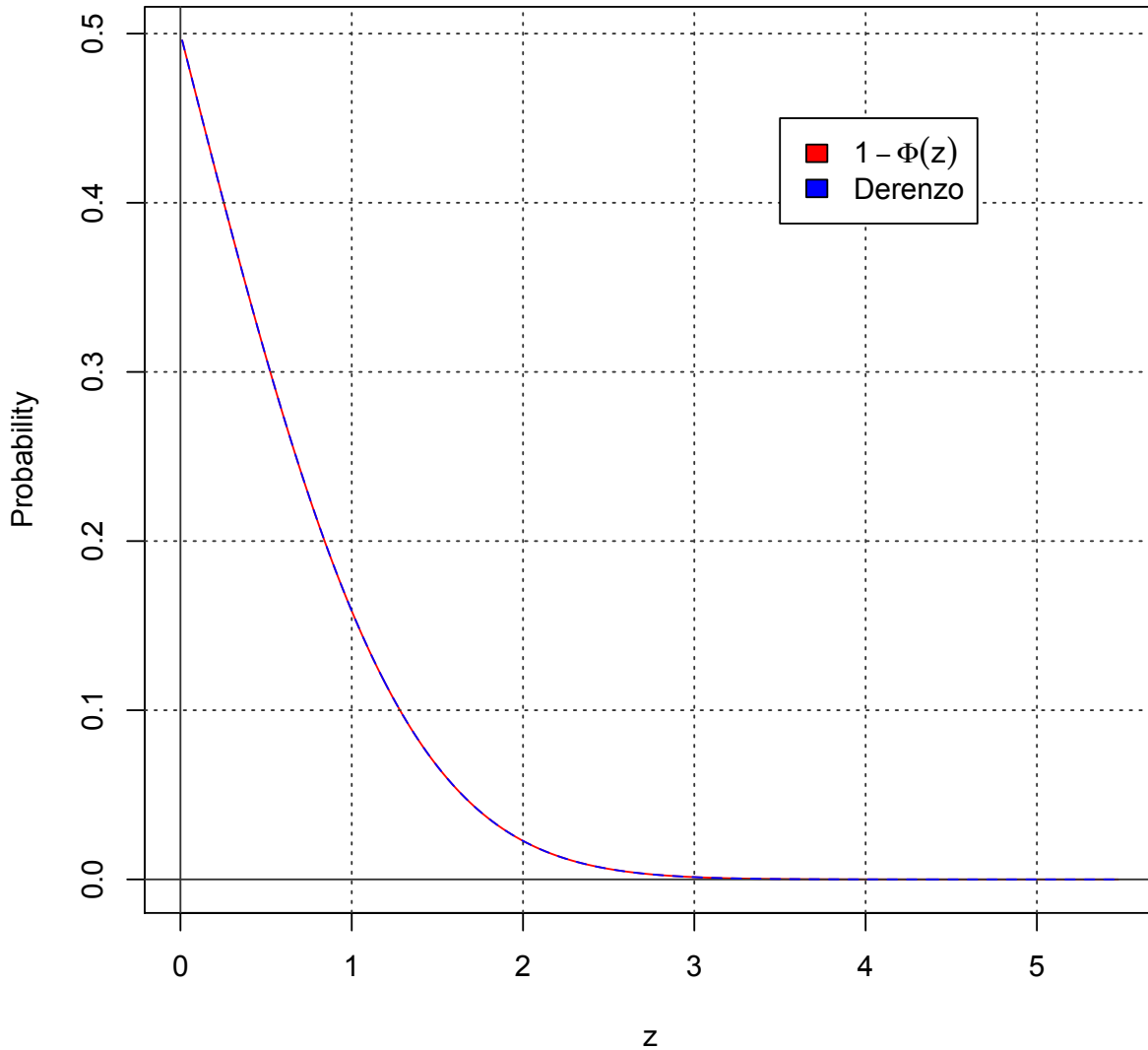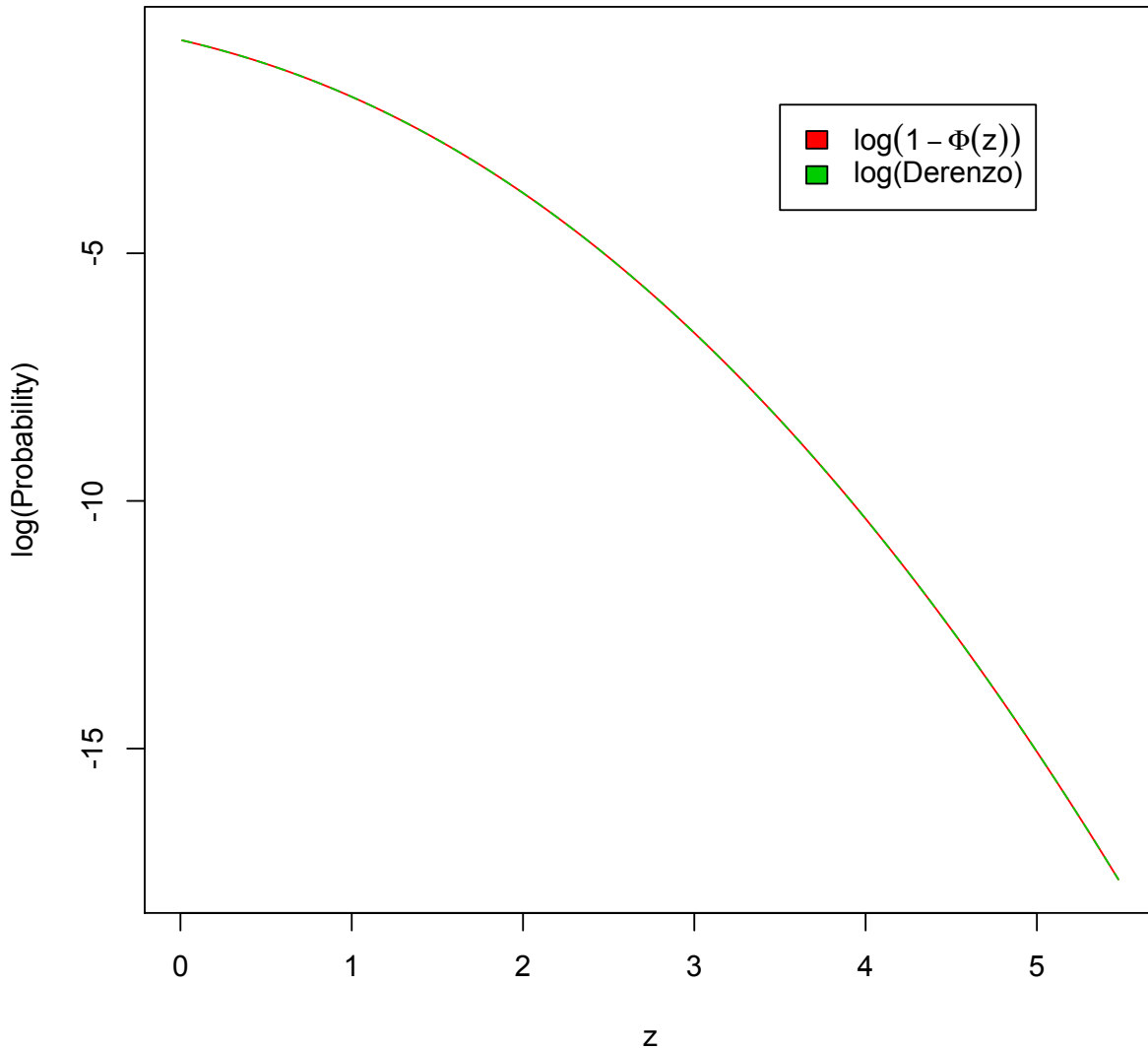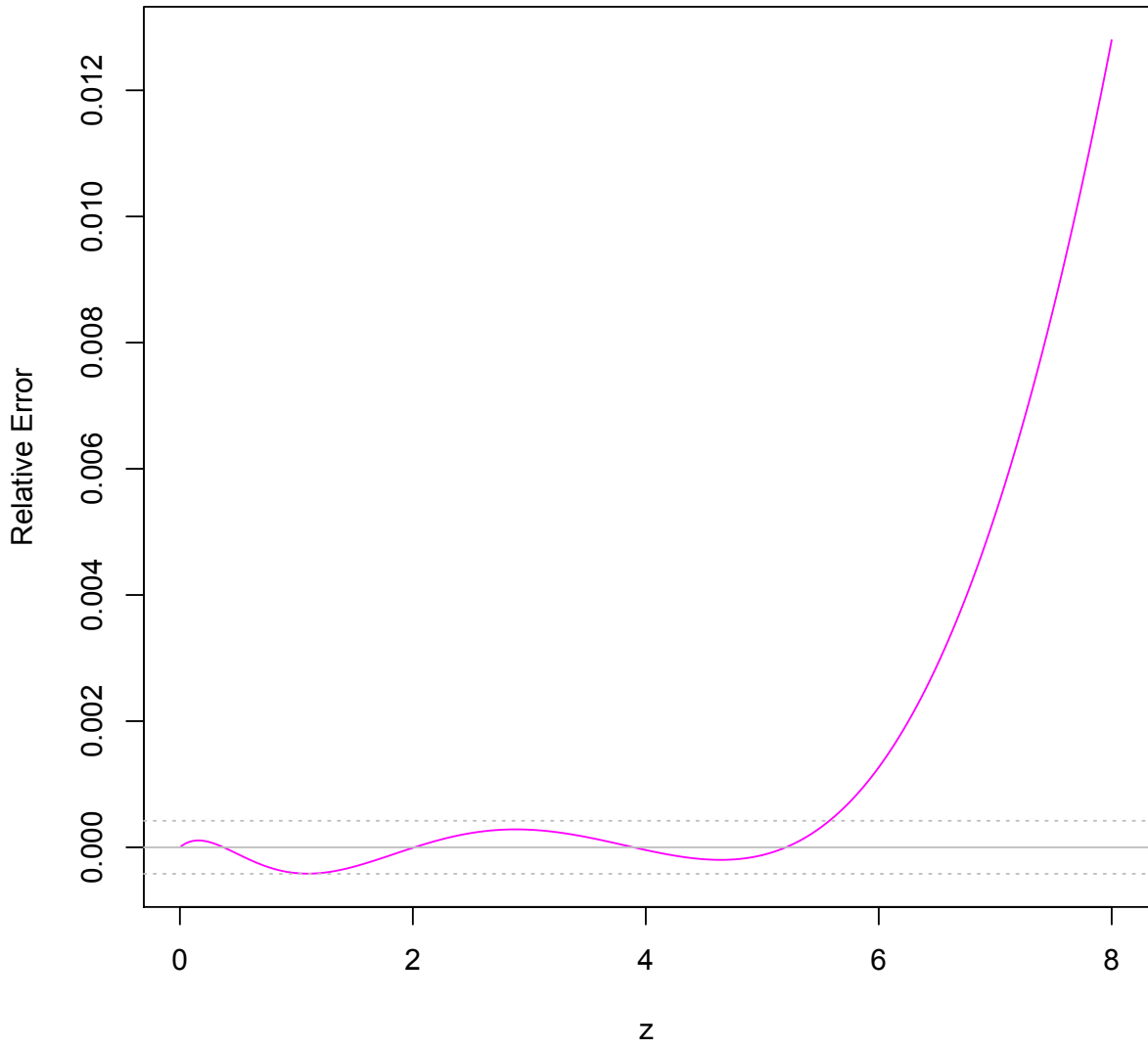
Derenzo's Approximation of $1 - \Phi(z)$


```
> ###################### DEFINE & PLOT LOG OF DERENZO'S FUNCTION ##############
>
> LFP <- function(z){log(.5)-((83*z+351)*z+562)/(703/z+165)}
> plot(x,pnorm(x,lower.tail=FALSE,log=TRUE),col=2,type="l",ylab="log(Probability)",xlab="z")
> title(expression(paste("Log Plot of Derenzo's Approximation of ",1-Phi(z))))
> lines(x,LFP(x),col=3,lty=2)
> legend(3.5,-2,legend=c(expression(log(1-Phi(z))),"log(Derenzo)"),fill=c(2,3),bg="white")
> # M3074CalcApprox2.pdf
```

Log Plot of Derenzo's Approximation of $1 - \Phi(z)$


```
> ########################### PLOT RELATIVE ERROR ###########################
> # Relative error is difference divided by magnitude. In our case,
> # RE = (1 - Phi(z) - FP(z)) / (1 - Phi(z))
> x=seq(.01,8,.01)
> RE <- function(z){ temp <- pnorm(z,lower.tail=FALSE);(temp-FP(z))/temp}
> plot(x,RE(x),col=6,type="l",ylab="Relative Error",xlab="z")
> abline(h=c(0,.00042,-.00042),col="gray",lty=c(1,3,3))
> title(expression(paste("Relative Error of Derenzo's Approximation of ",1-Phi(z))))
> # M3074CalcApprox3.pdf
> # I have added dotted lines .00045 above and below zero. The rel difference
> # stays within these in the range 0 < z <= 5.5  or so.
```

Relative Error of Derenzo's Approximation of $1 - \Phi(z)$

```
> #################### SOLVE PROBLEM 4.58 OF DEVORE ########################
> # I give the "canned" and the "Derenzo Function" solutions.
> # 165.58a.  Find   P(z >= 1).
> pnorm(1,lower.tail=FALSE)
[1] 0.1586553
> FP(1)
[1] 0.1587202
>
> # 165.58b  Find  P(z < -3) = 1 - Phi(3).
> pnorm(3,lower.tail=FALSE)
[1] 0.001349898
> FP(3)
[1] 0.001349521
>
> # 165.58c.  Find    P(-4 < z < 4) =  1 - 2*(1-Phi(4))
> paste("1 - ",2*pnorm(4,lower.tail=FALSE))
[1] "1 -  6.33424836662398e-05"
> paste("1 - ",2*FP(4))
[1] "1 -  6.33452125653876e-05"
>
> # 165.58d.  Find     P(z > 5)
> pnorm(5,lower.tail=FALSE)
[1] 2.866516e-07
> FP(5)
[1] 2.866870e-07
>
```