Math 3070 § 1. **Approximation Example:** Name: Example
Treibergs **Normal Approximation to Binomial.** June 10, 2011

The normal distribution can be used to approximate the binomial. Devore's rule of thumb is that if $np \geq 10$ and $n(1-p) \geq 10$ then this is permissible. In this example, I generate plots of the binomial pmf along with the normal curves that approximate it. I generate tables of binomial pmf, its normal approximation and the normal approximation with continuity correction.

Recall the mean and standard deviation of a binomial variable $X \sim \text{bin}(n, p)$ where X is the number of successes in n trials with probability of success p for each trial.

$$\mu = np; \qquad \sigma = \sqrt{np(1-p)}.$$

The cumulative density is approximated using the normal cdf $\Phi(z) = P(Z \leq z)$ where $Z \sim N(0,1)$ is a standard normal variable. The approximation, and approximation with continuity correction are given by

$$P(X \leq x) \approx \Phi\left(\frac{x-\mu}{\sigma}\right) \approx \Phi\left(\frac{x-\mu+.5}{\sigma}\right).$$

R Session:

```
R version 2.11.1 (2010-05-31)
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[R.app GUI 1.34 (5589) i386-apple-darwin9.8.0]

[Workspace restored from /home/1004/ma/treibergs/.RData]
```

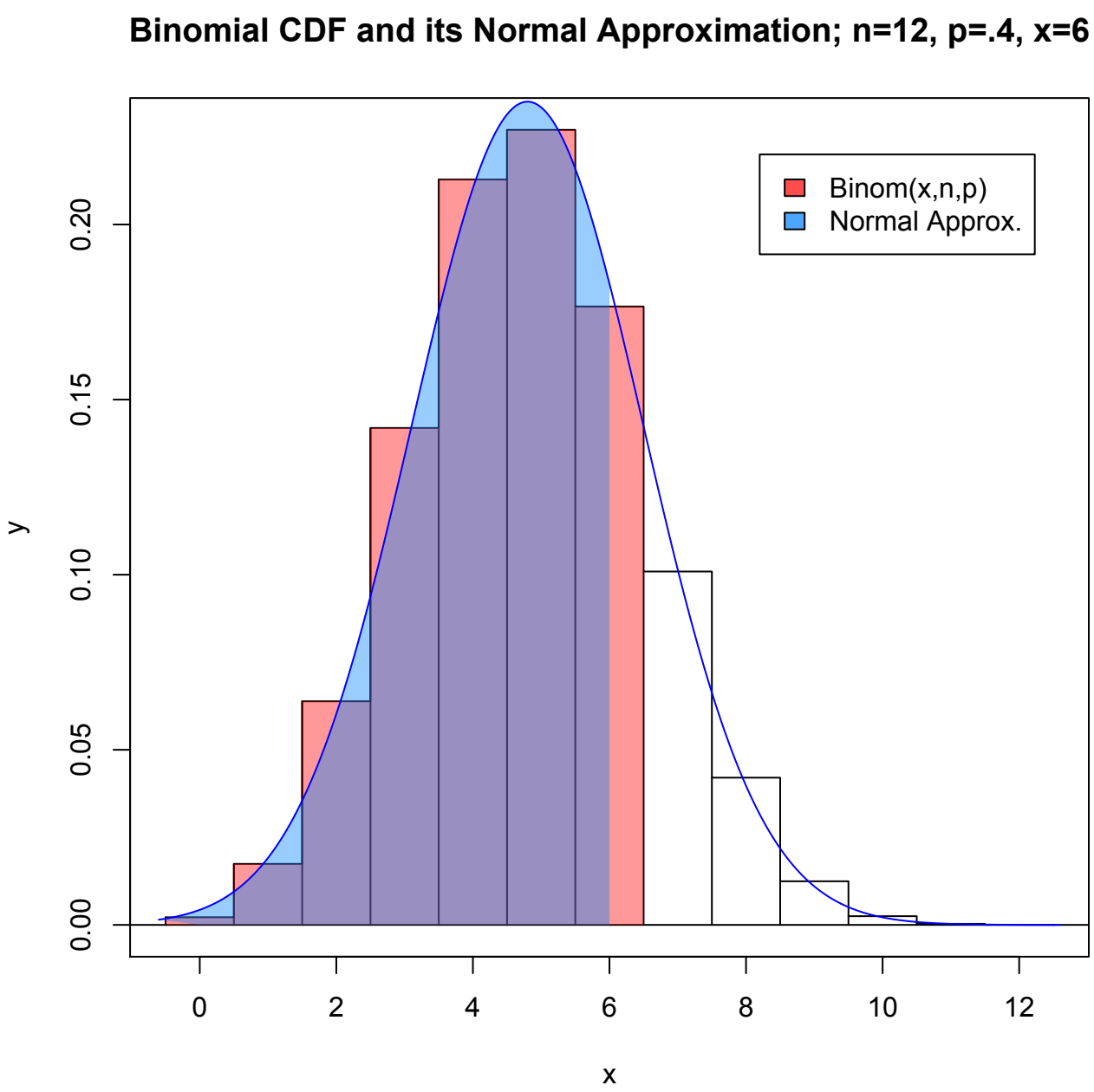
```
> ################ SET UP PLOTS OF Bin() AND ITS NORMAL APPROX. ###################
> N <- 12
> p <- .4
> me<- n*p
> se <- sqrt(n*p*(1-p))
> me;se
```

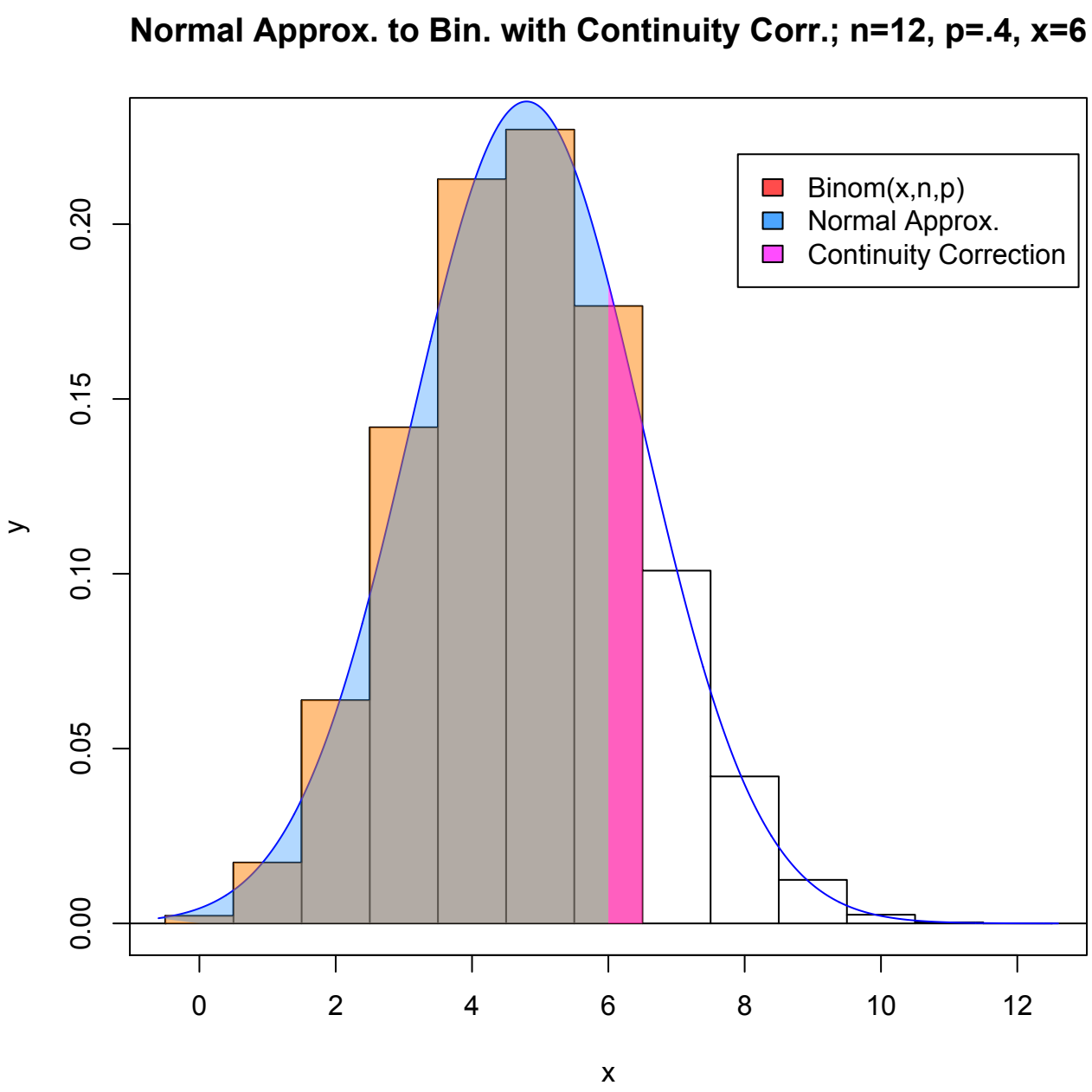
```
[1] 4.8
[1] 1.697056

> # To plot the pmf of binom(n,p), with unit bars of height    bin(x,n,p)
> x<- c(-.5,-.5,(0:n)+.5)
> y<- c(0,dbinom(0:n,n,p),0)

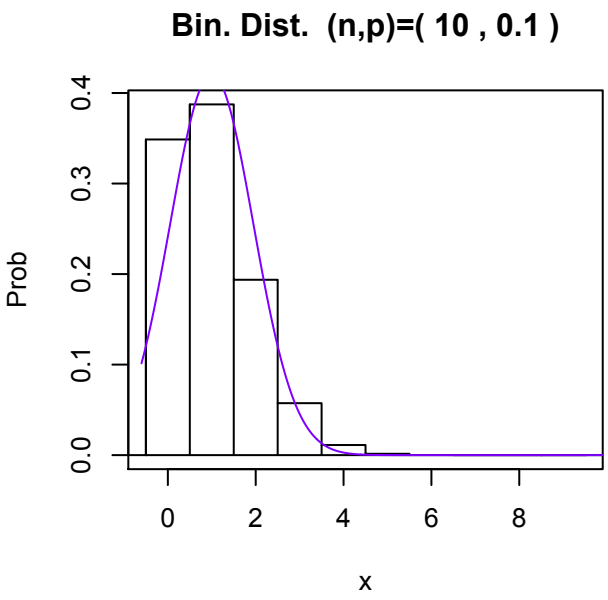

> # To plot the corresponding normal curve.
> xf <- seq(-.6,n+.6,.01)
> yf<- dnorm(xf,me,se)
>
> # To shade in the region  x <= z  under the normal curve.
> z <-6
> xf2 <- seq(-.6,z,.01)
> yf2 <- dnorm(xf2,me,se)

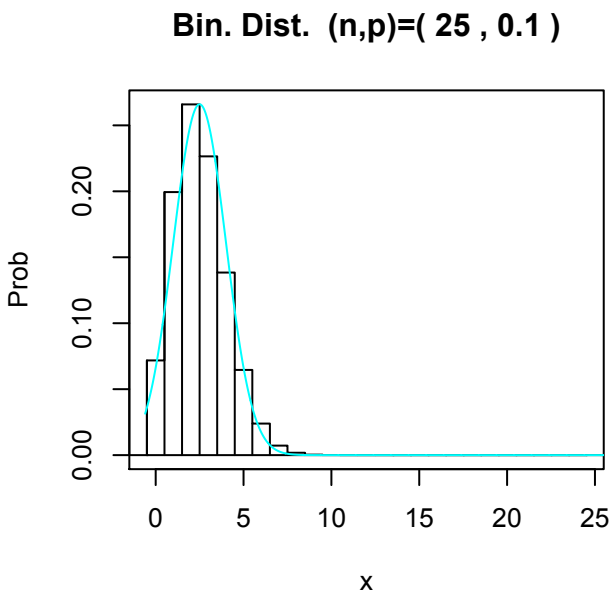
> # To shade in the region  x <= z  under the binomial step function.
> y3<-c(0,rep(dbinom(0:z,n,p),each=2),0,0)
> x3<-c(rep(c(-.5,(0:z)+.5),each=2),0);x3
 [1] -0.5 -0.5  0.5  0.5  1.5  1.5  2.5  2.5  3.5  3.5  4.5  4.5  5.5  5.5  6.5  6.5  0.0
>
>
> ############### PLOT NORMAL APPROX WITHOUT CONTINUITY CORRECTION ######################
> plot(x,y,type="s")
> abline(h=0)
> lines(x,y,type="h")
> polygon(x3,y3,col=rainbow(12,alpha=.4)[1],border=NA)
> polygon(c(0,xf2,z,0),c(0,yf2,0,0),col=rainbow(12,alpha=.4)[8],border=NA)
> lines(xf,yf,col=4)
> legend(8.2,.22,legend=c("Binom(x,n,p)","Normal Approx."),
+ fill=c(rainbow(12,alpha=.7)[1],rainbow(12,alpha=.7)[8]),bg="white")
> title("Binomial CDF and its Normal Approximation; n=12, p=.4, x=6")
> # M3074ApproxBin1.pdf
>
>
>
>
> ################## PLOT NORMAL APPROX WITH CONTINUITY CORRECTION ######################
> plot(x,y,type="s")
> abline(h=0)
> lines(x,y,type="h")
> lines(xf,yf,col=4)
> polygon(x3,y3,col=rainbow(12,alpha=.5)[2],border=NA)
> polygon(c(0,xf2,z,0),c(0,yf2,0,0),col=rainbow(12,alpha=.3)[8],border=NA)
> polygon(c(z,xf4,z+.5,0),c(0,yf4,0,0),col=rainbow(12,alpha=.5)[11],border=NA)
> title("Normal Approx. to Bin. with Continuity Corr.; n=12, p=.4, x=6")
> legend(7.9,.22,legend=c("Binom(x,n,p)","Normal Approx.","Continuity Correction"),
+ fill=c(rainbow(12,alpha=.7)[1],rainbow(12,alpha=.7)[8],
+ rainbow(12,alpha=.7)[11]),bg="white")
> # M3074ApproxBin2.pdf
```

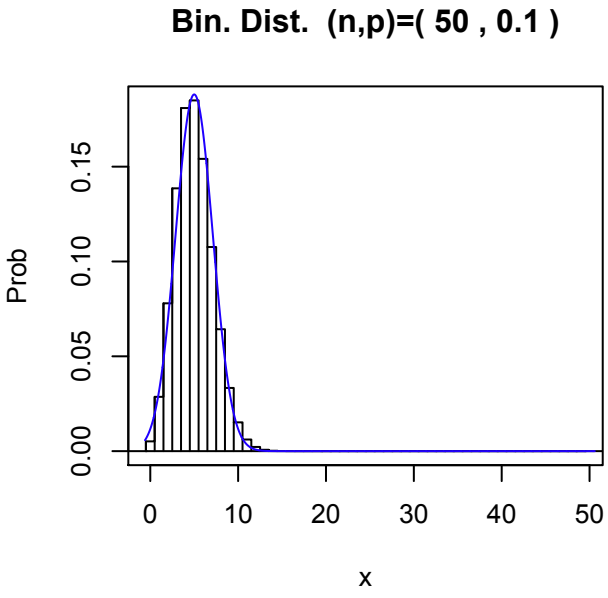

Binomial CDF and its Normal Approximation; n=12, p=.4, x=6
Binom(x,n,p)
Normal Approx.
y
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
12
x

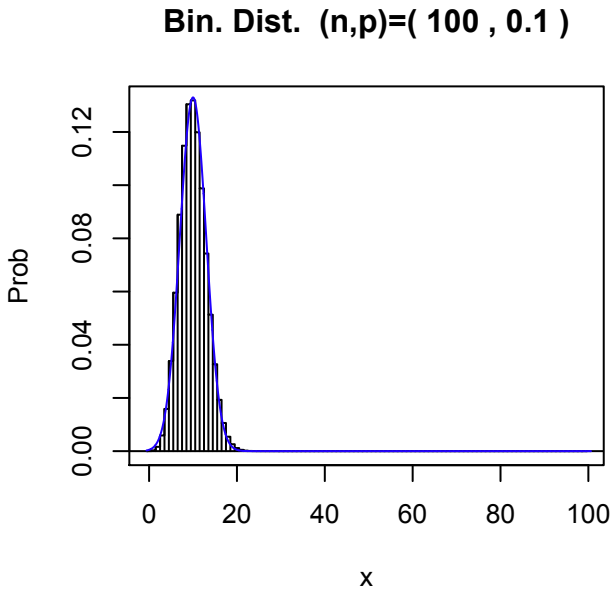

Normal Approx. to Bin. with Continuity Corr.; n=12, p=.4, x=6
Binom(x,n,p)
Normal Approx.
Continuity Correction
y
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
12
x

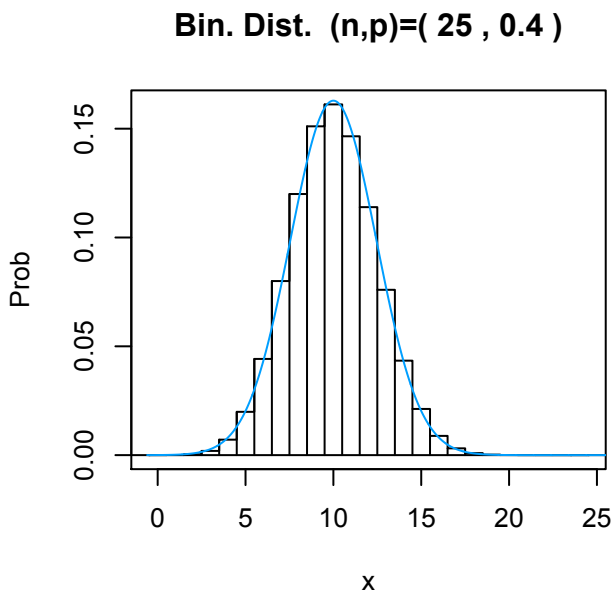
```
> ############## PICTURES OF HOW NORMAL FAILS TO APPROX IF np OR n(1-p) ARE SMALL #########
> # Write a function subprogram that does the repetitive plots.
> pl <- function(n,p){
+                      mt <- n*p
+                      st <- sqrt(mt*(1-p))
+                      x <- c(-.5, (0:n)-.5)
+                      Prob <- c(0, dbinom(0:n,n,p))
+                      plot(x,Prob,type="s", main=paste("Bin. Dist.  (n,p)=(",n,",",p,")"))
+                      lines(x, Prob, type="h")
+                      abline(h=0)
+                      xft<-seq(-.6, n+.6, .01)
+                      yft<-dnorm(xft, mt, st)
+                      lines(xft, yft, col=rainbow(16)[5+ceiling(10*runif(1))])
+                     }
>
> opar <- par(mfrow=c(2,2))
> pl(10,.1)
> pl(25,.1)
> pl(50,.1)
> pl(100,.1)
> # M3074ApproxBin3.pdf
>
> pl(25,.4)
> pl(25,.2)
> pl(25,.1)
> pl(25,.05)
> # M3074ApproxBin4.pdf
> par(opar)
```

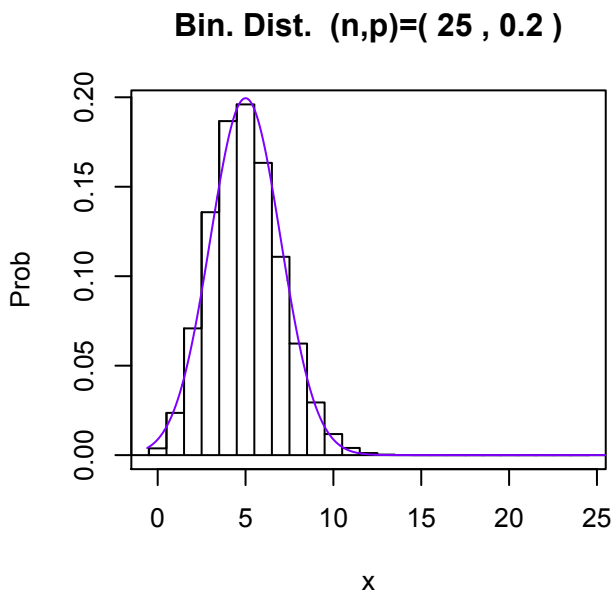

Bin. Dist. (n,p)=(10 , 0.1)
Prob
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
x

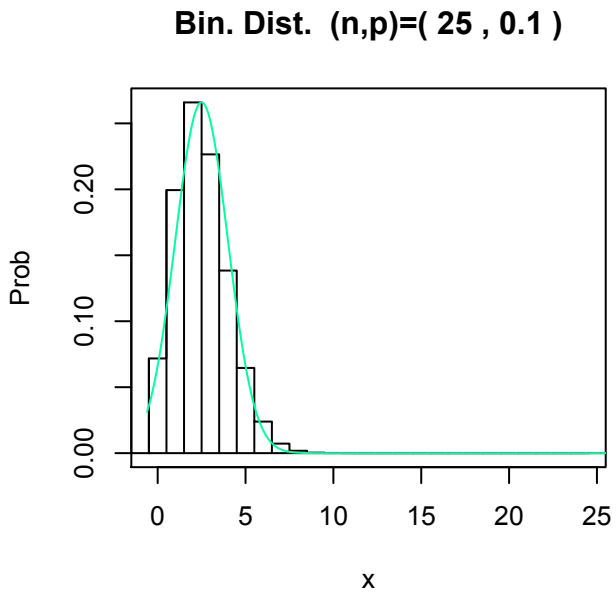

Bin. Dist. (n,p)=(25 , 0.1)
Prob
0.00
0.10
0.20
0
5
10
15
20
25
x

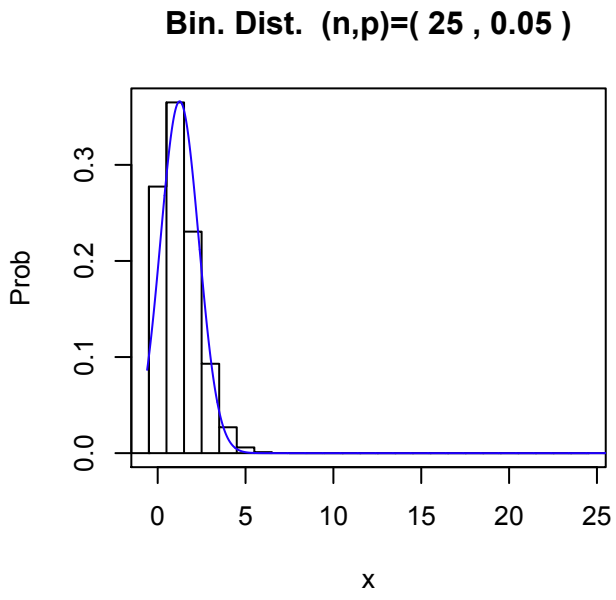

Bin. Dist. (n,p)=(50 , 0.1)
Prob
0.00
0.05
0.10
0.15
0
10
20
30
40
50
x


Bin. Dist. (n,p)=(100 , 0.1)
Prob
0.00
0.04
0.08
0.12
0
20
40
60
80
100
x


Bin. Dist. (n,p)=(25 , 0.4)
Prob
0.00
0.05
0.10
0.15
0
5
10
15
20
25
x


Bin. Dist. (n,p)=(25 , 0.2)
Prob
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
25
x


Bin. Dist. (n,p)=(25 , 0.1)
Prob
0.00
0.10
0.20
0
5
10
15
20
25
x


Bin. Dist. (n,p)=(25 , 0.05)
Prob
0.0
0.1
0.2
0.3
0
5
10
15
20
25
x

```
> ################## FUNCTION SUBPROGRAM TO PRINT TABLE FOR GIVEN n AND p ##############
> # Column three is difference between Bin(x,n,p) and Normal Approx(x)
> # Column five is difference between Bin(x,n,p) and normal approx. with cont. corr.
> # I used  fix(tb)  to develop the subprogram.
>
> tb <- function(n,p){
+                     m <- matrix(1:(n*6+6),ncol=6,
+                     dimnames=list(0:n,c("Bin(x,n,p)    ","Norm. Appr.","Error          ",
+                                  "w/Cont.Corr.","Error           ","Ratio of Errors")))
+                      cat("\n Normal Approximation to Binomial CDF  (n,p)=(",
+                          paste(n,",",p,")"),"\n\n")
+                      mt<- p*n
+                      st <- sqrt(mt*(1-p))
+                      for(i in 0:(n+1)){
+                                       m[i,1] <- pbinom(i-1,n,p)
+                                       m[i,2] <- pnorm((i-mt-1)/st)
+                                       m[i,3] <- m[i,1]-m[i,2]
+                                       m[i,4] <- pnorm((i-mt-.5)/st)
+                                       m[i,5] <- m[i,1]-m[i,4]
+                                       m[i,6] <- m[i,5]/m[i,3]
+                                      }
+                      print(format(round(m,digits=8),scientific=FALSE),quote=FALSE)
+                     }

> tb(20,.5)

 Normal Approximation to Binomial CDF  (n,p)=( 20 , 0.5 )

   Bin(x,n,p)     Norm. Appr. Error           w/Cont.Corr. Error            Ratio of Errors
0   0.00000095     0.00000387 -0.00000292      0.00001076  -0.00000981       3.35976699
1   0.00002003     0.00002850 -0.00000847      0.00007196  -0.00005194       6.13204207
2   0.00020123     0.00017331  0.00002792      0.00039812  -0.00019689      -7.05303706
3   0.00128841     0.00087256  0.00041585      0.00182522  -0.00053680      -1.29084332
4   0.00590897     0.00364518  0.00226379      0.00695315  -0.00104418      -0.46125469
5   0.02069473     0.01267366  0.00802107      0.02208567  -0.00139094      -0.17341068
6   0.05765915     0.03681914  0.02084001      0.05876243  -0.00110328      -0.05294070
7   0.13158798     0.08985625  0.04173173      0.13177624  -0.00018826      -0.00451111
8   0.25172234     0.18554668  0.06617565      0.25116748   0.00055486       0.00838463
9   0.41190147     0.32736042  0.08454105      0.41153164   0.00036984       0.00437465
10  0.58809853     0.50000000  0.08809853      0.58846836  -0.00036984      -0.00419799
11  0.74827766     0.67263958  0.07563809      0.74883252  -0.00055486      -0.00733570
12  0.86841202     0.81445332  0.05395870      0.86822376   0.00018826       0.00348890
13  0.94234085     0.91014375  0.03219710      0.94123757   0.00110328       0.03426659
14  0.97930527     0.96318086  0.01612440      0.97791433   0.00139094       0.08626303
15  0.99409103     0.98732634  0.00676469      0.99304685   0.00104418       0.15435768
16  0.99871159     0.99635482  0.00235677      0.99817478   0.00053680       0.22777120
17  0.99979877     0.99912744  0.00067133      0.99960188   0.00019689       0.29328140
18  0.99997997     0.99982669  0.00015328      0.99992804   0.00005194       0.33883687
19  0.99999905     0.99997150  0.00002754      0.99998924   0.00000981       0.35599322
20  1.00000000     0.99999613  0.00000387      0.99999867   0.00000133       0.34301686
> # np and n(1-p) equal 10 , ok by rule of thumb, and the errors are small.
```

```
> tb(10,.05)

 Normal Approximation to Binomial CDF  (n,p)=( 10 , 0.05 )

   Bin(x,n,p)     Norm. Appr. Error           w/Cont.Corr. Error           Ratio of Errors
0   0.59873694     0.23407995  0.36465698      0.50000000   0.09873694       0.27076662
1   0.91386164     0.76592005  0.14794160      0.92660346  -0.01274181      -0.08612731
2   0.98849644     0.98523839  0.00325805      0.99814550  -0.00964905      -2.96160299
3   0.99897150     0.99985684 -0.00088534      0.99999328  -0.00102178       1.15410582
4   0.99993631     0.99999981 -0.00006350      1.00000000  -0.00006369       1.00294709
5   0.99999725     1.00000000 -0.00000275      1.00000000  -0.00000275       1.00001192
6   0.99999992     1.00000000 -0.00000008      1.00000000  -0.00000008       1.00000001
7   1.00000000     1.00000000  0.00000000      1.00000000   0.00000000       1.00000000
8   1.00000000     1.00000000  0.00000000      1.00000000   0.00000000       1.00000000
9   1.00000000     1.00000000  0.00000000      1.00000000   0.00000000       1.00000000
10  1.00000000     1.00000000  0.00000000      1.00000000   0.00000000              NaN
> tb(10,.1)
> # np = 1 , not ok by rule of thumb, and the errors are large, even 10%.

 Normal Approximation to Binomial CDF  (n,p)=( 10 , 0.1 )

   Bin(x,n,p)     Norm. Appr. Error           w/Cont.Corr. Error           Ratio of Errors
0   0.34867844     0.14592027  0.20275817      0.29908073   0.04959771       0.24461512
1   0.73609893     0.50000000  0.23609893      0.70091927   0.03517966       0.14900388
2   0.92980917     0.85407973  0.07572945      0.94307685  -0.01326768      -0.17519839
3   0.98720480     0.98249251  0.00471229      0.99579600  -0.00859120      -1.82314698
4   0.99836506     0.99921730 -0.00085224      0.99988757  -0.00152251       1.78649051
5   0.99985310     0.99998759 -0.00013449      0.99999895  -0.00014585       1.08448691
6   0.99999088     0.99999993 -0.00000905      1.00000000  -0.00000912       1.00714357
7   0.99999963     1.00000000 -0.00000037      1.00000000  -0.00000037       1.00033022
8   0.99999999     1.00000000 -0.00000001      1.00000000  -0.00000001       1.00000864
9   1.00000000     1.00000000  0.00000000      1.00000000   0.00000000       1.00000000
10  1.00000000     1.00000000  0.00000000      1.00000000   0.00000000              NaN
> tb(10,.5)

 Normal Approximation to Binomial CDF  (n,p)=( 10 , 0.5 )

   Bin(x,n,p)     Norm. Appr. Error           w/Cont.Corr. Error           Ratio of Errors
0   0.00097656     0.00078270  0.00019386      0.00221326  -0.00123670      -6.37930302
1   0.01074219     0.00570602  0.00503617      0.01342835  -0.00268616      -0.53337370
2   0.05468750     0.02888979  0.02579771      0.05692315  -0.00223565      -0.08666074
3   0.17187500     0.10295161  0.06892339      0.17139086   0.00048414       0.00702438
4   0.37695313     0.26354463  0.11340850      0.37591482   0.00103831       0.00915547
5   0.62304687     0.50000000  0.12304687      0.62408518  -0.00103831      -0.00843831
6   0.82812500     0.73645537  0.09166963      0.82860914  -0.00048414      -0.00528140
7   0.94531250     0.89704839  0.04826411      0.94307685   0.00223565       0.04632115
8   0.98925781     0.97111021  0.01814760      0.98657165   0.00268616       0.14801740
9   0.99902344     0.99429398  0.00472946      0.99778674   0.00123670       0.26148896
10  1.00000000     0.99921730  0.00078270      0.99974789   0.00025211       0.32210138
```

```
> tb(20,.05)

 Normal Approximation to Binomial CDF  (n,p)=( 20 , 0.05 )

   Bin(x,n,p)     Norm. Appr. Error          w/Cont.Corr. Error           Ratio of Errors
0   0.35848592     0.15245089  0.20603503     0.30397945   0.05450648      0.26454956
1   0.73583952     0.50000000  0.23583952     0.69602055   0.03981897      0.16883926
2   0.92451633     0.84754911  0.07696722     0.93809389  -0.01357756     -0.17640707
3   0.98409847     0.97991306  0.00418541     0.99484041  -0.01074193     -2.56651877
4   0.99742606     0.99895780 -0.00153174     0.99983525  -0.00240919      1.57284443
5   0.99967071     0.99997969 -0.00030898     0.99999805  -0.00032735      1.05943177
6   0.99996605     0.99999986 -0.00003380     0.99999999  -0.00003394      1.00404107
7   0.99999714     1.00000000 -0.00000286     1.00000000  -0.00000286      1.00012618
8   0.99999980     1.00000000 -0.00000020     1.00000000  -0.00000020      1.00000170
9   0.99999999     1.00000000 -0.00000001     1.00000000  -0.00000001      1.00000001
10  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
...
> # "..." means I didn't copy some rows.
> tb(50,.05)

 Normal Approximation to Binomial CDF  (n,p)=( 50 , 0.05 )

   Bin(x,n,p)     Norm. Appr. Error          w/Cont.Corr. Error           Ratio of Errors
0   0.07694498     0.05237874  0.02456623     0.09718296  -0.02023798     -0.82381301
1   0.27943175     0.16519502  0.11423673     0.25820613   0.02122562      0.18580380
2   0.54053312     0.37280139  0.16773173     0.50000000   0.04053312      0.24165447
3   0.76040796     0.62719861  0.13320936     0.74179387   0.01861410      0.13973564
4   0.89638319     0.83480498  0.06157821     0.90281704  -0.00643385     -0.10448265
5   0.96222383     0.94762126  0.01460257     0.97421207  -0.01198824     -0.82096779
6   0.98821355     0.98842953 -0.00021598     0.99527792  -0.00706436     32.70809734
7   0.99681166     0.99824976 -0.00143811     0.99941157  -0.00259991      1.80787196
8   0.99924402     0.99982074 -0.00057672     0.99995056  -0.00070654      1.22509797
9   0.99984144     0.99998766 -0.00014623     0.99999722  -0.00015578      1.06533525
10  0.99997035     0.99999943 -0.00002908     0.99999990  -0.00002954      1.01592291
11  0.99999503     0.99999998 -0.00000496     1.00000000  -0.00000497      1.00298177
12  0.99999925     1.00000000 -0.00000075     1.00000000  -0.00000075      1.00041280
13  0.99999990     1.00000000 -0.00000010     1.00000000  -0.00000010      1.00004162
14  0.99999999     1.00000000 -0.00000001     1.00000000  -0.00000001      1.00000304
15  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000015
16  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
17  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
18  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
19  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
20  1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
...
```

```
> tb(100,.05)

 Normal Approximation to Binomial CDF  (n,p)=( 100 , 0.05 )

    Bin(x,n,p)     Norm. Appr. Error          w/Cont.Corr. Error           Ratio of Errors
0    0.00592053     0.01089073 -0.00497020     0.01947373  -0.01355320      2.72689081
1    0.03708121     0.03322871  0.00385250     0.05414683  -0.01706562     -4.42975253
2    0.11826298     0.08433431  0.03392867     0.12567455  -0.00741157     -0.21844572
3    0.25783866     0.17939768  0.07844098     0.24564856   0.01219010      0.15540470
4    0.43598130     0.32317760  0.11280370     0.40927290   0.02670840      0.23676879
5    0.61599913     0.50000000  0.11599913     0.59072710   0.02527203      0.21786398
6    0.76601398     0.67682240  0.08919158     0.75435144   0.01166255      0.13075837
7    0.87203952     0.82060232  0.05143720     0.87432545  -0.00228592     -0.04444107
8    0.93691041     0.91566569  0.02124472     0.94585317  -0.00894276     -0.42094051
9    0.97181171     0.96677129  0.00504042     0.98052627  -0.00871457     -1.72893796
10   0.98852759     0.98910927 -0.00058168     0.99419155  -0.00566396      9.73727356
11   0.99572582     0.99704730 -0.00132149     0.99857003  -0.00284421      2.15228300
12   0.99853565     0.99934052 -0.00080486     0.99971047  -0.00117482      1.45964639
13   0.99953673     0.99987903 -0.00034231     0.99995192  -0.00041519      1.21291839
14   0.99986415     0.99998182 -0.00011768     0.99999346  -0.00012932      1.09894225
15   0.99996295     0.99999777 -0.00003482     0.99999927  -0.00003633      1.04331410
16   0.99999057     0.99999978 -0.00000920     0.99999993  -0.00000936      1.01720994
17   0.99999776     0.99999998 -0.00000223     1.00000000  -0.00000224      1.00605984
18   0.99999950     1.00000000 -0.00000050     1.00000000  -0.00000050      1.00186331
19   0.99999989     1.00000000 -0.00000011     1.00000000  -0.00000011      1.00049612
20   0.99999998     1.00000000 -0.00000002     1.00000000  -0.00000002      1.00011384
21   1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00002247
22   1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000389
23   1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000097
24   1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
25   1.00000000     1.00000000  0.00000000     1.00000000   0.00000000      1.00000000
...
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```
> tb(200,.05)

 Normal Approximation to Binomial CDF  (n,p)=( 200 , 0.05 )

    Bin(x,n,p)     Norm. Appr.  Error          w/Cont.Corr. Error           Ratio of Errors
0     0.00003505     0.00058843  -0.00055338     0.00102736  -0.00099231      1.79317341
1     0.00040403     0.00175024  -0.00134621     0.00290991  -0.00250588      1.86143375
2     0.00233629     0.00472208  -0.00238579     0.00748051  -0.00514421      2.15619042
3     0.00904838     0.01157047  -0.00252209     0.01747754  -0.00842916      3.34213387
4     0.02644680     0.02578793   0.00065887     0.03717645  -0.01072965    -16.28497581
5     0.06234250     0.05237874   0.00996375     0.07214603  -0.00980353     -0.98391997
6     0.12374303     0.09718296   0.02656007     0.12807248  -0.00432946     -0.16300624
7     0.21330470     0.16519502   0.04810968     0.20865208   0.00465262      0.09670861
8     0.32702446     0.25820613   0.06881833     0.31324814   0.01377633      0.20018397
9     0.45470981     0.37280139   0.08190841     0.43556575   0.01914406      0.23372523
10    0.58306718     0.50000000   0.08306718     0.56443425   0.01863293      0.22431154
11    0.69975570     0.62719861   0.07255710     0.68675186   0.01300384      0.17922217
12    0.79648435     0.74179387   0.05469048     0.79134792   0.00513643      0.09391815
13    0.87010777     0.83480498   0.03530279     0.87192752  -0.00181975     -0.05154693
14    0.92186558     0.90281704   0.01904854     0.92785397  -0.00598839     -0.31437517
15    0.95564437     0.94762126   0.00802312     0.96282355  -0.00717918     -0.89481153
16    0.97620054     0.97421207   0.00198847     0.98252246  -0.00632192     -3.17928702
17    0.98791056     0.98842953  -0.00051898     0.99251949  -0.00460893      8.88078919
18    0.99417644     0.99527792  -0.00110148     0.99709009  -0.00291365      2.64522496
19    0.99733542     0.99824976  -0.00091434     0.99897264  -0.00163722      1.79059992
20    0.99884009     0.99941157  -0.00057148     0.99967117  -0.00083108      1.45427293
21    0.99951889     0.99982074  -0.00030185     0.99990467  -0.00038578      1.27807294
22    0.99980957     0.99995056  -0.00014099     0.99997499  -0.00016542      1.17329616
23    0.99992797     0.99998766  -0.00005969     0.99999407  -0.00006609      1.10726776
24    0.99997393     0.99999722  -0.00002328     0.99999873  -0.00002479      1.06487610
25    0.99999096     0.99999943  -0.00000847     0.99999975  -0.00000879      1.03787583
26    0.99999699     0.99999990  -0.00000290     0.99999996  -0.00000296      1.02114438
27    0.99999904     0.99999998  -0.00000094     0.99999999  -0.00000095      1.01120393
28    0.99999971     1.00000000  -0.00000029     1.00000000  -0.00000029      1.00560313
29    0.99999991     1.00000000  -0.00000009     1.00000000  -0.00000009      1.00263366
30    0.99999998     1.00000000  -0.00000002     1.00000000  -0.00000002      1.00116000
31    0.99999999     1.00000000  -0.00000001     1.00000000  -0.00000001      1.00047776
32    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00018375
33    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00006592
34    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00002249
35    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000409
36    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000000
37    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000000
38    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000000
39    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000000
40    1.00000000     1.00000000   0.00000000     1.00000000   0.00000000      1.00000000
...
> # np = 10 < n(1-p) , ok by rule of thumb, and the errors are sort of small (at most .02)
```