Math 3070 § 1. Treibergs $\sigma\tau$ | **Second Midterm Exam** | Name: Solutions October 20, 2010

1. *Ball Bearings of Blandings Company manufactures ball bearings whose diameters are normally distributed with mean 2.5060 cm and standard deviation of .0125 cm. Specifications call for the diameter to be at least 2.4900 cm. What proportion of the ball bearings will meet the specifications? What diameter is the 80% percentile of diameters?*

   Let X be the diameter which is normally distributed with mean $\mu = 2.5060$ and standard deviation $\sigma = .0125$. Then the probability that the bearing meets spec is (from table A3 on p. 668)

$$P(X \ge 2.4900) = 1 - P(X < 2.4900) = 1 - P\left(Z = \frac{X-\mu}{\sigma} < \frac{2.4900 - 2.5060}{.0125}\right)$$
$$= 1 - \Phi(-1.28) = 1.0000 - .1003 = .8997.$$

   The 80th percentile z-score is defined by $.800 = \Phi(z_{80\%})$. Since the same table gives $\Phi(0.84) = .7995$ and $\Phi(0.85) = .8023$, by interpolating, $z_{80\%} = 0.840 + (0.85 - 0.84)(.8000 - .7995)/(.8023 - .7995) = .842$ so $x_{80\%} = \mu + \sigma z_{80\%} = 2.5060 + (.0125)(.842) = \boxed{2.5165}$.

2. *Suppose that a random full time student at the U has registered for X courses this semester. What is the probability that a student takes fewer than five courses? What is the probability that a student is not taking at least four couses? Find $E(X)$. If the registrar charges a surcharge of \$10 to any student taking six or more courses, what is the average surcharge paid? Assume that X has the following pmf*

x	3	4	5	6	7
$p(x)$	.30	.40	.15	.10	.05

   The probability that the student takes fewer than five courses is $P(X < 5) = p(3) + p(4) = .30 + .40 = \boxed{0.70}$. The probability that the student does not take at least four courses is $P(\{X \ge 4\}^c) = P(X < 4) = p(3) = \boxed{0.30}$. $E(X) = \sum_{x=3}^{7} xp(x) = 3(.30) + 4(.40) + 5(.15) + 6(.10) + 7(.05) = \boxed{4.2}$. The surcharge function is $f(x) = 0$ if $x \le 5$ and $f(x) = 10$ if $x \ge 6$. The expected surcharge is $E(f(X)) = \sum_{x=3}^{7} f(x)p(x) = 0(.30) + 0(.40) + 0(.15) + 10(.10) + 10(.05) = \boxed{1.50}$ dollars.

3. *The probability that a student pilot passes the written test for a private pilot's license is 0.7. What is the probability that the student will pass the test on the third try? Before the fourth try?*

   Let X be the number of failed attempts at passing the test before the student passes. This is a negative binomial or geometric variable with $x \in \{0, 1, 2, 3, \ldots\}$, $p = P(\text{success}) = 0.7$ and the number of successes to to observe $r = 1$. Thus the pmf is $\text{nb}(x; 1, p) = (1-p)^x p$. The probability P that the student passes on the third try means that there were $x = 2$ failed attempts or $P = nb(2, ; 1, .7) = (.3)^2(.7) = \boxed{0.063}$. The probability that the student passes before the third try is that there were two or fewer failed attmpts, so $P = P(X \le 2) = nb(0, ; 1, .7) + nb(1, ; 1, .7) + nb(2, ; 1, .7) = (.3)^0(.7) + (.3)^1(.7) + (.3)^2(.7) = \boxed{0.973}$.

4. *Let X be the proportion of mail order customers that respond to a certain mail order solicitation. Suppose that its pdf is given by*

$$f(x) = \begin{cases} \frac{3}{2} - x, & \textit{if } 0 < x < c; \\ \\ 0, & \textit{otherwise.} \end{cases}$$

For what c is $f(x)$ a pdf? What is the median proportion of respondents? What is the expected proportion of respondents?

To be a pdf, we need $f(x) \geq 0$ so $c \leq \frac{3}{2}$ and the total probability is one. Thus

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^c \frac{3}{2} - x\, dx = \left[\frac{3x}{2} - \frac{x^2}{2}\right]_0^c = \frac{3c}{2} - \frac{c^2}{2}.$$

Equivalently

$$0 = c^2 - 3c + 2 = (c-2)(c-1)$$

so $c \in \{1, 2\}$. Only $c = 1$ gives $f(x) \geq 0$ so $c = 1$.

The cdf is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$ so

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ \int_0^x \frac{3}{2} - x\, dx = \frac{3x}{2} - \frac{x^2}{2}, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x \geq 1. \end{cases}$$

The median $\tilde{\mu}$ satisfies

$$\frac{1}{2} = F(\tilde{\mu}) = \frac{3\tilde{\mu}}{2} - \frac{\tilde{\mu}^2}{2}$$

so $\tilde{\mu}^2 - 3\tilde{\mu} + 1 = 0$ so $\tilde{\mu} = \dfrac{3 - \sqrt{5}}{2}$. (The "+" root is greater than one.)

The expected proportion is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}.$$

5. *A common practice among airline companies is to sell more tickets than actual seats to a particular flight because customers who buy tickets do not always show up for the flight. Suppose that the percentage of no shows at flight time is 2%. For a particular flight with 197 seats, a total of 200 tickets were sold. What is the exact probability that the airline overbooked the flight (more ticket purchasers showed up for the flight than the number of seats)? (You don't need to evaluate.) Use an approximation to compute the probability. Explain why your approximation is valid.*

The probability of a success (no show) is .02. If X is the number of no shows, it is a binomial random variable with $p = 0.02$, and $n = 200$ the number of tickets sold. The plane is overbooked if the number of people who show up exceed the number of seats $200 - X > 197$, or $X < 3$. The probability is then the cdf

$$P(X < 3) = P(X \leq 2) = \text{Bin}(2, .02, 200) = \sum_{i=0}^{2} \binom{200}{i} (0.02)^i (0.98)^{200-i}.$$

In this case, $\lambda = np = 4$. We may use the Poisson approximation, according to the rule of thumb on p. 122, because $n = 200 > 50$ and $np = 4 < 5$. Thus, according to the Poisson cdf on p. 667, $P(X \leq 2) \approx F(2, 4) = \boxed{.238}$.