Math 3070 § 1. **First Midterm Exam** Name: SAMPLE

Treibergs $\sigma\tau$ January 31, 2000

(1.) Compute the sample mean $\bar{\mathbf{x}}$ and sample standard deviation $\mathbf{s}$ for the January mean temperatures (in F°) for Seattle from 1900 to 1907 (tabulated below.) Compute the sample median $\tilde{\mathbf{x}}$ and sample fourth spread $\mathbf{f}_s$. What is the advantage of using $\tilde{\mathbf{x}}$ and $\mathbf{f}_s$ over using $\bar{\mathbf{x}}$ and $\mathbf{s}$? Without resumming, what would converted **median** $\tilde{\mathbf{x}}$ and sample standard deviation $\mathbf{s}$ be if the same data were coverted to Celsius (C°)? $\left[T_C = \frac{5}{9}(T_F - 32)\right]$

43.8 40.2 39.8 42.9 41.8 40.8 42.4 33.8

(2.) An urn contains 26 tiles labelled A through Z. How many different three letter words can be made by choosing three tiles in order from the urn without replacement? (Each word would be made up of three different letters.) What is the probability that there is at least one vowel $\{A, E, I, O, U\}$ in a random three letter word chosen this way?

(3.) A homework assignment consits of 24 problems. In order to save time, the instructor corrects only eight of them, which he selects at random. If in your assignment only one problem is wrong, what is the probability that it would be included among those selected for correction?

(4.) A system can experience two types of defects. Let A_i, $(i = 1, 2)$ denote the event that the system has a defect of type i. Suppose that

$$P(A_1) = .5, \qquad P(A_2) = .6, \qquad P(A_1 \cup A_2) = .9r$$

What is the probability that the system has both type 1 and type 2 defects? What is the probability that there are no defects? Given that the system has at least one type of defect, what is the probability that it has exactly one type of defect?

(5.) The Go-Utes company, which manufactures widgets, has three assembly plants, one in Provo, one in Price and one in Payson. Of those widgets manufactured in Provo, 3% needed rework to correct a defect, whereas 5% of those manufactured in Price needed rework and 8% of those made in Payson needed rework. Suppose 45% of all widgets are manufactured in the Provo plant, 35% in Price, and 20% in Payson. If a randomly selected widget needed rework, what is the probability that it came from the Provo plant?

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Solutions to the Sample Exam.

(1.) Compute the sample mean $\bar{\mathbf{x}}$ *and sample standard deviation* $\mathbf{s}$ *for the January mean temperatures (in* F°*) for Seattle from 1900 to 1907 (sorted data below.) Compute the sample median* $\tilde{\mathbf{x}}$ *and sample fourth spread* $\mathbf{f}_s$*. What is the advantage of using* $\tilde{\mathbf{x}}$ *and* $\mathbf{f}_s$ *over using* $\bar{\mathbf{x}}$ *and* $\mathbf{s}$*? Without resumming, what would converted* **median** $\tilde{\mathbf{x}}$ *and sample standard deviation* $\mathbf{s}$ *be if the same data were coverted to Celsius (*C°*)?*
$\left[T_C = \frac{5}{9}(T_F - 32)\right]$ 33.8 39.8 40.2 40.8 41.8 42.4 42.9 43.8
There are $n = 8$ observations. The sample mean is $\bar{x} = \frac{1}{n}\sum_{k=1}^n x_k = \frac{1}{8}(43.8 + \cdots + 33.8) = 325.5/8 = \boxed{40.6875}$. The sample variance

$$s^2 = \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{1}{n}\left(\sum_{k=1}^n x_k\right)^2\right) = \frac{1}{7}\left(13311.01 - \frac{(325.5)^2}{8}\right) = 9.6041.$$

Thus, the sample standard deviation is $s = \boxed{3.099}$. The sample median is $\tilde{x} = \frac{1}{2}(x_4 + x_5) = \frac{1}{2}(40.8 + 41.8) = \boxed{41.3}$. The quartile scores are the medians of the first half and the second half of the data. As there are four in each half, the quartile scores are $q_1 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(39.8 + 40.2) = \boxed{40.0}$ and $q_3 = \frac{1}{2}(x_6 + x_7) = \frac{1}{2}(42.4 + 42.9) = \boxed{42.65}$. The fourth spread $f_s = q_3 - q_1 = 42.65 - 40.0 = \boxed{2.65}$. The median and fourth spread have the advantage of not being so sensitive to outliers. Under the transformation $y = \frac{5}{9}x - \frac{160}{9}$, the sample mean is $\bar{y} = \frac{5}{9}(\bar{x} - 32) = 4.826$ and $s_y = \frac{5}{9}s_x = \frac{5}{9}(3.099) = \boxed{1.722}$ As the change of variables is linear with positive coefficient, it preserves order and linear combination. Thus $\tilde{y} = \frac{5}{9}(\tilde{x} - 32) = \frac{5}{9}(41.3 - 32) = \boxed{5.167}$.
(2.)*An urn contains 26 tiles labelled A through Z. How many different three letter words can be made by choosing three tiles in order from the urn without replacement? (Each word would be made up of three different letters.) What is the probability that there is at least one vowel* $\{A, E, I, O, U\}$ *in a random three letter word chosen this way?*

The number of three letter words is the number of permutations of 26 tiles taken three at a time, or $P_{26,3} = 26 \cdot 25 \cdot 24 = \boxed{15600}$. The event E that "there is at least one vowel" is complementary to the event E' that "there are no vowels." But $P(E')$ is the number of words without a vowel (the number of permutations of the consonants taken three at a time) divided by the number of all words. Thus $P(E) = 1 - P(E') = 1 - \frac{P_{21,3}}{P_{26,3}} = 1 - \frac{7980}{15600} = \boxed{.333}$.
(3.) *A homework assignment consits of 24 problems. In order to save time, the instructor corrects only eight of them, which he selects at random. If in your assignment only one problem is wrong, what is the probability that it would be included among those selected for correction?*

The event A that an incorrect problem is chosen is complementary to the event A' that only correct problems are chosen. Since you have 23 correct problems, there are $C_{23,8} = \binom{23}{8} = \frac{23\cdot 22\cdot\ldots\cdot 16}{8!} = 490314$ ways to do this. The number of ways of choosing problems is $C_{24,8} = \binom{24}{8} = \frac{24!}{8!16!} = 735471$. thus $P(A) = 1 - P(A') = 1 - \frac{C_{23,8}}{C_{24,8}} = 1 - \frac{490314}{735471} = \boxed{0.333}$. (As expected: she's grading 8 of 24 so the chances she grades the bad problem is 8/24.)
(4.) *A system can experience two types of defects. Let* A_i*, (*$i = 1, 2$*) denote the event that the system has a defect of type* i*. Suppose that*

$P(A_1) = .5, \quad P(A_2) = .6, \quad P(A_1 \cup A_2) = .9$

What is the probability that the system has both type 1 and type 2 defects? What is the probability that there are no defects? Given that the system has at least one type of defect, what is the probability that it has exactly one type of defect?

The event of both defects is $A_1 \cap A_2$. Using the formula for union, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Plugging values, $.9 = .5 + .6 - P(A_1 \cap A_2)$ so $P(A_1 \cap A_2) = \boxed{0.2}$. The event that there are no defects is $A_1' \cap A_2' = (A_1 \cup A_2)'$, by deMorgan's Law. Thus $P(A_1' \cap A_2') = P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2) = 1 - 0.9 = \boxed{0.1}$. At least one defect is the event $B = A_1 \cup A_2$. The event that there is exactly one defect is $C = (A_1 \cap A_2') \cup (A_1' \cap A_2)$. Since these are mutually exclusive, $P((A_1 \cap A_2') \cup (A_1' \cap A_2) = P(A_1 \cap A_2') + P(A_1' \cap A_2) = .3 + .4 = .7$ since $P(A_1 \cap A_2') = P(A_1) - P(A_1 \cap A_2) = .5 - .2 = .3$ and $P(A_1' \cap A_2) = P(A_2) - P(A_1 \cap A_2) = .6 - .2 = .4$. Note $C \subset B$ so $C \cap B = C$. Finally $P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C)}{P(B)} = \frac{.7}{.9} = \boxed{.778}$.
(5.) *The Go-Utes company, which manufactures widgets, has three assembly plants, one in Provo, one in Price and one in Payson. Of those widgets manufactured in Provo, 3% needed rework to correct a defect,*

whereas 5% of those manufactured in Price needed rework and 8% of those made in Payson needed rework. Suppose 45% of all widgets are manufactured in the Provo plant, 35% in Price, and 20% in Payson. If a randomly selected widget needed rework, what is the probability that it came from the Provo plant?

We seek $P($ from Provo $|$ defective $)$. Use Bayes formula (or draw the chart.)

$$P(\text{ Provo } | \text{ def. }) = \frac{P(\text{ def } | \text{ Provo })P(\text{ Provo })}{P(\text{ def } | \text{ Provo })P(\text{ Provo }) + P(\text{ def } | \text{ Payson })P(\text{ Payson }) + P(\text{ def } | \text{ Price })P(\text{ Price })}$$
$$= \frac{(.03)(.45)}{(.03)(.45) + (.05)(.35) + (.08)(.20)} = \boxed{0.287}$$

Some solved problems similar to ones that may occur on the exam.

(6.) *Four married couples bought 8 seats in the same row for a concert. In how many different ways can they be seated? In how many different ways can they be seated so that the men and women alternate seats? (No woman sits next to a woman and no man sits next to a man?) If the eight people take their seats randomly, what is the probability that either some woman ends up sitting next to a woman or a man next to a man?*

The number of seating orderings is the number of permutations of 8 or $N = 8! = 40320$. The only way to take alternate seats is if the women sit in positions 1,3,5,7 and the men in 2,4,6,8 or the other way around. Thus there is either a man or woman in the first seat; there are all possible permutations of the four women in their four seats and there are all permutations of the four men for the remaininig seats. Thus by the product formula there are $n = 2 \times 4! \times 4! = 1152$ ways to take alternate seats. The event E that either some woman ends up next to a woman or a man next to a man or both is complementary to the event E' that the men and women alternate. Thus, assuming that each of the permutations is equally likely, $P(E) = 1 - P(E') = 1 - n/N = 1 - (1152/40320) = .971$.

(7.) *Suppose that* 80% *of Utah voters are Republicans and the rest are Democrats. If* 95% *of the Republicans voted for Governor Leavitt and* 40% *of the Democrats voted for Gov. Leavitt, what is the probability that a random Utah voter voted for Gov. Leavitt? If you know that a voter voted for the governor, what is the probability that she is a Republican?*

Let A denote the event that a Utah voter is Republican and B the event that the voter voted for the governor. We are given that $P(A) = .8$ so $P(A') = .2$. We are also given that $P(B|A) = .95$ and $P(B|A') = .4$. Thus the union $B = (A \cap B) \cup (A' \cap B)$ consists of mutually exclusive events, so the probability that a random voter voted for the governor is $P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B|A) + P(A')P(B|A') = (.8)(.95) + (.2)(.4) = .84$. If we are given that the voter voted for the governor, the chances that she is Republican is $P(A|B) = P(A \cap B)/P(B) = P(A)P(B|A)/P(B) = (.8)(.95)/(.84) = .905$.

(8.) *A standard deck of cards has 52 cards, thirteen cards* $\{2 - 10, J, Q, K, A\}$ *in each of four suits* $\clubsuit$, $\diamondsuit$, $\heartsuit$ *and* $\spadesuit$. *Five cards are drawn at random without repalcement. What is the probability drawing aRoyal Flush? (10,J,Q,K,A of the same suit.) What is the probability of drawing a flush? (All five cards of the same suit.) What is the probability of of drawing a full house? (three of one kind and a pair of another, e.g.* $\{8\clubsuit, 8\heartsuit, 8\spadesuit, K\diamondsuit, K\heartsuit\}$*.)*

We assume that each hand is equally likely. There are a total of $N = \binom{52}{5} = 2,598,960$ five card hands (order not important), drawn without replacement. There are $n_1 = 4$ royal flushes, only one $A, K, Q, J, 10$ per suit, so the probability of drawing a royal flush is $P = n_1/N = 1.539 \times 10^{-6}$. The number of flushes is $n_2 = 4\binom{13}{5} = 5,148$, the number of choices of suit times number of five card hands in a suit so the probability of a flush is $P = n_2/N = .00198$. The number of hull houses is $n_3 = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$, the number of ways of choosing a kind for the three times the number of subsets of three in four suits times the number of remaining kinds for the pair times the number of pairs in four suits, so the probability of a hull house is $n_3/N = .00144$.

(9.) *The following data are weights (correct to the nearest .001 lb.) of 27 "one pound" packages of grapes. (after sorting. Call the sorted observations* x_i, $i = 1, \ldots, 27$.*) (a.) Find reasonable class boundaries for this data and construct a histogram. (b.) Find the sample mean, sample standard deviation, sample median, sample range, sample lower fourth, sample upper fourth, sample fourth spread of the data. (c.) Determine whether any observations from this data are mild outliers, or extreme outliers. Draw a box plot for the data. Be sure to indicate the mild and extreme outliers (if any) on your box plot.*

0.973	0.981	0.985	0.988	0.990	0.994	0.996	0.997	1.000
1.002	1.002	1.003	1.007	1.009	1.009	1.010	1.012	1.013
1.013	1.017	1.018	1.020	1.024	1.025	1.028	1.031	1.043

Since there are $n = 27$ observations, the best histogram would have about $\sqrt{27} = 5.20$ classes. Since the observations vary from .973 to 1.043, then it is reasonable to divide the data according to $.97- < .98- < .99- < 1.00- < 1.01- < 1.02- < 1.03 < -1.04- < 1.05$

The sample mean $= \bar{\mathbf{x}} = \frac{1}{n}\sum_i^n x_i = \frac{27.19}{27} = 1.0070$ rounded from 1.007037037. The sample variance $\mathbf{s}^2 = \frac{1}{n-1}\left(\sum_{i=1}^n x_i^2 - \frac{1}{n}\left[\sum_{i=1}^n x_i\right]^2\right) = \frac{1}{26}\left(27.388238 - \frac{(27.19)^2}{27}\right) = 0.000265422$ so $s = 0.0163$. The sample median is the middle observation $\tilde{\mathbf{x}} = x_{14} = 1.009$. The sample range $\mathbf{r} = \max - \min = x_{27} - x_1 = 1.043 - .973 = .070$. Since n is odd, the sample lower fourth and the sample upper fourth are defined as the medians of the lower and upper $\frac{n+1}{2} = 14$ observations. Namely, $\mathbf{f}_\ell = \text{median}\{x_1, \dots, x_{14}\} = \frac{1}{2}(x_7 + x_8) = \frac{0.996+0.997}{2} = .9965$ and $\mathbf{f}_u = \text{median}\{x_{14}, \dots, x_{27}\} = \frac{1}{2}(x_{20} + x_{21}) = \frac{1.017+1.018}{2} = 1.0175$. Thus the sample fourth spread of the data is $\mathbf{f}_s = \mathbf{f}_u - \mathbf{f}_\ell = 1.0175 - .9965 = .0210$.

An extreme outlier is any observation such that $x > 3\mathbf{f}_s + \mathbf{f}_u = (3)(.0210) + 1.0175 = 1.0805$ or $x < -3\mathbf{f}_s + \mathbf{f}_\ell = -(3)(.0210) + .9965 = .9335$. None of the observations in our data set are extreme outliers. A mild outlier is an observation which is not extreme but which satisfies $x > 1.5\mathbf{f}_s + \mathbf{f}_u = (1.5)(.0210) + 1.0175 = 1.049$ or $x < -1.5\mathbf{f}_s + \mathbf{f}_\ell = -(1.5)(.0210) + .9965 = .965$. None of our observations are mild outliers either. Thus a box plot has a box from $\mathbf{f}_\ell$ to $\mathbf{f}_u$, a line at $\tilde{\mathbf{x}}$ and whiskers extending left to $x_1 = .973$ and right to $x_{27} = 1.043$.

(10.) *Using interpolation, find the 10% trimmed mean of the sample in Problem 1.*

Since $\frac{2}{27} < \frac{1}{10} < \frac{3}{27}$ we find the trimmed means by discarding the two and three lowest and highest values to get $\bar{\mathbf{x}}_{\text{tr}[2/27]} = \text{mean}\{x_3, \dots, x_{25}\} = \frac{1}{n-4}\sum_{i=3}^{25} x_i = \frac{23.162}{23} = 1.007043478$ and $\bar{\mathbf{x}}_{\text{tr}[3/27]} = \text{mean}\{x_4, \dots, x_{24}\} = \frac{1}{n-6}\sum_{i=4}^{24} x_i = \frac{21.149}{21} = 1.007095238$. To do the interpolation, since $\frac{1}{10} = \frac{3}{10}\frac{2}{27} + \frac{7}{10}\frac{3}{27}$ it follows that $\bar{\mathbf{x}}_{\text{tr}[1/10]} = (.3)\bar{\mathbf{x}}_{\text{tr}[2/27]} + (.7)\bar{\mathbf{x}}_{\text{tr}[3/27]} = (.3)(1.007043478) + (.7)(1.007095238) = 1.00707971$.

(11) *Suppose for a sample of $n = 11$ observations, $\sum_{i=1} x_i = 165$ and $\sum_{i=1} x_i^2 = 2585$, what is the sample mean and sample variance? If all the observations are multiplied by 2, what are the resulting $\sum_{i=1} x_i$, $\sum_{i=1} x_i^2$, the sample mean and sample variance? Suppose a new data set consists of the same numbers as from part a.) but repeated twice making the sample size 22. What are the resulting $\sum_{i=1} x_i$, $\sum_{i=1} x_i^2$, the sample mean and sample variance for the new data set?*

The sample mean $= \bar{\mathbf{x}} = \frac{1}{n}\sum_i^n x_i = \frac{165}{11} = 15.0$. The sample variance

$\mathbf{s}^2 = \frac{1}{n-1}\left(\sum_{i=1}^n x_i^2 - \frac{1}{n}\left[\sum_{i=1}^n x_i\right]^2\right) = \frac{1}{10}\left(2585 - \frac{165^2}{11}\right) = 11.0$. If $y_i = 2x_i$ then $\sum_i^n y_i = \sum_i^n 2x_i = 2\times 165 = 330$ so the sample mean $= \bar{\mathbf{y}} = \frac{1}{n}\sum_i^n y_i = \frac{2\times 330}{11} = 30.0$ and $\sum_i^n y_i = \sum_i^n (2x_i)^2 = 4\times 2585 = 10340$ and the sample variance

$\mathbf{s}^2 = \frac{1}{n-1}\left(\sum_{i=1}^n y_i^2 - \frac{1}{n}\left[\sum_{i=1}^n y_i\right]^2\right) = \frac{1}{n-1}\left(4\sum_{i=1}^n x_i^2 - \frac{1}{n}\left[2\sum_{i=1}^n x_i\right]^2\right) = 4\times 11.0 = 44.0$. If each observation is repeated twice (call it z_i, $i = 1, \dots, 22$) then $\sum_i^{2n} z_i = 2\sum_i^n x_i = 330$ and $\sum_i^{2n} z_i^2 = 2\sum_i^n x_i^2 = 5170$. The resulting sample variance becomes

$\mathbf{s}^2 = \frac{1}{2n-1}\left(\sum_{i=1}^{2n} z_i^2 - \frac{1}{2n}\left[\sum_{i=1}^{2n} z_i\right]^2\right) = \frac{1}{21}\left(5170 - \frac{330^2}{22}\right) = 10.4762.$

(12) *Four coins, a penny, a nickle, a dime and a quarter are tossed simultaneously resulting in a head (H) or a tail (T) for each. What is the sample space? Give an example of two mutually exclusive events. What is the probability that the coins that land (H) add up to at least 14¢ ?*

The sample space is the totality of coin outcomes $q_1q_2q_3q_4$ where q_i is (H) or (T) and $i = 1$ for penny, $i = 2$ for nickle, $i = 3$ for dime and $i = 4$ for quarter. We list the sample space

$$S = \begin{Bmatrix} HHHH & HHHT & HHTH & HHTT & HTHH & HTHT & HTTH & HTTT \\ THHH & THHT & THTH & THTT & TTHH & TTHT & TTTH & TTTT \end{Bmatrix}$$

An example of two mutually exclusive events are $A = \{\text{the penny is H } (q_1 = H)\}$ and $B = \{\text{all coins are T } (q_1 = q_2 = q_3 = q_4 = T)\}$ because the only way an observation is in both event A and event B is that the penny is H and that all coins are T, in particular the penny is not H, a contradiction. So no observation is in both events.

The event $C = \{\text{coins with H add up to at least 14¢}\} =$

$$C = \left\{ \begin{matrix} HHHH & HHHT & HHTH & HTHH & HTTH \\ THHH & THHT & THTH & TTHH & TTTH \end{matrix} \right\}$$

We assume that all outcomes are equally likely. Thus the probability is gotten by counting $P(C) = \frac{N(C)}{N(S)} = \frac{10}{16} = .625$.

(13.) *An urn contains three red balls and seven green balls. Suppose four are withdrawn at random without replacement. What is the probability that an odd number of red balls are drawn?*

We assume that any subset of four balls taken from the urn is equally likely. the number of subsets of four taken from all 3+7=10 balls is $\binom{10}{4}$. The subsets that contain an odd number of red balls contain either one or three red balls. The number of subsets of four that contain exactly one red ball is the number of ways that one red ball may be chosen from the three $\binom{3}{1}$ times the number of ways the remaining three balls may be chosen from the seven greens, or $\binom{7}{3}$. The number of subsets of four that contain exactly three red balls is the number of ways that three red ball may be chosen from the three $\binom{3}{3}$ times the number of ways the remaining ball may be chosen from the seven greens, or $\binom{7}{1}$. Hence

$$P = \frac{\binom{3}{1}\binom{7}{3} + \binom{3}{3}\binom{7}{1}}{\binom{10}{4}} = \frac{3 \cdot 35 + 1 \cdot 7}{210} = .533333$$

(14) *Using just the axioms of probability and deMorgan's laws, show that*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cup B \cup C).$$

We begin by finding the formula for the union of two sets. Because $A \cup B = A \cup (A' \cap B)$ is the union of mutually exclusive events, $P(A \cup B) = P(A \cup (A' \cap B)) = P(A) + P(A' \cap B)$. Because $B = (A \cap B) \cup (A' \cap B)$ is the union of mutually exclusive events, $P(B) = P((A \cap B) \cup (A' \cap B)) = P(A \cap B) + P(A' \cap B)$. Eliminating $P(A' \cap B)$ from the last two equations gives $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Hence, viewing $B \cap C$ as one set, $P(A \cup B \cup C) = P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$. deMorgans law says $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Thus using the two set formula,

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C)) = P(A) + P(B) + P(C) - P(B \cap C) - \\ & - \{P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))\} = \\ =& P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C). \end{aligned}$$

(15.) *On my drive to work along the usual route, there is a probability of .4 that I have to stop at a signal light at 17th street and a probability of .5 that I have to stop at the light at 5th street, and a probability of .6 that I have to stop at least at one of these two streets. What is the probability that I have to stop at both signals? At the second but not at the first? At exactly one of the two signals?*

Let the events $A = \{\text{stopped at 17th st.}\}$ and $B = \{\text{stopped at 5th st.}\}$. We are given that $P(A) = .4$, $P(B) = .5$ and $P(A \cup B) = .6$. Solving the two set union formula, and substituting, the probability of stopping at both signals is

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = .4 + .5 - .6 = .3$.

The probability of stopping at the second but not the first is $P(A' \cap B)$. Because $B = (A \cap B) \cup (A' \cap B)$ is the union of mutually exclusive events, we find $P(B) = P(A \cap B) + P(A' \cap B)$ or $P(A' \cap B) = P(B) - P(A \cap B) = .5 - .3 = .2$. Similarly, the event of being stopped first but not at the second is $P(A \cap B') = P(A) - P(A \cap B) = .4 - .3 = .1$. Finally since the event of being stopped at exactly one signal is the union of mutually exclusive events of being stopped at the second but not the first with being stopped at the first but not the second, its probability is $P((A' \cap B) \cup (A \cap B')) = P(A' \cap B) + P(A \cap B') = .2 + .1 = .3$.

(16.) *70% of the light aircraft that disappear while in flight are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared. If it has an emergency locator, what is the probability that it will not be discovered? If it does not have an emergency locator, what is the probability it will be discovered?*

Let S be the sample space consisting of light aircraft which disappear while in flight. Let A denote the event that the disappeared aircraft is discovered and B the event that the disappeared aircraft has an emergency locator. We are given that $P(A) = .7$ so $P(A') = .3$, $P(B|A) = .6$ so $P(B'|A) = .4$ and $P(B'|A') = .9$, hence $P(B|A') = .1$. Given that a disappeared aircraft has an emergency locator, the probability that it will not be discovered is $P(A'|B)$. Bayes' theorem says

$$P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B|A')P(A')}{P(B|A)P(A) + P(B|A')P(A')} = \frac{(.1)(.3)}{(.6)(.7) + (.1)(.3)} = .0667.$$

Given that a dissappeared aircraft does not have an emergency locator, the probability it will be discovered is

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(B'|A)P(A)}{P(B'|A)P(A) + P(B'|A')P(A')} = \frac{(.4)(.7)}{(.4)(.7) + (.9)(.3)} = .5091.$$

(17.) *One year when Utah was still on the quarter system, 500 students took Math 307 in the fall, 200 in the winter and 300 in the spring. Endquarter surveys revealed that 220 students were satisfied with the course in the fall, 170 in the winter and 180 in the spring. If a student who took Math 307 that year is selected at random and admits to having been satisfied by the course, is the student most likely to have taken the course in the fall, winter or spring?*

Let the sample space consist of all students who took Math 307 that year. Let A_i denote the event that the student took Math 307 in the ith quarter, $i = 1, 2, 3$. These A_i are mutually exclusive and exhustive events. Let B be the event that the student is satisfied with the course. We are given that 500+200+300=1000 students took the course, and so $P(A_1) = .5$, $P(A_2) = .2$ and $P(A_3) = .3$. We are also given that $P(B|A_1) = \frac{220}{500} = .44$, $P(B|A_2) = \frac{170}{200} = .85$ and $P(B|A_3) = \frac{180}{300} = .6$. We are asked to compute and compare $P(A_j|B)$, the conditional probability that the student took the course in the jth quarter, given that the student was satisfied. Bayes formula says

$$\begin{aligned}(P(A_1|B), P(A_2|B), P(A_3|B)) =& \frac{(P(B|A_1)P(A_1), P(B|A_2)P(A_2), P(B|A_3)P(A_3))}{\sum_{i=1}^{3} P(B|A_i)P(A_i)} \\ =& \frac{((.44)(.5), (.85)(.2), (.6)(.3))}{(.44)(.5) + (.85)(.2) + (.6)(.3)} = \frac{(.22, .17, .18)}{.57} = (.386, .298, .316);\end{aligned}$$

so that the student is most likely to have taken the course in fall ($j = 1$) which has the largest conditional probability.

(18.) *A boiler has five identical relief valves. The probability that any one valve will open is .95. Assuming independent operation of valves, compute* $P($*at least one valve opens*$)$.

If A_i denotes the event that the ith valve opens, for $i = 1, \ldots, 5$, then we are asked to compute $P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$, the probability that at least one valve opens. The complementary event (de Morgan's laws again!) is

$(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)' = A_1' \cap A_2' \cap A_3' \cap A_4' \cap A_5'$

that none of the valves open. We are assuming $P(A_i) = .95$ so $P(A_i') = .05$. We are assuming that the Ai' are independent events. It follows that the A_i' are independent and we may compute

$$\begin{aligned}P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) =& 1 - P\left((A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)'\right) \\ =& 1 - P\left(A_1' \cap A_2' \cap A_3' \cap A_4' \cap A_5'\right) \\ =& 1 - P(A_1')P(A_2')P(A_3')P(A_4')P(A_5') \\ =& 1 - (.05)^5 \\ =& 1 - .0000003125 = .9999996875.\end{aligned}$$